BONDO UNIVERSITY COLLEGE

SCHOOL OF MATHEMATICS AND ACTUARIAL SCIENCE

FIRST YEAR FIRST SEMESTER EXAMINATION FOR THE DEGREE OF MASTER OF SCIENCE IN PURE MATHEMATICS

SMA 811: GROUP THEORY

INSTRUCTIONS:

1. This paper consists of FIVE questions

2. Attempt any THREE questions.

3. Observe further instructions on the answer booklet.

QUESTION 1 **[20 Marks]**

(a) Describe the structure of the quaternion group. **[5 mks]**

(b) Verify that the dihedral group is of order $2n$, for $n \geq 3$. **[5 mks]**

(c) Show that there is a canonical injective homomorphism $\alpha : G \rightarrow \text{Sym}(G)$. **[5 mks]**

(d) With the aid of an example, show that there exists a subgroup H of a group G and an element $g \in G$ such that $gHg^{-1} \subset H$ but $gHg^{-1} \neq H$. **[5 mks]**

QUESTION 2 **[20 Marks]**

(a) **i)** Let a and b be elements of a group G. If a has order m and b has order n, what can we say about the order of ab? **[8 mks]**

ii) Consider the elements $a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $a = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ in $\text{GL}_2(\mathbf{Z})$. Show that $a^4 = 1$ and $b^3 = 1$, but that ab has infinite order, and hence that the group $< a, b >$ is infinite.

[3 mks]

(b) Let G be a group and X be a subset of G. Show that for any $g \in G$, and $x \in X$, $\text{Stab}(gx) = g.\ \text{Stab}(x).g^{-1}$. **[3 mks]**

(c) Show that every group of order $2p$, p an odd prime, is cyclic or dihedral. **[6 mks]**

QUESTION 3 **[20 Marks]**

(a) Let $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$, the field with p elements, and let $G = \text{GL}_n(\mathbf{F}_p)$. Find a Sylow p-subgroup of G. **[4 mks]**

(b) Let p and q be prime integers such that $p < q$. Describe the groups of order pq. **[8 mks]**

(c) **i)** With the aid of an example, describe a metabelian group **[3 mks]**

ii) Show that a subgroup of a nilpotent group is nilpotent. **[5 mks]**

QUESTION 4 **[20 Marks]**

(a) Consider the subgroups $B = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\}$ and $U = \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\}$ of $\mathrm{GL}_2(\mathbf{F})$, for some field **F.** Show that U is a normal subgroup of B and that B is solvable. **[4 mks]**

(b) State and prove the Jordan-Holder Theorem. **[7 mks]**

(c) Classify all groups of order 30. **[9 mks]**

QUESTION 5 **[20 Marks]**

(a) With the aid of at least two examples, give an account of matrix representation of a finite group G over a field **F.**

[6 mks]

(b) Show that the dimension of the center of $\mathbf{F}[G]$ as an $\mathbf{F}-$ vector space is the number of conjugacy classes in G. **[7 mks]**

(c) Show that two $\mathbf{F}[G]$ modules are isomorphic iff their characters are equal. Is this result true if $\mathbf{F}$ is allowed to have characteristic $p \neq 0$? Explain. **[7 mks]**