

# Numerical Analysis of Prey Refuge Effects on the Stability of Holling Type III Four-Species Predator-Prey System

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## Research Article

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# Numerical Analysis of Prey Refuge Effects on the Stability of Holling Type III Four-Species Predator-Prey System

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## Abstract

The dynamic behavior of a multi-species system that includes a prey refuge and a Holling type III functional response is examined in this work. The pre-requisites for the presence of the equilibrium points, as well as their local and global stabilities, for the suggested system are analyzed and derived. The Routh-Hurwitz criterion and the eigenvalue technique are used to study the local stabilities. On the other hand, the global stabilities have been studied using the Lyapunov technique. Numerical simulations have been carried out using the Matlab ode45 solver software to verify the analytical results. The findings show that refuge plays a significant part in improving the dynamical stability of the system.

## Keywords

Dynamical system, Lyapunov technique, Routh-Hurwitz criterion, Eigenvalue

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## 1. Introduction

The behavior of any complex dynamic system such as, biological models that explain the properties of intricate interactions between specific organisms in a biological context is a natural outcome of interaction between its constituent parts [1]. Many ecological and biological processes are rooted on the complex features of species and their dynamic connections with one another. The interaction among species is one of these relationships and has been a central subject in mathematical biology due to its pervasiveness and significance [2]. Most scholars (see [3] and [4]), have extensively investigated the qualitative dynamic properties of interacting predator-prey models in order to understand the species' long-term behavior.

\*\*Special description of the title. (dispensable)

In the modelling of naturally occurring processes, dynamical systems are widely used [3]. The theory of dynamical systems is largely concerned with establishing qualitative predictions about the behavior of systems that evolve over time when the system’s control parameters and the initial state change [5]. Mathematical models of real-world phenomena are hence expressed as systems of nonlinear differential equations (DE), which are usually difficult to solve explicitly [6]. As a result, numerical methodologies need to be employed to evaluate the solutions of such systems, with stability analysis being used to illustrate the solutions’ trajectories [7]. This numerical characterization of the solution(s) behavior is carried out by first determining the system’s equilibrium points and then using stability analysis to comprehend the general behavior of solutions [4]. Consider a nonlinear system of ordinary differential equation of the form,

$$\frac{du_i}{dt} = g_i(u_i) \tag{1}$$

where  $u \in \mathbb{R}^m; i = 1, 2, \dots m$ .

Suppose that a point  $u(t) = u^*(t)$  is an equilibrium point of System 1 and let  $u_i(t)$  be its original solution. Then, the nature of  $u_i(t)$  near  $u_i^*(t)$  determines the local stability according to Ma and Wang [8].

Let’s represent the original solution to Equation 1 as  $u_i(t) = u_i^*(t) + \psi(t)$ , where  $u^*(t)$  stands for any alternative solution to Equation 1 and  $\psi(t)$  denotes any minor modification to the original solution  $u(t)$ . Consequently, according to Ma and Wang [8],  $\psi(t)$  can be linearly approximated as  $\frac{d\psi}{dt} = \zeta(t)\psi$ , where the matrix  $\zeta(t) = \left(\frac{\partial g_i}{\partial y_j}\right)_{u^*}$  is the Jacobian matrix at the equilibrium point,  $u^*(t)$ . The signs of the eigenvalues of this matrix dictate the linear stability of the equilibrium points,  $u_i^*(t)$  and the characteristic equation for the Jacobian matrix is given by,

$$Det(\zeta - \gamma I) = a_0\rho^n + a_1\rho^{n-1} + a_2\rho^{n-2} + \dots + a_n = 0.$$

The constraints on the coefficients  $a_i (i=1,2,\dots,n)$  are then obtained using Routh-Hurwitz criterion which assures local asymptotic stability at  $u^*(t)$  if the choice of parametric values is such that all the constraints are simultaneously satisfied [9]. If  $u_i(t)$  continue being close to  $u_i^*(t)$  as  $t \rightarrow \infty$ , it is considered stable otherwise, it is considered unstable. Furthermore, if the solutions approach the critical point over time rather than just staying within a set radius, the critical point is asymptotically stable [7].

A lot of work has been done in establishing the stability of the equilibrium points of dynamical systems, both locally and globally [10]. Shireen [9], studied a four-species model with prey refuge for its dynamical behavior including the stabilities of its equilibrium points and carried out numerical simulations to confirm his analytical results. However, his model incorporated a linear predator-prey functional response which is rare among most feeding patterns.

