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Derivation of black Scholes equation using Heston's model with dividend yielding asset

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Abstract
Black Scholes formula is crucial in modern applied finance. Since the introduction of Black – Scholes concept model that assumes volatility is constant; several studies have proposed models that address the shortcomings of Black – Scholes model. Heston's models stands out amongst most volatility models because the process of volatility is positive and is a process that obeys mean reversion and this is what is observed in the real market world. One of the shortcomings of Heston's model is that it doesn't incorporate dividend yielding asset. Black Scholes partial differential equation revolves around Geometric Brownian motion and its extensions. We therefore incorporate dividend yielding asset on Heston's model and use it to model a new Black Scholes equation using the knowledge of partial differential equations.

Keywords: Volatility, dividends, geometric brownian motion, black - Scholes formula, Heston's model

1. Introduction

From research done in financial literature, the pricing process that is applied in options is normally valued in reference to derivatives and securities. In pricing of options we have statistical models that use various independent variables to calculate the value of an option. We look at the following parameters in formulating mathematical equation for pricing of an option;

- Index price of the underlying asset.
- Option exercise price over a given time period.
- A given option expiry date.
- Dividend paying rate allocated during the life of an option.
- The risk free - interest rate of the given option.
- The volatility of the underlying asset at a given time period.

When we formulate these variables into a mathematical equation, the resulting equation is what gives the value of the option. The standard Black Scholes equation [2] gives one of the option pricing models and it is derived under the following assumptions;

- Constant volatility.
- No dividend yield.
- Interest rates are constant and known.
- The returns are log - normally distributed.
- No commission and transaction cost.

In real practice these assumptions are not sometimes applicable in the market. The standard pricing model of stock is a normally given as function of some time S_t that follows a diffusion process presented by the following Geometric Brownian motion;

$$dS_t = \mu S_t dt + \sigma S_t dZ_t, \tag{1}$$

where μ is refers to the growth rate of the option (drift rate), S_t if the price of the option at time t, σ is the constant volatility (volatility rate) and dZ_t is the standard wiener process also known as Brownian motion.

The studies by Hull and White [5], Stein and Stein [11] and Heston [3] derived stochastic volatilities which considered two factor processes where one of the factors become responsible in producing a dynamic of volatility coefficient. Heston's volatility model [3] stands out among these models because the process for volatility is greater than zero and follows a mean reverting process which is in contrary to Black Scholes model that assumes constant volatility. Heston's model [3] also has existence of closed – form solution of vanilla options. It assumes that the spot price gives a diffusion process derived by;

$$\frac{dS_t}{S_t} = \mu dt + \sqrt{\upsilon_t}\, dZ_t \tag{2}$$

where μ (linear drift rate) is a constant, S_t if the price of the option at time t, υ_t is a non – constant instantaneous volatility and Z_t is the standard Wiener process. From an overview, this is a process resembling Geometric Brownian motion with a proposed volatility given by a mean reverting diffusion process. It is from this Heston's volatility model that we propose a diffusion process with a dividend yielding asset that we use to derive a new Black Scholes differential equation.

2. Preliminaries

2.1 Itô Process

It is the Wiener diffusion process in with constant parameters a and b being functions of the price of an underlying option defined by the variables P and t. It can be written mathematically in form a diffusion process given by;

$$dP = a(P,t)dt + b(P,t)dZ_t \tag{3}$$

In this case the underlying linear drift rate and volatility (variance) rate of the Itô process are prone to change over a given time frame. Considering a small time interval between t and $t+\Delta t$, that changes the from X to $X+\Delta X$ is equation (3) is thus expressed as;

$$\Delta P = a(P,t)\Delta t + b(P,t)\varepsilon\sqrt{\Delta t} \tag{4}$$

This small approximation still results in to a relationship between the drift rate and the volatility rate. It assumes that the change of the variable P is normally distributed with the drift rate and the variance rate of P remaining constant which are proven to be equal to $a(P,t)\Delta t$ and $b^2(P,t)\Delta t$ respectively during the interval between t and $t+\Delta t$. This is also denoted by $\Delta P \sim N\left(a(P,t)\Delta t, b(P,t)\sqrt{\Delta t}\right)$.

