Volume 4, No. 9, August 2017

Journal of Global Research in Mathematical Archives

UGC Approved Journal

RESEARCH PAPER

Available online at http://www.jgrma.info

ISSN 2320 – 5822

NORM BOUNDS FOR CONTRACTIVE NORMALOID OPERATORS

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Abstract: In this paper we establish the upper and lower norm estimates of contractive normaloid operators using inner product, Schwarz inequality for non-negative real numbers and some operator inequalities.

Keywords: Normaloid operators, Contractive operators, inner product, Schwarz inequality and numerical radius.

INTRODUCTION

We establish some inequalities for positive contractive normaloid operators. For this purpose, some inequalities for vectors in inner product spaces due to [2, 4 and 6]. Let H a complex Hilbert space with an inner product $\langle .,. \rangle$ and $B(H)$ the algebra of all bounded linear operators on H. $\|\cdot\|$ will also denote the usual operator norm. An operator $S \in B(H)$ is said to be normaloid if
$\|S\| = sup\{|\langle Sx, x\rangle|: \|x\| = 1\}$ and contractive if $\|S\| \leq 1$.

BASIC CONCEPTS AND PRELIMINARIES

Here we start by defining some key terms that are useful in the sequel.

Definition 2.1. An inner product on a vector space V is a map $\langle .,. \rangle: V \times V \to \mathbb{K}$ such that $\forall\, x, y, z \in V$ and $\lambda \in \mathbb{K}$; the following properties are satisfied:
(i). $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.
(ii). $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$.
(iii). $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$.
(iv). $\langle x, y\rangle = \overline{\langle y, x\rangle}$.
The ordered pair $(V, \langle .,. \rangle)$ is called an inner product space.
Definition 2.2. **Schwarz inequality**; for $S \in B(H)$ and $x, y \in H$ then
$|\langle Sx, y\rangle|^2 \leq \langle Sx, x\rangle\langle Sy, y\rangle$.
Definition 2.3. Let $S: H \to H$ the adjoint of S is $S^*: H \to H$ such that:
$\langle Sx, y\rangle = \langle x, S^*y\rangle, x, y \in H$.

Definition 2.4. An operator S is said to be normal if $S\,S^* = S^*S$ i.e. it commutes with its adjoint and S is self-adjoint if $S=S^*$.

Definition 2.5. Numerical radius $r(S)$ is the supremum of set of scalars $r(S) = sup\{|\lambda|: \lambda \in W(S)\}$ with the following properties :

1. $r(S) = \|S\|$.
2. $r(S^*S) = r(SS^*)$.
3. $r(USU^*) = r(S)$.
4. $r(S_1 \oplus S_2 \oplus \dots \oplus S_n) = max\{r(S_i): i = 1,2,\dots,n\}$.

MAIN RESULTS

4 Upper Norm Estimates

Proposition 4.1. Let S_1, S_2, S_3 be contractive normaloid operators where S_1 and S_2 are positive then $\begin{bmatrix} S_1 & S_2^* \\ S_2 & S_3 \end{bmatrix}$ is positive contractive in $B(H \otimes H)$ if and only if

$$|\langle S_2x, y\rangle|^2 \leq \langle S_1x, x\rangle\langle S_3y, y\rangle, \forall\, x, y \in H. \quad (1)$$

Proof. Suppose $\begin{bmatrix} S_1 & S_2^* \\ S_2 & S_3 \end{bmatrix}$ is positive contractive in $B(H \otimes H)$ then from [5], Equation (1) i.e. $|\langle S_2x, y\rangle|^2 \leq \langle S_1x, x\rangle\langle S_3y, y\rangle, \forall\, x, y \in H$. We have

$$\left|\left\langle \begin{bmatrix} S_1 & S_2^* \\ S_2 & S_3 \end{bmatrix}\begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix} \right\rangle\right|^2 \leq \left\langle \begin{bmatrix} S_1 & S_2^* \\ S_2 & S_3 \end{bmatrix}\begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} x \\ 0 \end{bmatrix}\right\rangle \left\langle \begin{bmatrix} S_1 & S_2^* \\ S_2 & S_3 \end{bmatrix}\begin{bmatrix} 0 \\ y \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix}\right\rangle$$