In this paper therefore, we discuss the dynamical behavior of a set of predator-prey system formulated in [11], given by the following model.

$$\begin{aligned} \frac{du_1}{dt} &= \tau_1 u_1 \left( 1 - \frac{u_1}{\kappa_1} \right) - \gamma_1 u_1 + \gamma_2 u_2 - \eta_1 \left( \frac{u_1^2 v}{h + u_1^2} \right) - \tilde{\eta}_1 u_1 w, \\ \frac{du_2}{dt} &= \tau_2 u_2 \left( 1 - \frac{u_2}{\kappa_2} \right) + \gamma_1 u_1 - \gamma_2 u_2, \\ \frac{dv}{dt} &= \eta_2 \left( \frac{u_1^2 v}{h + u_1^2} \right) - \mu_1 v - \xi_1 v w, \\ \frac{dw}{dt} &= \tilde{\eta}_2 u_1 w + \xi_2 v w - \mu_2 w, \end{aligned} \tag{2}$$

where  $\tau_1$  and  $\tau_2$ ,  $\kappa_1$  and  $\kappa_2$ ,  $\gamma_1$ ,  $\gamma_2$ ,  $\eta_1$ ,  $\eta_2$ ,  $\mu_1$ ,  $\mu_2$ ,  $\tilde{\eta}_1$  and  $\tilde{\eta}_2$ ,  $\xi_1$  and  $\xi_2$  are positive constants.

## 2. Existence of the steady states

The System 2 is shown to have a total of five non-negative equilibrium points namely: The vanishing fixed point;  $E_0(0,0,0,0)$ , the planar fixed point;  $E_1(u_1^*, u_2^*, 0, 0)$ , the specialist predator-free fixed point;  $E_2(u_1^*, u_2^*, 0, w^*)$ , the generalist predator-free fixed point;  $E_3(u_1^*, u_2^*, v^*, 0)$  and finally the coexistence fixed point;  $E_4(u_1^*, u_2^*, v^*, w^*)$ . Clearly, the existence of  $E_0(0, 0, 0, 0)$  is trivial and hence ignored. The conditions that determine the existence of the remaining equilibrium points are derived as shown below.

### 2.1. The Planar Fixed Point

**Proposition 2.1:** *The planar fixed point  $E_1$  only exists if the conditions  $\tau > \gamma$ ,*

$$\frac{\tau_1(\tau_2 - \gamma_2)}{\kappa_1 \gamma_2} > \frac{\tau_2(\tau_1 - \gamma_1)}{\kappa_2 \gamma_2} \text{ and } (\tau_2 - \gamma_2)(\tau_1 - \gamma_1) < \gamma_1 \gamma_2 \text{ hold.}$$

**Proof:** Let  $u_1^*$  and  $u_2^*$  be non-negative solutions to

$$\tau_1 u_1 \left( 1 - \frac{u_1}{\kappa_1} \right) - \gamma_1 u_1 + \gamma_2 u_2 = 0, \tag{3}$$

$$\tau_2 u_2 \left( 1 - \frac{u_2}{\kappa_2} \right) + \gamma_1 u_1 - \gamma_2 u_2 = 0. \tag{4}$$

From Equation 3, we have

$$u_2 = \frac{1}{\gamma_2} \left[ \frac{\tau_1 u_1^2}{\kappa_1} - (\tau_1 - \gamma_1) u_1 \right]. \tag{5}$$

Substituting Equation 5 into Equation 4, we obtain

$$A u_1^3 + B u_1^2 + C u_1 + D = 0 \tag{6}$$

Where,  $A = -\frac{\tau_2 \tau_1^2}{\gamma_2^2 \kappa_1^2} < 0$ ,  $B = \frac{2\tau_1 \tau_2}{\gamma_2^2 \kappa_1} (\tau_1 - \gamma_1)$ ,  $C = \left[ \frac{\tau_1 \kappa_2 (\tau_2 - \gamma_2)}{\kappa_1 \gamma_2} - \frac{\tau_2 (\tau_1 - \gamma_1)^2}{\gamma_2^2} \right]$ ,  
 and  $D = \left[ \gamma_1 \kappa_2 - \frac{\kappa_2 (\tau_2 - \gamma_2) (\tau_1 - \gamma_1)}{\gamma_2} \right]$ .

Therefore, given the value of  $u_1 = u_1^*$ , the corresponding value of  $u_2 = u_2^*$  can be found from Equation 5 for which the inequality  $u_1^* > \frac{\kappa_1}{\tau_1} (\tau_1 - \gamma_1)$  holds for  $u_2^*$  to be positive

and  $u_1^*$  is the positive root of the Polynomial 6. Therefore, Equation 6 has a unique positive solution, that is  $u_1 = u_1^*$  if by Descartes rule of signs, the following conditions hold:

$$\tau > \gamma, \quad \frac{\tau_1 (\tau_2 - \gamma_2)}{\kappa_1 \gamma_2} > \frac{\tau_2 (\tau_1 - \gamma_1)^2}{\kappa_2 \gamma_2^2}, \quad \text{and} \quad (\tau_2 - \gamma_2) (\tau_1 - \gamma_1) < \gamma_1 \gamma_2.$$

### 2.2. The Specialist Predator-free Fixed Point

**Proposition 2.2** *The specialist predator-free fixed point  $E_2$  only exists under a certain value of  $\kappa_1$  in the unprotected area, below which it fails to exist.*

