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Formulating Black Scholes equation using a jump diffusion Heston's model

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Abstract
In modern financial mathematics, accurate values are obtained by taking into account a considerable number of more realistic assumptions in logistic Black Scholes equation. The aspects considered here are cost of transactions in trading, perfect illiquid markets and risks that occur from non – protected portfolio or large investments that have a lot of impact on price of the assets, volatility, the percentage drift and the life of the portfolio itself. In modern world of finance, Jump diffusion process is used to assess the behavior of non – continuous asset when pricing of options. Since the introduction of Black – Scholes concept model that assumes volatility is constant; several studies have proposed models that address the shortcomings of Black – Scholes model. Heston's models stands out amongst most volatility models because the process of volatility is greater the zero and follows mean reversion and this is what is observed in the market world. One of the shortcomings of Heston's model is that it doesn't incorporate the aspect jump diffusion process. The Black – Scholes partial differential equation that has been studied so far revolves around Geometric Brownian motion and its extensions. We therefore incorporate jump diffusion process on Heston's model and use it to formulate a new Black – Scholes equation using the knowledge of partial differential equations.

Keywords: Option, Jump diffusion, poisson distribution, volatility, geometric brownian motion, Black-Scholes formula, Heston's Model

1. Introduction

A jump diffusion model is composed of two parts; the first part is the diffusion process component that is modeled by a Brownian motion that describes the instantaneous price vibration of life of portfolio and a jump component modeled by the process of Poisson distribution describing the abnormality that can be observed in vibration of price of the portfolio. The standard Black – Scholes – equation [2] is one of the option pricing models and it is derived under the following assumptions;

- Constant volatility.
- No dividend yield.
- Interest rates are constant and known.
- The returns are log - normally distributed.
- No commission and transaction cost.
- The existence of a liquid market.

In the modern world, some of these assumptions cannot be applied in the market sector where the price of an option depends on dynamics that exist from the underlying asset to be sold incase its price increases in the market. The pricing process of stock is carried out by a standard model that follows a Geometric Brownian motion which involves a function of time given by;

$$dS_t = \mu S_t dt + \sigma S_t dZ_t \,, \qquad (1)$$

where μ is the percentage growth of the underlying portfolio, σ is the constant volatility, S_t is the price of the asset at time t and dZ_t is the Brownian motion.

A study on stochastic volatilities that involves two factor models has been carried out by Hull and White [5], Stein and Stein [11] and Heston [3] among others. Heston's model [3] stands out among these models because the processes for volatility is greater than zero and processes mean reversion which is in contrary to Black – Scholes model that assumes constant volatility. Heston's model [3] also has existence of closed form solution of vanilla options. It assumes that spot price depicts a diffusion process of the form;

$$dS_t = \mu S_t dt + \sqrt{\upsilon_t} S_t dZ_t , \tag{2}$$

where μ (linear drift rate) is a constant, υ_t is a non – constant instantaneous volatility and Z_t is a standard wiener process. Equation (2) is a process resembling Geometric Brownian Motion with a proposed volatility formed from the process of mean reversion. It is from this Heston's volatility model that we propose a diffusion process with a jump component that we use to formulate a new Black Scholes equation.

2. Preliminaries

2.1 Itô Process

It is the Wiener diffusion process in with constant parameters a and b being functions of the price of an underlying option defined by the variables P and t. It can be written mathematically in form a diffusion process given by;

$$dP = a(P,t)dt + b(P,t)dZ_t \tag{3}$$

In this case the underlying linear drift rate and volatility (variance) rate of the Itô process are prone to change over a given time frame. Considering a small time interval between t and $t+\Delta t$, that changes the from X to $X+\Delta X$ is equation (3) is thus expressed as;

$$\Delta P = a(P,t)\Delta t + b(P,t)\varepsilon\sqrt{\Delta t} \tag{4}$$

This small approximation still results in to a relationship between the drift rate and the volatility rate. It assumes that the change of the variable P is normally distributed with the drift rate and the variance rate of P remaining constant which are proven to be equal to $a(P,t)\Delta t$ and $b^2(P,t)\Delta t$ respectively during the interval between t and $t+\Delta t$. This is also denoted by $\Delta P \sim N\left(a(P,t)\Delta t, b(P,t)\sqrt{\Delta t}\right)$.