Simplifying yields

$$\left|\left\langle \begin{bmatrix} S_1 & S_2^* \\ S_2 & S_3 \end{bmatrix}\begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix} \right\rangle\right|^2 = |\langle S_2x, y\rangle|^2 \quad (2)$$

and

$$\left\langle \begin{bmatrix} S_1 & S_2^* \\ S_2 & S_3 \end{bmatrix}\begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} x \\ 0 \end{bmatrix} \right\rangle = \langle S_1x, x\rangle \quad (3)$$

and

$$\left\langle \begin{bmatrix} S_1 & S_2^* \\ S_2 & S_3 \end{bmatrix}\begin{bmatrix} 0 \\ y \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix} \right\rangle = \langle S_3y, y\rangle \quad (4)$$

Combining Equations 2, 3 and 4 yield

$|S_2x, y|^2 \leq \langle S_1x, x\rangle\langle S_3y, y\rangle\ \forall\, x, y \in H$.

Conversely, assume that Equation (1) holds, then $\forall\, x, y \in H$

$$\begin{aligned}\left\langle \begin{bmatrix} S_1 & S_2^* \\ S_2 & S_3 \end{bmatrix}\begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix} \right\rangle &= \langle S_1x, x\rangle + \langle S_2^*y, x\rangle + \langle S_2x, y\rangle + \langle S_3y, y\rangle \\ &= \langle S_1x, x\rangle + \langle S_3y, y\rangle + 2Re\langle S_2x, y\rangle \\ &\geq 2\langle S_1\, x, x\rangle^{\frac{1}{2}}\langle S_3\, y, y\rangle^{\frac{1}{2}} + 2Re\langle S_2x, y\rangle \\ &\geq\ 2|\langle S_2x, y\rangle| - 2|\langle S_2x, y\rangle| \\ &\geq 0.\end{aligned}$$

Therefore $\begin{bmatrix} S_1 & S_2^* \\ S_2 & S_3 \end{bmatrix}$ is positive.

Theorem 4.2. Let S_1 and S_2 be contractive normaloid operators belonging to the norm ideal associated with the Hilbert Schmidt norm $\|\cdot\|_{HS}$ then

$$\left\|S_1^{\frac{1}{2}}S_2^{\frac{1}{2}}\right\|_{HS}^2 \le \|Re(S_1S_2)\|_{HS}^{\frac{1}{2}}.$$

Proof. From [15, Lemma 1]

$$\|S_1S_2\| \le \|Re(S_1S_2)\|. \qquad (5)$$

Then it follows that using Equation 5, we have

$$\left\|S_1^{\frac{1}{2}}S_2^{\frac{1}{2}}\right\|_{HS}^2 = \left\|\left(S_1^{\frac{1}{2}}S_2^{\frac{1}{2}}\right)^*\left(S_1^{\frac{1}{2}}S_2^{\frac{1}{2}}\right)\right\|_{HS}$$

$$= \left\|S_1^{\frac{1}{2}}S_1S_2^{\frac{1}{2}}\right\|_{HS}$$

$$\le \|Re(S_1S_2)\|_{HS}^{\frac{1}{2}}$$

Corollary 4.3. Let $S_1, \dots, S_n$ for all $n \in \mathbb{N}$ be positive contractive normaloid operators belonging to the norm ideal associated norm $\|\cdot\|_{HS}$ then

$$\left\|\sum_{i=1}^{n} S_i^{\frac{1}{2}}\right\|_{HS}^2 \le \sum_{i,j}^{n}\left\|S_i^{\frac{1}{2}}S_j^{\frac{1}{2}}\right\|_{HS}.$$

Proof. From Theorem 2, it follows that

$$\left\|\sum_{i=1}^{n} S_i^{\frac{1}{2}}\right\|_{HS}^2 = \left\|\sum_{i=1}^{n}\left(S_1^{\frac{1}{2}}S_2^{\frac{1}{2}}\dots S_n^{\frac{1}{2}}\right)^*\left(S_1^{\frac{1}{2}}S_2^{\frac{1}{2}}\dots S_n^{\frac{1}{2}}\right)\right\|_{HS}$$