**Proof:** Suppose that  $u_1^*$ ,  $u_2^*$  and  $w^*$  are the non-negative roots of the algebraic equations

$$\tau_1 u_1 \left( 1 - \frac{u_1}{\kappa_1} \right) - \gamma_1 u_1 + \gamma_2 u_2 - \tilde{n}_1 u_1 w = 0, \tag{7}$$

$$\tau_2 u_2 \left( 1 - \frac{u_2}{\kappa_2} \right) + \gamma_1 u_1 - \gamma_2 u_2 = 0, \tag{8}$$

$$\tilde{n}_2 u_1 w - \mu_2 w = 0 \tag{9}$$

From Equation 9, we have

$$u_1^* = \frac{\mu_2}{\tilde{n}_2} \tag{10}$$

Substituting Equation 10 into Equation 8 gives

$$u_2^* = \frac{1}{2\tau_2 \tilde{n}_1} \left[ \kappa_2 \tilde{n}_2 (\tau_2 - \gamma_2) + \sqrt{\kappa_2^2 \tilde{n}_2^2 (\tau_2 - \gamma_2)^2 + 4\tau_2 \tilde{n}_2 \gamma_1 \kappa_2 \mu_2} \right]. \tag{11}$$

Substituting Equation 10 and Equation 11 into Equation 7 leads to

$$w^* = \frac{1}{\tilde{n}_1} \left[ (\tau_1 - \gamma_1) - \frac{\tau_1 \mu_2}{\kappa_1 \tilde{n}_2} + \frac{\gamma_2 u_2^* \tilde{n}_2}{\mu_2} \right]. \tag{12}$$

This implies that  $w^*$  is positive if and only if

$$(\tau_1 - \gamma_1) + \frac{\gamma_2 u_2^* \tilde{n}_2}{\mu_2} > \frac{\tau_1 \mu_2}{\kappa_1 \tilde{n}_2} \tag{13}$$

**Remark 2.2:** Condition 13 clearly specifies a carrying capacity  $\kappa_1$  threshold value in the unprotected region. As a result, generalist predators continue to exist. However, Condition 13 may fail if  $\kappa_1$  becomes too small.

### 2.3. The Generalist Predator-free Fixed Point

**Proposition 2.3** *The generalist predator-free fixed point  $E_3$  only exists when  $\eta_2 > \mu_1$ , but fails to exist when the value of  $\kappa_1$  in the unprotected zone falls below its threshold value.*

**Proof:** Let the non-negative solutions of the algebraic equations below

$$\begin{aligned} \tau_1 u_1 \left( 1 - \frac{u_1}{\kappa_1} \right) - \gamma_1 u_1 + \gamma_2 u_2 - \frac{\eta_1 u_1^2 v}{h + u_1^2} &= 0, \\ \tau_2 u_2 \left( 1 - \frac{u_2}{\kappa_2} \right) + \gamma_1 u_1 - \gamma_2 u_2 &= 0, \\ \frac{\eta_2 u_1^2 v}{h + u_1^2} - \mu_1 v &= 0, \end{aligned} \tag{14}$$

be represented by  $u_1^*$ ,  $u_2^*$  and  $v^*$ . From the third equation of System 14, we have

$$u_1^* = \sqrt{\frac{h\mu_1}{\eta_2 - \mu_1}} \tag{15}$$

Substituting Equation 15 into the second equation of System 14 and doing a little simplification gives an expression for  $u_2^*$  as,

$$u_2^* = \frac{1}{2\tau_2} \left[ \kappa_2 (\tau_2 - \gamma_2) + \sqrt{\kappa_2^2 (\tau_2 - \gamma_2)^2 + 4\tau_2 \gamma_1 \kappa_2 u_1^*} \right]. \tag{16}$$

From the first equation of System 14, it follows that

$$v^* = \frac{\eta_2}{\mu_1 \eta_1} \left[ \gamma_2 u_2^* + (\tau_1 - \gamma_1) \sqrt{\frac{h\mu_1}{\eta_2 - \mu_1} - \frac{\tau_1 h \mu_1}{\kappa_1 (\eta_2 - \mu_1)}} \right] \tag{17}$$

The following conditions must be satisfied for the positivity of the solutions of  $u_1^*$ ,  $u_2^*$  and  $v^*$ .

$$\eta_2 > \mu_1 \tag{18}$$

$$\gamma_2 u_2^* + (\tau_1 - \gamma_1) \sqrt{\frac{h\mu_1}{\eta_2 - \mu_1}} > \frac{\tau_1 h \mu_1}{\kappa_1 (\eta_2 - \mu_1)}. \tag{19}$$

**Remark 2.3:** It can be seen from Condition 19 that it gives a threshold value of  $\kappa_1$  in the unreserved region at which the specialist predators continue to survive.

Unfortunately, when  $\kappa_1$  becomes too small, the generalist predator-free fixed point could fail making its existence violated.