2.2 Itô Lemma

Stochastic differential equations are best solved by Itô Lemma, where wiener - like differential process are put into mathematical formulation of partial differential equations to obtain solutions of stochastic differential equations. In deriving Itô lemma we consider value of a variable P that follows an Itô process from equation (3), where P is said to have a linear drift rate of a and a percentage variance of b^2 such that from Itô lemma, it is stated that a function $G(P,t)$ that can be differentiable twice in P and once in t forms an Itô process of the form;

$$dG = \left(\frac{\partial G}{\partial P}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial P^2}b^2\right)dt + \frac{\partial G}{\partial P}bdZ\,, \tag{5}$$

where $\left(\frac{\partial G}{\partial P}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial P^2}b^2\right)$ is the percentage drift and a percentage variance derived by $\left(\frac{\partial G}{\partial P}\right)^2 b^2 dt$

2.3 Geometric Brownian Motion

Geometric Brownian motion is a specific Itô Process following a diffusion process given by

$$dF = aFdt + bFdZ \tag{6}$$

where $a(F,t) = aF$, $b(F,t) = bF$ and Z is the standard wiener process. A geometric Brownian motion used in the application of stock pricing is given by;

$$dS = \mu Sdt + \sigma SdZ\,, \tag{7}$$

where S is the price of the underlying asset, μ is the expected growth rate or the rate of return of the underlying asset and σ is the percentage volatility of price of the underlying asset. Re - written equation (7) results to the following;

$$dS = \mu S dt + \sigma S dZ \tag{8}$$

A discrete time model over a given time frame t to Δt that is normally used in pricing of the underlying asset is given by;

$$\Delta S = \mu S \Delta t + \sigma S \varepsilon \sqrt{\Delta t}\,, \tag{9}$$

where ΔS is the rate of change in price of the asset S within a time interval Δt and $Z = \varepsilon\sqrt{\Delta t}$ such that ε is a random variable derived by a standardized normal distribution with mean zero and standard deviation of one [5].

2.4 Stochastic Process

Stochastic processes are activities or events whose occurrences are random over a given period of time; this makes them obey the law of probability. Mathematically it can be defined as a process represented by X_t which is a collection of random variables $[X_t : t \in T]$ found in a probability space that varies over a given set T. We define various types of stochastic processes below.

2.4.1 Markov Process

It is a stochastic process where factors from the past history of the asset does not influence the behavior of the current asset price since it is believed that the current price already contain relevant information from the past history that could affect the new price of the underlying stock.

2.4.2 Wiener Process

It is a Brownian motion with mean rate of change zero and variance rate of one. It can also be defined as a random variable of value Z that follows a Wiener process with the following properties;

Property 1: Over a small period of time ΔZ is defined by;

$$\Delta Z = \varepsilon\sqrt{\Delta t} \tag{10}$$

where ε is normally distributed by mean of zero and variance of 1. That is $\varepsilon \sim N(0,1)$

Property 2: Over a two varied short time periods, ΔZ are independent. That is $\operatorname{cov}(\Delta Z_i, \Delta Z_j) = 0$, $i \neq j$ [5].

2.5 Dividend

Dividends are rates of payments in form of returns given to stockholders after a given time period normally a year as a reward of investments to distribute profits earned by a company or a corporation.

2.6 Dividend yielding Heston Model

We incorporate dividends on Heston's model of closed – form solution of vanilla options. It will therefore follow a diffusion process having a dividend yielding asset given by;

$$dS_t = S_t\left((\mu - y_t)dt + \sqrt{\upsilon_t}\, dZ_t\right), \tag{11}$$

where μ (linear drift rate) is a constant, y_t is the rate of dividend yield, υ_t is a non – constant instantaneous volatility and Z_t is a Wiener process. Oduor [10] and Opondo [7].