2.2 Itô Lemma

Stochastic differential equations are best solved by Itô Lemma, where wiener - like differential process are put into mathematical formulation of partial differential equations to obtain solutions of stochastic differential equations. In deriving Itô lemma we consider value of a variable P that follows an Itô process from equation (3), where P is said to have a linear drift rate of a and a percentage variance of b^2 such that from Itô lemma, it is stated that a function $G(P,t)$ that can be differentiable twice in P and once in t forms an Itô process of the form;

$$dG = \left(\frac{\partial G}{\partial P}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial P^2}b^2\right)dt + \frac{\partial G}{\partial P}bdZ , \tag{5}$$

where $\left(\frac{\partial G}{\partial P}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial P^2}b^2\right)$ is the percentage drift and a percentage variance derived by $\left(\frac{\partial G}{\partial P}\right)^2 b^2 dt$

2.3 Geometric Brownian motion

Geometric Brownian motion is a specific Itô Process following a diffusion process given by

$$dF = aFdt + bFdZ \tag{6}$$

where $a(F,t) = aF$, $b(F,t) = bF$ and Z is the standard wiener process. A geometric Brownian motion used in the application of stock pricing is given by;

$$dS = \mu Sdt + \sigma SdZ , \tag{7}$$

where S is the price of the underlying asset, μ is the expected growth rate or the rate of return of the underlying asset and σ is the percentage volatility of price of the underlying asset. Re - written equation (7) results to the following;

$$dS = \mu S dt + \sigma S dZ \tag{8}$$

A discrete time model over a given time frame t to Δt that is normally used in pricing of the underlying asset is given by;

$$\Delta S = \mu S \Delta t + \sigma S \varepsilon \sqrt{\Delta t}\,, \tag{9}$$

where ΔS is the rate of change in price of the asset S within a time interval Δt and $Z = \varepsilon\sqrt{\Delta t}$ such that ε is a random variable derived by a standardized normal distribution with mean zero and standard deviation of one [5].

2.4 Stochastic Process

Stochastic processes are activities or events whose occurrences are random over a given period of time; this makes them obey the law of probability. Mathematically it can be defined as a process represented by X_t which is a collection of random variable $[X_t : t \in T]$ found in a probability space that varies over a given set T. We define various types of stochastic processes below

2.4.1 Markov Process

It is a stochastic process where factors from the past history of the asset does not influence the behavior of the current asset price since it is believed that the current price already contain relevant information from the past history that could affect the new price of the underlying stock.

2.4.2 Wiener Process

It is a Brownian motion with mean rate of change zero and variance rate of one. It can also be defined as a random variable of value Z that follows a Wiener process with the following properties;

Property 1: Over a small period of time ΔZ is defined by;

$$\Delta Z = \varepsilon\sqrt{\Delta t} \tag{10}$$

where ε is normally distributed by mean of zero and variance of 1. That is $\varepsilon \sim N(0,1)$

Property 2: Over a two varied short time periods, ΔZ are independent. That is $\mathrm{cov}(\Delta Z_i, \Delta Z_j) = 0$, $i \neq j$ [5].

2.5 Jump Process

This occurs when the assets deviate abruptly or suddenly from its normal path due to non-systematic risk. This happens due to new information which causes negative or positive effects in asset price. The information may be firm or sector oriented. Such information represents non – systematic risk meaning that the jumps are uncorrelated with the market.

2.6 Heston's Model with Jump diffusion model

We incorporate jump diffusion process onto Heston's model of closed – form solution of vanilla options. It will therefore follow a diffusion process given by;

$$dS_t = S_t\left((\mu - \lambda k)dt + \sqrt{\upsilon_t}\, dZ_t\right) + (g-1)dN\,, \tag{11}$$

where μ (linear drift rate) is a constant, υ_t is a non – constant instantaneous volatility, λ is the rate at which jumps occurs per unit time, k is the proportional increase in price of the asset measuring the jump size, N is Poisson process that generates the jumps diffusion process and Z_t is the standard Wiener process. Andanje [1], Opondo [7] and Oduor [10].