### 2.4. The Interior Equilibrium Point

**Proposition 2.4** *The coexistence equilibrium point  $E_4$  exists whenever the conditions*

$$\tau > \gamma, \quad u_1^* < \frac{\mu_2}{\tilde{n}_2} \text{ and } u_1^* > \frac{h\mu_1}{(\eta_2 - \mu_1)} \text{ hold.}$$

**Proof** Let the non-negative roots of the system of equations below be denoted by  $u_1^*, u_2^*, v^*$  and  $w^*$ . Then,

$$\begin{aligned} \tau_1 u_1 \left( 1 - \frac{u_1}{\kappa_1} \right) - \gamma_1 u_1 + \gamma_2 u_2 - \frac{\eta_1 u_1^2 v}{h + u_1^2} - \tilde{n}_1 u_1 w &= 0, \\ \tau_2 u_2 \left( 1 - \frac{u_2}{\kappa_2} \right) + \gamma_1 u_1 - \gamma_2 u_2 &= 0, \\ \frac{\eta_2 u_1^2 v}{h + u_1^2} - \mu_1 v - \xi_1 v w &= 0, \\ \tilde{n}_2 u_1 w + \xi_2 v w - \mu_2 w &= 0. \end{aligned} \tag{20}$$

From the fourth Equation of System 20, we have

$$v^* = \left( \frac{\mu_2 - \tilde{n}_2 u_1^*}{\xi_2} \right). \tag{21}$$

Given the third Equation of System 20, we can get

$$w^* = \frac{(\eta_2 - \mu_1) u_1^{*2} - h\mu_1}{\xi_1 (h + u_1^{*2})}. \tag{22}$$

The value of  $u_2^*$ , can be expressed as shown below upon mathematical manipulations on the second equation of System 20

$$u_2^* = \frac{1}{2\tau_2} \left[ \kappa_2 (\tau_2 - \gamma_2) + \sqrt{\kappa_2^2 (\tau_2 - \gamma_2)^2 + 4\tau_2 \gamma_1 \kappa_2 u_1^*} \right]. \tag{23}$$

Substituting the values of  $v^*, w^*$  and  $u_2^*$ , from Equation 21, Equation 22 and Equation 23 respectively into the first equation of System 20 gives

$$\begin{aligned} -\eta_1 u_1^2 + \kappa_1 (\tau_1 - \gamma_1) u_1 + \kappa_1 \gamma_2 \left[ \frac{\kappa_2 (\tau_2 - \gamma_2) + \sqrt{\kappa_2^2 (\tau_2 - \gamma_2)^2 + 4\tau_2 \gamma_1 \kappa_2 u_1^*}}{2\tau_2} \right] - \\ \left[ \frac{\kappa_1 \eta_1 (\mu_2 - \tilde{n}_2 u_1) u_1^2}{\xi_2 (h + u_1^2)} \right] - \left[ \frac{\kappa_1 \tilde{n}_1 (\eta_2 - \mu_1) u_1^3 - h \kappa_1 \tilde{n}_1 \mu_1 u_1}{\xi_1 (h + u_1^2)} \right] = 0 \end{aligned} \tag{24}$$

Multiplying every term of Equation 24 by  $2\tau_2\xi_1\xi_2(h+u_1^2)$  leads to

$$\begin{aligned}
 & -2\tau_1\tau_2\xi_1\xi_2(h+u_1^2)u_1^2 + 2\tau_2\xi_1\kappa_2(h+u_1^2)(\tau_1-\gamma_1)u_1 \\
 & \kappa_1\gamma_2\xi_1\xi_2(h+u_1^2)\left[\kappa_1(\tau_2-\gamma_2) + \sqrt{\kappa_1^2(\tau_2-\gamma_2)^2 + 4\tau_2\gamma_2\kappa_1u_1}\right] \\
 & 2\tau_2\xi_1\kappa_1\eta_1(\mu_2-\tilde{n}_2u_1)u_1^2 - 2\tau_2\xi_2\kappa_1\tilde{n}_1(\eta_2-\mu_1)u_1^3 + 2\tau_2\xi_2h\kappa_1\tilde{n}_1\mu_1u_1 = 0
 \end{aligned} \tag{25}$$

Having known the value of  $u_1^*$  from Equation 25, the other values of  $v^*$ ,  $w^*$  and  $u_2^*$ , can easily be computed from Equation 21, Equation 22 and Equation 23 respectively, when-

ever  $\tau > \gamma$ ,  $u_1^* < \frac{\mu_2}{\tilde{n}_2}$  and  $u_1^* > \frac{h\mu_1}{(\eta_2-\mu_1)}$  respectively; and  $u_1^*$  is the positive root of

Equation 25.