$$= \left\|\sum_{i,j}^{n} S_i^{\frac{1}{2}}S_j^{\frac{1}{2}}\right\|_{HS}$$

$$\le \sum_{i=1}^{n}\left\|S_i^{\frac{1}{2}}S_j^{\frac{1}{2}}\right\|_{HS}.$$

Theorem 4.4. Let S_1, S_2 be contractive normaloid operators on a Hilbert space then

$$\|S_1S_2 - S_2S_1\| \le \|S_1\|\,\|S_2\|.$$

Proof. Suppose $S_1$is positive and S_1 is self-adjoint. From [6, Theorem 2.9], $r(S_1S_2 - S_2S_1) \le \frac{1}{2}\|S_1\|\left(\|S_2\| + \|S_2^2\|^{\frac{1}{2}}\right)$ since $r(S) = \|S\|$ by [1, Theorem 5]. This implies that

$$\|S_1S_2 - S_2S_1\| \le \frac{1}{2}\|S_1\|\left(\|S_2\| + \|S_2^2\|^{\frac{1}{2}}\right)$$

Since $S_1, S_2 \in B(H)$ such that S_2 is a projection from [6, Theorem 2.8] then

$$\|S_1S_2 - S_2S_1\| \le \frac{1}{2}\left(\|S_1\| + \|S_1^2\|^{\frac{1}{2}}\right) \qquad (6)$$

If $A = \sqrt{S_1 - S_1^2}$, the operator $E = \begin{bmatrix} S_1 & A \\ A & I - S_1 \end{bmatrix}$ is a projection on $H \oplus H$, since $S_1\sqrt{S_1 - S_1^2}S_1$. If $B = \begin{bmatrix} S_2 & 0 \\ 0 & 0 \end{bmatrix}$, then $EB - BE = \begin{bmatrix} S_1S_2 - S_2S_1 & -S_2A \\ AS_2 & 0 \end{bmatrix}$. By Equation 6, we have

$$\|EB - BE\| \le \frac{1}{2}\left(\|B\| + \|B^2\|^{\frac{1}{2}}\right).$$

Now let $C = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, where I is the identity operator. Then $\begin{bmatrix} S_1S_2 - S_2S_1 & 0 \\ 0 & 0 \end{bmatrix} = C(EB - BE)C^*$ if we let $S_1E \in B(H)$ such that E is a projection, then

$$\|S_1B - BS_1\| \leq \frac{1}{2}\left(\|S_1\| + \|S_1^2\|^{\frac{1}{2}}\right) \text{ by Equation 6} \quad (7)$$

So

$$\left\|\begin{bmatrix} S_1S_2 - S_2S_1 & 0 \\ 0 & 0 \end{bmatrix}\right\| = \|EB - BE\| \text{ by property (iii) of Definition 2.5}$$
$$\leq \frac{1}{2}\left(\|B\| + \|B^2\|^{\frac{1}{2}}\right) \text{ by Equation 7}$$
$$= \frac{1}{2}\left(\|S_2\| + \|S_2^2\|^{\frac{1}{2}}\right) \text{ by Equation 6}$$

hence

$$\|S_1S_2 - S_2S_1\| \leq \frac{1}{2}\left(\|S_2\| + \|S_2^2\|^{\frac{1}{2}}\right) \quad (8)$$

Since S_1 is positive, then it follows from Equation 7 that

$$\left\|\frac{S_1}{\|S_1\|}S_2 - S_2\frac{S_1}{\|S_1\|}\right\| \leq \frac{1}{2}\left(\|S_2\| + \|S_2^2\|^{\frac{1}{2}}\right).$$

Then

$$\|S_1S_2 - S_2S_1\| \leq \frac{1}{2}\|S_1\|\left(\|S_2\| + \|S_2^2\|^{\frac{1}{2}}\right)$$
$$\leq \|S_1\|\|S_2\|.$$

Theorem 4.4. Let $S_1, S_2 \in B(H)$ be contractive normaloid operators. Then

$$\left\|\frac{S_1 + S_2}{2}\right\|^2 \leq \frac{\|S_1S_1^* + S_2S_2^*\|}{2} + r(S_2^*S_1).$$

Proof. Since

$$\|S_1x + S_2x\|^2 = \|S_1x\|^2 + \|S_2x\|^2 + 2Re\langle S_1x, S_2x\rangle$$
$$\leq \langle (S_1S_1^* + S_2S_2^*)x, x\rangle + 2|\langle (S_2^*S_1)x, x\rangle|, \forall\, x \in H.$$