3. Main results

3.1 Derivation of Black Scholes equation using a dividend yielding Heston's model

We consider a Heston's stochastic volatility model with a dividend yielding asset that follows a diffusion process given in equation (11). We take a variable G that is a function of S and t which form a price of any call option that is differentiable twice in S and once in t. Then using Itồ lemma we have;

$$\partial G = \left(\frac{\partial G}{\partial S}(\mu - y_t)S_t + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\upsilon_t S_t^2\right)dt + \frac{\partial G}{\partial S}\sqrt{\upsilon_t}\, S_t dZ_t \tag{12}$$

Taking the discrete version of equation (11) and (12) we have that;

$$\Delta S = S_t\left((\mu - y_t)\Delta t + \sqrt{\upsilon_t}\,\Delta Z_t\right) \tag{13}$$

and

$$\Delta G = \left(\frac{\partial G}{\partial S}(\mu - y_t)S_t + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\upsilon_t S_t^2 \right)\Delta t + \frac{\partial G}{\partial S}\sqrt{\upsilon_t}S_t \Delta Z_t \tag{14}$$

where ΔS and ΔG denote the changes in S and G in over a given time period defined by Δt. Using Itồ lemma in section 2.2 both G and S in equation (13) and (14) depend of the same factors that causes the ΔZ. We need to eliminate the Wiener process by choosing a portfolio of an asset and derivative. We consider a portfolio that is short of one derivative and takes $+\frac{\partial G}{\partial S}$: Shares. We also define Π as the value of the portfolio such that the portfolio holder will have both short and long option position in acquiring quantity of shares. By definition.

$$\Pi = -G + \frac{\partial G}{\partial S}S \tag{15}$$

The discrete version of equation (15), $\Delta\Pi$ from the value of the portfolio over interval Δt is given by [1].

$$\Delta\Pi = -\Delta G + \frac{\partial G}{\partial S}\Delta S \tag{16}$$

Substituting equations (13) and (14) in (16), we obtain

$$\Delta\Pi = -\left(\left(\frac{\partial G}{\partial S}(\mu - y_t)S_t + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\upsilon_t S_t^2 \right)\Delta t + \frac{\partial G}{\partial S}\sqrt{\upsilon_t}S_t\Delta Z_t \right) + \frac{\partial G}{\partial S}\left(S_t\left((\mu - y_t)\Delta t + \sqrt{\upsilon_t}\Delta Z_t\right)\right)$$

This simplifies to

$$\Delta\Pi = \left(-\frac{\partial G}{\partial t} - \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\upsilon_t S_t^2 \right)\Delta t \tag{17}$$

Since equation (17) does not involve ΔZ and is no longer stochastic, this makes the portfolio to be risk free over the time Δt. The assumptions listed in the Black Scholes model shows that the portfolio is likely to earn a uniform rate of return during the short - term life of the riskless assets. In case it earns more, the arbitrageurs will make a risk free profit by borrowing money from financial institutions to buy this risk free portfolio. Contrary if it earned less, they could make a risk free profit by shortening the life portfolio and buying a risk free asset. It follows that

$$\Delta\Pi = r\Pi\Delta t, \tag{18}$$

where r, is the percentage risk – free interest denoted by $(\mu - y_t)$ as from Heston's model where μ is the underlying growth rate and y_t is dividend yielding rate at time t. Substituting equations (15) and (17) into (18) we obtain the following.

$$\left(-\frac{\partial G}{\partial t} - \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\upsilon_t S_t^2 \right)\Delta t = (\mu - y_t)\left(-G + \frac{\partial G}{\partial S}S \right)\Delta t$$

This simplifies to

$$\frac{\partial G}{\partial t} + (\mu - y_t)\frac{\partial G}{\partial S}S_t + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\upsilon_t S_t^2 = (\mu - y_t)G \tag{19}$$

Equation (20) is the Black Scholes equation with a dividend yielding Heston Stochastic volatility model. If no dividends are earned, the equation reduces to the following

$$\frac{\partial G}{\partial t} + \mu\frac{\partial G}{\partial S}S_t + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\upsilon_t S_t^2 = \mu G \tag{20}$$

4. Conclusion

In this paper we have derived a Black Scholes equation using a dividend yielding Heston's stochastic volatility model. This can be used to help investors on analyzing their investment strategies to make viable decisions.

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