### 3. Stability Analysis of the Equilibrium Points

#### 3.1. Local Stability of $E_0, E_1, E_2, E_3$ and $E_4$

The local asymptotic stability of each equilibrium point is examined by constructing the Jacobian matrix and obtaining the eigenvalues determined at each equilibrium point. The stability, instability, or saddle of the equilibrium point is determined by the signs of the real parts of the eigenvalues of the Jacobian matrix. The eigenvalues of the equilibrium points must be negative in order for them to be stable. By differentiating the right hand side of System 2 with respect to  $u_1, u_2, v$ , and  $w$ , the entries of the universal variational matrix are constructed as shown below.

$$J = \begin{pmatrix} A^* & \gamma_2 & -\frac{\eta_1u_1^2}{h+u_1^2} & -\tilde{n}_1u_1 \\ \gamma_1 & \tau_2 - \frac{2\tau_2u_2}{\kappa_2} - \gamma_2 & 0 & 0 \\ A^{**} & 0 & \frac{\eta_2u_1^2}{h+u_1^2} - \mu_1 - \xi_1w & -\xi_1v \\ \tilde{n}_2w & 0 & \xi_2w & \tilde{n}_2u_1 + \xi_2v - \mu_2 \end{pmatrix} \tag{26}$$

where,

$$A^* = \tau_1 - \gamma_1 - \frac{2\tau_1}{\kappa_1} - \left\{ \frac{2\eta_1u_1v(h+u_1^2) - 2\eta_1u_1^3v}{(h+u_1^2)^2} \right\} - \tilde{n}_1w,$$

$$A^{**} = \frac{2\eta_2u_1v(h+u_1^2) - 2\eta_2u_1^3v}{(h+u_1^2)^2}.$$

At this point, we determine whether the equilibrium points in Section 2 are stable, unstable or saddle as discussed in the propositions that follow.



**Proposition 3.1** *The vanishing equilibrium point  $E_0$  is a saddle point whenever  $\tau \leq \gamma$  or  $\tau \geq \gamma$*

**Proof:** The Jacobian matrix evaluated at the vanishing equilibrium point,  $J(E_0)$  is given by

$$J(E_0) = \begin{pmatrix} \tau_1 - \gamma_1 & \gamma_2 & 0 & 0 \\ \gamma_1 & \tau_2 - \gamma_2 & 0 & 0 \\ 0 & 0 & -\mu_1 & 0 \\ 0 & 0 & 0 & -\mu_2 \end{pmatrix}.$$

Finding the roots of  $J(E_0)$  involves solving the auxiliary equation,  $|J(E_0) - \lambda_i I| = 0$ , where  $\lambda_i$  ( $i = 1, 2, 3, 4$ ) denote the eigenvalues of  $J(E_0)$ . Therefore,  $|J(E_0) - \lambda_i I| = 0$  can be simplified to

$$(-\mu_2 - \lambda_1)(-\mu_1 - \lambda_2) \left| \begin{pmatrix} (\tau_1 - \gamma_1) - \lambda_4 & \gamma_2 \\ \gamma_1 & (\tau_2 - \gamma_2) - \lambda_3 \end{pmatrix} \right| = 0,$$

from which we have  $(-\mu_2 - \lambda_1) = 0$  and  $(-\mu_1 - \lambda_2) = 0$ . So,  $\lambda_1 = -\mu_2 < 0$  and  $\lambda_2 = -\mu_1 < 0$ . The remaining two eigenvalues are obtained by solving the equation

$$\left| \begin{pmatrix} (\tau_1 - \gamma_1) - \lambda & \gamma_2 \\ \gamma_1 & (\tau_2 - \gamma_2) - \lambda \end{pmatrix} \right| = 0,$$

which can be expressed in the form

$$\lambda^2 + e_1 \lambda + e_2 = 0, \tag{27}$$

where,  $e_1 = -(\tau_1 - \gamma_1) - (\tau_2 - \gamma_2)$  and  $e_2 = (\tau_1 - \gamma_1)(\tau_2 - \gamma_2) - \gamma_1 \gamma_2$ .

According to Routh-Hurwitz criterion [12], all the roots of the polynomial 27 will have negative real parts if the conditions  $e_1 > 0$  and  $e_2 > 0$  hold.

Let us then consider the following two cases of Equation 27

- (i). *Case 1:* If  $\tau = \gamma$ , then  $e_2 < 0$ , which implies that one of the eigenvalues will have a positive real part.
- (ii). *Case 2:* If  $\tau > \gamma$ , then  $e_1 < 0$  and if  $\tau < \gamma$ , then  $e_2 < 0$ , implying that in each scenario, one of the eigenvalues is positive.

Therefore, since at least one of the eigenvalues of  $E_0$  is positive and the rest are negative, then  $E_0$  is a saddle point whenever  $\tau \leq \gamma$  or  $\tau \geq \gamma$ .

**Proposition 3.2:** *The planar equilibrium point  $E_1$  is locally asymptotically stable if  $\mu_2 > \rho_2$ ,  $\mu_1 > \eta_2$  and  $\kappa_1 (\tau_1 - \gamma_1) < 2\tau_1 \mu_1$ , otherwise unstable.*

**Proof:** The  $J(E_1)$  is given as

$$J = \begin{pmatrix} \tau_1 - \gamma_1 - \frac{2\tau_1 u_1^*}{\kappa_1} & \gamma_2 & -\frac{\eta_1 u_1^{*2}}{h + u_1^{*2}} & -\tilde{n}_1 u_1^* \\ \gamma_1 & \tau_2 - \gamma_2 - \frac{2\tau_2 u_2^*}{\kappa_2} & 0 & 0 \\ 0 & 0 & \frac{\eta_2 u_1^{*2}}{h + u_1^{*2}} - \mu_1 & 0 \\ 0 & 0 & 0 & \tilde{n}_2 u_1^* - \mu_2 \end{pmatrix}$$