Taking the supremum over $x \in H, \|x\| = 1$, we obtain

$$\|S_1 + S_2\|^2 \leq r(S_1S_1^* + S_2S_2^*) + 2r(S_2^*S_1)$$
$$= \|S_1S_1^* + S_2S_2^*\| + 2r(S_2^*S_1)$$

Therefore

$$\left\|\frac{S_1 + S_2}{2}\right\|^2 \leq \frac{\|S_1S_1^* + S_2S_2^*\|}{2} + r(S_2^*S_1).$$

5 Lower Norm Estimates

Lemma 5.1. Let ψ and ϕ be non-negative continuous functions on $[0,\infty)$ satisfying $\varphi(z)$ $\phi(z) = z\ \forall\ z \in[0,\infty)$. Let S_1, S_2 and S_3 be as in Proposition 4.1 and $S_1S_2 = S_2S_3$. Then

$$\left\|\begin{bmatrix} \varphi(S_1)^2 & S_2^* \\ S_2 & \phi(S_3)^2 \end{bmatrix}\right\| \text{ is also positive and}$$

$$\left\|\begin{bmatrix} \varphi(S_1)^2 & S_2^* \\ S_2 & \phi(S_3)^2 \end{bmatrix}\right\| \geq \|\varphi(S_1)^2\|\|\phi(S_3)^2\| + \|S_2\|^2.$$

Proof. Suppose S_1 and S_3 are both invertible. Since $S_1S_2 = S_2S_3$, for any function that is continuous on $[0,\infty)$, then $k(S_3)S_2 = S_2k(S_1)$. But $\varphi(z)\ \phi(z) = z$ for $z \in[0,\infty)$, then $\varphi(S)\ \phi(S) = S$ for any positive operator $S \in B(H)$. The two facts imply that $\varphi(S_3)S_3^{-\frac{1}{2}}\ S_2\varphi(S_1)S_1^{-\frac{1}{2}} = S_2$. Consequently,

$$\begin{bmatrix} \varphi(S_1)^2 & S_2^* \\ S_2 & \phi(S_3)^2 \end{bmatrix} = \begin{bmatrix} \varphi(S_1)^2 S_1^{-\frac{1}{2}} & 0 \\ 0 & \phi(S_3) S_3^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} S_1 & S_2^* \\ S_2 & S_3 \end{bmatrix} \begin{bmatrix} \varphi(S_1) S_1^{-\frac{1}{2}} & 0 \\ 0 & \phi(S_3) S_3^{-\frac{1}{2}} \end{bmatrix}.$$

Invoking Proposition 4.1 completes the proof.

Theorem 5.2. Let $S \in B(H)$ be positive contracting operators. If $\alpha \in \mathbb{C} \setminus \{0\}$ and $m > 0$ are such that $\|S - \alpha S^*\| < m$ then

$$\|Sx\|^2 > \frac{|\alpha|^2}{m^2} (\|Sx\|^4 - |\langle S^2 x, x\rangle|^2).$$

Proof. Assume without loss of generality that S is normal. We employ the reverse of quadratic Schwarz inequality in [1] i.e.

$$0 \leq \|a\|^2 \|b^2\| - |\langle a, b\rangle|^2 \leq \frac{1}{|a|^2} \|a\|^2 \|a - \alpha b\|^2$$

For every $a, b \in H$ and $\alpha \in \mathbb{C} \setminus \{0\}$ let $a = Sx$, $b = S^* x$ we get

$$\|Sx\|^2 \|S^*\|^2 - |\langle Sx, S^* x\rangle|^2 \leq \frac{1}{|a|^2} \|Sx\|^2 \|Sx - \alpha S^*\|^2$$

$$\|Sx\|^4 - |S^2 x, x|^2 - \frac{m^2}{|a|^2} \|Sx\|^2$$

$$\|Sx\|^2 > \frac{|a|^2}{m^2} (\|Sx\|^4 - \|Sx\|^4).$$

Lemma 5.3. Let $S_1, S_2 \in B(H)$ be positive contractive operators. If $n > 0$ and $\|S_1 - S_2\| \leq n$. Then

$$r(S_2^* S_1) \geq \frac{1}{2} [\|S_1 S_1^* + S_2 S_2^*\| - n^2]. \quad (9)$$

Proof. Without loss of generality assume that $S_1 S_1^* + S_2 S_2^*$ is self adjoint.