3. Main results

3.1 Formulation of Black Scholes equation using a Heston's Jump diffusion model

We take a Heston's jump model of a diffusion process of the form;

$$dS_t = S_t((\mu - \lambda k)dt + \sqrt{\upsilon_t}dZ_t) + (g-1)S_t dN \qquad (12)$$

where μ is the percentage growth, υ_t is non - constant instantaneous volatility, λ is jump rate that occurs per unit time, k is the proportional increase in price of the asset measuring the jump size, S_t is the asset price at time t and N is Poisson process that generates the jumps diffusion process . We therefore have that dZ_t and dN are taken to be independently and identically distributed.

Assuming that over a small change of time interval dt the price of the underlying option moves from S to gS where g is absolute price jump size such that the relative jump of price of the asset is given by

$$\frac{dS}{S} = \frac{gS - S}{S} = g - 1 \qquad (13)$$

The relative price jump size of S_t and $g-1$ are log – normally distributed with a derived mean as follows;

$$E(g-1) = \exp\left(\mu + \frac{1}{2}\sigma^2\right) - 1 \equiv k$$

and a derived variance as follows;

$$E\left(g - 1 - E(g-1)^2\right)^2 = \exp\left(2\mu + \sigma^2\right)\exp(\left(\sigma^2\right) - 1)$$

Merton [6]

The assumption that the absolute price jump size $(g-1)$ is lognormal random variable is that;

$$(g-1) \sim i.i.d \log normal\left[k = \exp\{\mu + \frac{1}{2}\sigma^2\} - 1, \exp\{2\mu + \sigma^2\}\left(\exp\{\sigma^2\} - 1\right)\right]$$

Equivalently, in the log – run the jump size gives a return as $\ln\left(\frac{gS}{S}\right)$ which is a random variable described by;

$$\ln\left(\frac{gS}{S}\right) = \ln(g) \sim i.i.dN\left(\mu, \sigma^2\right)$$

We also note that;

$$\ln E[(g-1)] = E[\ln(g)] \neq E[\ln(g-1)]$$

Using the jump diffusion component dN , the expected relative price change $E\left[\frac{dS_t}{S_t}\right]$ over the time interval dt is $\lambda k dt$ derived as;

$$E[(g-1)dN] - E[(g-1)]E[dN] = k\lambda dt \qquad (14)$$

The resulting solution is a predictable function of the jump diffusion process. In order to have the jump diffusion component non - predictable, the return expected from the underlying asset μdt is computed by $-\lambda k dt$ on the adjusted drift term on the jump diffusion. This is given by;

$$E\left[\frac{dS_t}{S_t}\right] = E[(\mu - \lambda k)]dt + E\left[\sqrt{\upsilon_t}dZ_t\right] + E[(g-1)dN] \qquad (15)$$

Simplifying equation (15) gives

$$E\left[\frac{dS_t}{S_t}\right]=\mu dt \quad (16)$$

We take a variable G that is a function of S and t which form a price of any call option that is differentiable twice in S and once in t. Then using Itồ lemma we have;

$$\partial G=\left(\frac{\partial G}{\partial S}(\mu-\lambda k)S_t+\frac{\partial G}{\partial t}+\frac{1}{2}\frac{\partial^2 G}{\partial S^2}\upsilon_2 S_t^2\right)dt+\frac{\partial G}{\partial S}\sqrt{\upsilon_t}S_t dZ_t+\left(GgS_t-GS_t\right)dN \quad (17)$$

where $\left(GgS_t-GS_t\right)$ shows that there is occurrence in jumps. We take the discrete version of equation (12) and (17) given as;

$$\Delta S_t=S_t((\mu-\lambda k)\Delta t+\sqrt{\upsilon_t}\Delta Z_t)+(g-1)S_t\Delta N \quad (18)$$