The roots  $\lambda_i$  ( $i = 1, 2, 3, 4$ ) of  $J(E_1)$  are obtained by solving the characteristic equation defined by  $|J(E_1) - \lambda_i I| = 0$  while upon mathematical computations leads to

$$\left\{ (\tilde{n}_2 u_1^* - \mu_2) - \lambda_1 \right\} \left\{ \left( \frac{\eta_2 u_1^{*2}}{h + u_1^{*2}} - \mu_1 \right) - \lambda_2 \right\} \begin{vmatrix} A_{11} - \lambda_4 & \gamma_2 \\ \gamma_1 & A_{22} - \lambda_3 \end{vmatrix} = 0,$$

Where the entries  $A_{11} = \frac{\kappa_1 \tau_1 - \kappa_1 \gamma_1 - 2\tau_1 u_1^*}{\kappa_1}$  and  $A_{22} = \frac{\kappa_2 \tau_2 - \kappa_2 \gamma_2 - 2\tau_2 u_2^*}{\kappa_2}$ , from which

$$\left( \tilde{n}_2 u_1^* - \mu_2 \right) - \lambda_1 = 0 \quad \text{and} \quad \left( \frac{\eta_2 u_1^{*2}}{h + u_1^{*2}} - \mu_1 \right) - \lambda_2 = 0 \quad \text{hence} \quad \lambda_1 = \tilde{n}_2 u_1^* - \mu_2 \quad \text{and}$$

$$\lambda_2 = \frac{\eta_2 u_1^{*2}}{h + u_1^{*2}} - \mu_1. \text{ Similarly, the other eigenvalues of } J(E_1) \text{ are obtained by solving}$$

$$\left\{ \frac{\kappa_1 \tau_1 - \kappa_1 \gamma_1 - 2\tau_1 u_1^*}{\kappa_1} - \lambda_4 \right\} \left\{ \frac{\kappa_2 \tau_2 - \kappa_2 \gamma_2 - 2\tau_2 u_2^*}{\kappa_2} - \lambda_3 \right\} - \gamma_1 \gamma_2 = 0. \tag{28}$$

Making  $\lambda_3$  the subject of the formula from Equation 28 leads to

$$\lambda_3 = \left\{ \frac{\kappa_2 \tau_2 - \kappa_2 \gamma_2 - 2\tau_2 u_2^*}{\kappa_2} \right\} - \left\{ \frac{\kappa_1 \gamma_1 \gamma_2}{\kappa_1 \tau_1 - \kappa_1 \gamma_1 - 2\tau_1 u_1^* - \kappa_1 \lambda_4} \right\}.$$

If we consider  $\lambda_3$  as being negative, that is  $\lambda_3 = -x$ , then  $\lambda_4$  can be obtained from

$$-x = \left\{ \frac{\kappa_2 \tau_2 - \kappa_2 \gamma_2 - 2\tau_2 u_2^*}{\kappa_2} \right\} - \left\{ \frac{\kappa_1 \gamma_1 \gamma_2}{\kappa_1 \tau_1 - \kappa_1 \gamma_1 - 2\tau_1 u_1^* - \kappa_1 \lambda_4} \right\}, \text{ for which,}$$

$$\lambda_4 = \frac{\kappa_1 \tau_1 - \kappa_1 \gamma_1 - 2\tau_1 u_1^*}{\kappa_1} - \frac{\kappa_1 \gamma_1 \gamma_2}{\kappa_2 x + \kappa_2 \left\{ \frac{\kappa_2 \tau_2 - \kappa_2 \gamma_2 - 2\tau_2 u_2^*}{\kappa_2} \right\} - \kappa_1 \lambda_4}.$$

Local asymptotic stability of  $E_1$  requires that all the eigenvalues  $\lambda_1, \lambda_2, \lambda_3$  and  $\lambda_4$  of  $J(E_1)$  be negative. Therefore, since  $\lambda_3 < 0$ , it implies that  $\lambda_4 < 0$  if  $\kappa_1(\tau_1 - \gamma_1) < 2\tau_1 u_1^*$ . Moreover,

for the negativity of the eigenvalues  $\lambda_1 = \tilde{n}_2 u_1^* - \mu_2$  and  $\lambda_2 = \frac{\eta_2 u_1^{*2}}{h + u_1^{*2}} - \mu_1$ , then the

conditions  $\mu_1 > \eta_2$  and  $\mu_2 > \varrho_2$  must hold. Hence,  $E_1$  is locally asymptotically stable when;  $\mu_2 > \varrho_2$ ,  $\kappa_1(\tau_1 - \gamma_1) < 2\tau_1 u_1$  and  $\mu_1 > \eta_2$ , otherwise is unstable.