For any $x \in H$, $\|x\| = 1$ and from $\|S_1 - S_2\| \leq n$ we have

$$\|S_1 x - S_2 x\| \leq n = |\langle S_1 x - S_2 x, S_1 x - S_2 x\rangle| \leq n^2.$$

$$= \|S_1 x\|^2 + \|S_2 x\|^2 - 2Re\langle S_1 x, S_2 x\rangle \leq n^2.$$

Thus

$$\|S_1 x\|^2 + \|S_2 x\|^2 \leq 2Re\langle S_1 x, S_2 x\rangle + n^2 \quad (10)$$

But

$$\|S_1 x\|^2 + \|S_2 x\|^2 = \langle (S_1^* S_1)x, x\rangle + \langle (S_2^* S_2)x, x\rangle$$
$$= \langle (S_1^* S_1 + S_2^* S_2)x, x\rangle.$$

Using Equation 10 we obtain

$$\langle (S_1 S_1^* + S_2^* S_2)x, x\rangle \leq 2|\langle (S_2^* S_1)x, x\rangle| + n^2 \quad (11)$$

Taking the supremum over $x \in H$, $\|x\| = 1$ in (11) we get

$$r(S_1^* S_1) \leq 2r(S_2^* S_1) + n^2 \quad (12)$$

Since the operator $S_1 S_1^* + S_2 S_2^*$ is self-adjoint, hence

$$r(S_1 S_1^* + S_2 S_2^*) = \|S_1 S_1^* + S_2 S_2^*\|.$$

Therefore Equation 12 becomes

$$r(S_2^* S_1) \geq \frac{1}{2} [\|S_1 S_1^* + S_2 S_2^*\| - n^2].$$

Corollary 5.4. Let S_1, S_2 be positive contractive operators on Hilbert space then

$\|S_1 \otimes S_2 + S_2 \otimes S_1\| \geq 2\|S_1\|\|S_2\|$

Proof.

$$\|S_1 \otimes S_2 + S_2 \otimes S_1\|^2 = \langle S_1 \otimes S_2 + S_2 \otimes S_1, S_1 \otimes S_2 + S_2 \otimes S_1 \rangle$$
$$= \langle S_1 \otimes S_2, S_1 \otimes S_2 \rangle + \langle S_1 \otimes S_2, S_2 \otimes S_1 \rangle + \langle S_2 \otimes S_1, S_1 \otimes S_2 \rangle + \langle S_2 \otimes S_1, S_2 \otimes S_1 \rangle$$
$$= \langle S_1, S_1 \rangle \langle S_2, S_2 \rangle + \langle S_1, S_2 \rangle \langle S_2, S_1 \rangle + \langle S_2, S_1 \rangle \langle S_1, S_2 \rangle + \langle S_2, S_2 \rangle \langle S_1, S_1 \rangle$$
$$= \|S_1\|^2\|S_2\|^2 + \|S_2\|^2\|S_1\|^2 + 2Re\langle S_1, S_2 \rangle$$

So by Cauchy-Schwarz inequality

$$\|S_1 \otimes S_2 + S_2 \otimes S_1\|^2 \leq \|S_1\|^2\|S_2\|^2 + \|S_2\|^2\|S_1\|^2 + 2\|S_1\|\|S_2\|$$
$$= (\|S_1\|\|S_2\| + \|S_2\|\|S_1\|)^2$$

i.e

$$\|S_1 \otimes S_2 + S_2 \otimes S_1\|^2 \leq (\|S_1\|\|S_2\| + \|S_2\|\|S_1\|)^2$$

Taking square roots on both sides we obtain

$$\|S_1 \otimes S_2 + S_2 \otimes S_1\| \leq \|S_1\|\|S_2\| + \|S_2\|\|S_1\|$$
$$\leq 2\|S_1\|\|S_2\|$$

But $\|S_1 \otimes S_2\| = \|S_1\|\|S_2\|$. Since $\|S_1 \otimes S_2 + S_2 \otimes S_1\| \geq 0$, this implies that $\|S_1 \otimes S_2 + S_2 \otimes S_1\| \geq 2\|S_1\|\|S_2\|$.

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