And

$$\Delta G=\left(\frac{\partial G}{\partial S}(\mu-\lambda k)S_t+\frac{\partial G}{\partial t}+\frac{1}{2}\frac{\partial^2 G}{\partial S^2}\upsilon_2 S_t^2\right)\Delta t+\frac{\partial G}{\partial S}\sqrt{\upsilon_t}S_t\Delta Z_t+\left(GqS_t-GS_t\right)\Delta N \quad (19)$$

where ΔS and ΔG gives the rate of changes in S and G over a small rate of change of time, Δt. Using Itồ lemma mentioned in section 2.2 both G and S in equation (18) and (19) have the same effect of uncertainty on ΔZ. We need to eliminate the Wiener process by choosing a portfolio of an asset and derivative. We consider a portfolio that is short of one derivative and takes $+\frac{\partial G}{\partial S}$: Shares. We also define Π as the value of the portfolio such that the portfolio holder will have both short and long option position in acquiring quantity of shares. By definition

$$\Pi=G-\frac{\partial G}{\partial S}S \quad (20)$$

The discrete value of equation (20) in the interval Δt is given by;

$$\Delta\Pi=\Delta G-\frac{\partial G}{\partial S}\Delta S \quad (21)$$

When we substitute equation (18) and equation (19) onto equation (21) and removing the discrete aspect, we obtain

$$d\Pi=\left(\frac{\partial G}{\partial S}+\frac{1}{2}\frac{\partial^2 G}{\partial S^2}\upsilon_t S_t^2\right)dt+\left(GqS_t-GS_t-\frac{\partial G}{\partial S}S_t(g-1)\right)dN \quad (22)$$

Equation (22) shows the elimination growth rate of asset, μ. This shows that the value of the option is not affected by the growth rate of the price of the asset. There is no arbitrage principle asserting that the risk free portfolio return is equivalent to the risk free rate. This implies that the time interval dt is equivalent to r for the percentage rate of return on the underlying portfolio. This is given by;

$$E\left(d\Pi\right)=r\Pi dt, \quad (23)$$

Where r is the interest rate that is risk free? Andanje [1]
Substituting equation (20) and equation (21) onto equation (23) gives;

$$E\left[\left(\frac{\partial G}{\partial t}+\frac{1}{2}\frac{\partial^2 G}{\partial S^2}\upsilon_t S_t^2\right)dt+\left[GqS_t-GS_t-\frac{\partial G}{\partial S}S_t(g-1)\right]dN\right]=r\left(G-\frac{\partial G}{\partial S}S\right)dt \quad (24)$$

This simplifies to;

$$\left(\frac{\partial G}{\partial t}+\frac{1}{2}\frac{\partial^2 G}{\partial S^2}\upsilon_t S_t^2\right)dt+\left[(g-1)GS_t-\frac{\partial G}{\partial S}S_t(g-1)\right]EdN=r\left(G-\frac{\partial G}{\partial S}S\right)dt \tag{25}$$

The equivalence of equation (27) is expressed as;

$$\left(\frac{\partial G}{\partial t}+\frac{1}{2}\frac{\partial^2 G}{\partial S^2}\upsilon_t S_t^2\right)dt+\left[(g-1)GS_t-\frac{\partial G}{\partial S}S_t(g-1)\right]\lambda dt=r\left(G-\frac{\partial G}{\partial S}S\right)dt \tag{26}$$

Simplifying equation (26) further gives;

$$\frac{\partial G}{\partial t}+\frac{1}{2}\frac{\partial^2 G}{\partial S^2}\upsilon_t S_t^2+(g-1)\lambda GS_t-\frac{\partial G}{\partial S}S_t\lambda(g-1)=rG \tag{27}$$

Equation (27) is the formulated Black Scholes equation using Heston's Jump Diffusion Process Model.

4. Conclusion

In this paper we have derived a Black – Scholes – Merton partial differential equation using Heston's Jump Diffusion Process model. Historical data can be applied in this model to compare the results with other existing extended Black – Scholes – Merton models which can help investors on analyzing their investment strategies and make viable decisions.

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