**Proposition 3.3** *The specialist predator-free fixed point  $E_2$  is locally asymptotically stable if  $\eta_2 < (\mu_1 + \xi_1)$ ,  $e_i > 0$  ( $i = 1, 2, 3$ ) and  $e_1 e_2 - e_3 > 0$ ; where  $e$ 's are defined in the proof that follow, otherwise,  $E_2$  is unstable.*

**Proof:** The  $J(E_2)$  is given as

$$J(E_2) = \begin{pmatrix} \tau_1 - \gamma_1 - \frac{2\tau_1 u_1^*}{\kappa_1} - \tilde{n}_1 w^* & \gamma_2 & -\frac{\eta_1 u_1^{*2}}{h + u_1^{*2}} & -\tilde{n}_1 u_1^* \\ \gamma_1 & \tau_2 - \gamma_2 - \frac{2\tau_2 u_2^*}{\kappa_2} & 0 & 0 \\ 0 & 0 & \frac{\eta_2 u_1^{*2}}{h + u_1^{*2}} - \mu_1 - \xi_1 w^* & 0 \\ \tilde{n}_2 w^* & 0 & \tilde{n}_2 w^* & \tilde{n}_2 u_1^* - \mu_2 \end{pmatrix}$$

For simplicity, the above matrix can as well be written as

$$J(E_2) = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & 0 & 0 \\ 0 & 0 & b_{33} & 0 \\ b_{41} & 0 & b_{43} & b_{44} \end{pmatrix},$$

where  $b_{ij}$  ( $i, j = 1, 2, 3, 4$ ) are the entries corresponding to the relevant rows and columns respectively. Finding  $\lambda_i$  ( $i = 1, 2, 3, 4$ ) of  $J(E_2)$  involves solving the characteristic equation  $|J(E_2) - \lambda_i I| = 0$ , which can easily be simplified to

$$(b_{33} - \lambda_1) = \begin{vmatrix} b_{11} - \lambda & b_{12} & b_{14} \\ b_{21} & b_{22} - \lambda & 0 \\ b_{41} & 0 & b_{44} - \lambda \end{vmatrix} = 0. \tag{29}$$

Hence, one of the eigenvalues of  $J(E_2)$  is determined by solving the equation  $b_{33} - \lambda_1 = 0$ . So,

$$\lambda_1 = b_{33} = \frac{\eta_2 u_1^{*2}}{h + u_1^{*2}} - \mu_1 - \xi_1 w^*$$

and the remaining roots of  $J(E_2)$  are obtained by

simplifying the second part of Equation 29 to obtain

$$b_{41}[-b_{14}(b_{22} - \lambda)] + (b_{44} - \lambda)[(b_{11} - \lambda)(b_{22} - \lambda) - b_{12}b_{21}] = 0. \tag{30}$$

Equation 30 can further be simplified to  $\lambda^3 + e_1 \lambda^2 + e_2 \lambda + e_3 = 0$  where the coefficients  $e_i$  ( $i = 1, 2, 3$ ) are defined as;  $e_1 = -(b_{11} + b_{22} + b_{44})$ ,  $e_2 = (b_{11}b_{22} + b_{11}b_{44} + b_{22}b_{44} - b_{12}b_{21} - b_{14}b_{41})$  and  $e_3 = (b_{14}b_{41}b_{22} - b_{11}b_{22}b_{44} + b_{12}b_{21}b_{44})$ .

By Routh-Hurwitz criterion (See [12], and [13],  $\lambda_i$  have negative real parts if  $e_3 > 0$ ,  $e_1 > 0$ ,  $e_2 > 0$  and  $e_1e_2 - e_3 > 0$ , and. In addition, given the first eigenvalue  $\frac{\eta_2 u_1^{*2}}{h + u_1^{*2}} - \mu_1 - \xi_1 v^*$ ,  $\eta_2 < (\mu_1 + \xi_1)$ . Thus,  $E_2$  is locally asymptotically stable if these conditions are satisfied, otherwise unstable.

**Proposition 3.4** *The generalist predator-free fixed point  $E_3$  is locally asymptotically stable if  $\mu_2 > (\rho_2 + \xi_2)$ ,  $e_i > 0$  ( $i = 1, 2, 3$ .) and  $e_1e_2 - e_3 > 0$ ; where  $e$ 's are defined in the following proof, otherwise unstable.*

**Proof:** The Matrix  $J$  evaluated at  $E_3$  is expressed as

$$J(E_3) = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & 0 & 0 \\ b_{31} & 0 & b_{33} & b_{34} \\ 0 & 0 & 0 & b_{44} \end{pmatrix}$$

where,

$$b_{11} = \tau_1 - \gamma_1 - \frac{2\tau_1 u_1^*}{\kappa_1} - \frac{2\eta_1 u_1^* v^* (h + u_1^{*2}) - 2\eta_1 u_1^{*3} v^*}{(h + u_1^{*2})^2}, \quad b_{22} = \tau_2 - \gamma_2 - \frac{2\tau_2 u_2^*}{\kappa_2},$$

$$b_{33} = \frac{\eta_2 u_1^{*2}}{h + u_1^{*2}} - \mu_1, \quad b_{31} = \frac{2\eta_2 u_1^* v^* (h + u_1^{*2}) - 2\eta_2 u_1^{*3} v^*}{(h + u_1^{*2})^2}, \quad b_{44} = \tilde{n}_2 u_1^* + \xi_2 v^* - \mu_2,$$

$$b_{12} = \gamma_2, \quad b_{13} = -\frac{\eta_1 u_1^{*2}}{h + u_1^{*2}}, \quad b_{14} = -\tilde{n}_1 u_1^*, \quad b_{21} = \gamma_1, \quad b_{34} = -\xi_1 v^*$$

The eigenvalues  $\lambda_i$  ( $i = 1, 2, 3, 4$ ) of  $J(E_3)$  are obtained by solving the auxiliary equation  $|J(E_3) - \lambda_i I| = 0$ , which simplifies to

$$(b_{44} - \lambda_1) = \begin{vmatrix} (b_{11} - \lambda) & b_{12} & b_{13} \\ b_{21} & b_{22} - \lambda & 0 \\ b_{31} & 0 & b_{33} - \lambda \end{vmatrix} = 0, \tag{31}$$

from which,  $(b_{44} - \lambda_1) = 0, \Rightarrow \lambda_1 = b_{44} = \tilde{n}_2 u_1^* + \xi_2 v^* - \mu_2$ , and the three remaining eigenvalues are determined by simplifying the second part of Equation 31 to get,  $b_{31} [-b_{13}(b_{22} - \lambda)] + (b_{33} - \lambda)[(b_{11} - \lambda)(b_{22} - \lambda) - b_{12}b_{21}] = 0$ , whose expansion can be expressed in the form of  $\lambda^3 + e_1\lambda^2 + e_2\lambda + e_3 = 0$ , where the coefficients  $e_i$  ( $i = 1, 2, 3$ ) are defined as;  $e_1 = -(b_{11} + b_{22} + b_{33})$ ,  $e_2 = (b_{11}b_{33} + b_{22}b_{33} + b_{11}b_{22} - b_{12}b_{21} - b_{13}b_{31})$  and  $e_3 = (b_{13}b_{31}b_{22} - b_{11}b_{22}b_{33} + b_{12}b_{21}b_{33})$ . All the roots of the cubic equation  $\lambda^3 + e_1\lambda^2 + e_2\lambda + e_3 = 0$  will have negative real parts if by Routh-Hurwitz criterion [14], [15],  $e_1 > 0$ ,  $e_2 > 0$ ,  $e_3 > 0$  and  $e_1e_2 - e_3 > 0$ . Thus, the

Routh-Hurwitz conditions ( $e_1 > 0, e_2 > 0, e_3 > 0$  and  $\varphi = e_1 \times e_2 - e_3 > 0$ ) for the negativity of the real parts of all the eigenvalues of  $J(E_3)$  have to be satisfied if in addition from the first eigenvalue  $\tilde{n}_2 u_1^* + \xi_2 v^* - \mu_2, \mu_2 > (\tilde{n}_2 + \xi_2)$  for the local asymptotic stability of  $E_3$ . Hence,  $E_3$  is locally asymptotically stable if these conditions hold, otherwise unstable.

**Proposition 3.5** *The coexistence equilibrium point  $E_4$  is locally asymptotically stable whenever  $e_1 > 0, e_3 > 0$  and  $e_1 e_2 e_3 - e_3^2 - e_1^2 e_4 > 0$ , where  $e$ 's are discussed in the following proof.*

**Proof:** Consider the Jacobian Matrix 26 evaluated at  $E_4$  below

$$J(E_2) = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & 0 & 0 \\ 0 & 0 & b_{33} & 0 \\ b_{41} & 0 & b_{43} & b_{44} \end{pmatrix},$$

where,

$$b_{11} = \frac{\kappa_1 \tau_1 - \kappa_1 \gamma_1 - 2\tau_1 u_1^*}{\kappa_1} - \frac{2\eta_1 u_1^* v^* (h + u_1^{*2}) - 2\eta_1 u_1^{*3} v^*}{(h + u_1^{*2})^2} - \tilde{n}_1 w^*, b_{12} = \gamma_2$$

$$b_{13} = -\frac{\eta_1 u_1^{*2}}{h + u_1^{*2}}, b_{14} = -\tilde{n}_1 u_1^*, b_{44} = \tilde{n}_2 u_1^* + \xi_2 v^* - \mu_2, b_{21} = \gamma_1, b_{34} = -\xi_1 v^*$$

$$b_{22} = \frac{\kappa_2 \tau_2 - \kappa_2 \gamma_2 - 2\tau_2 u_2^*}{\kappa_2}, b_{31} = \frac{2\eta_2 u_1^* v^* (h + u_1^{*2}) - 2\eta_2 u_1^{*3} v^*}{(h + u_1^{*2})^2}, b_{41} = \tilde{n}_2 w^*$$

$$b_{43} = \tilde{n}_2 w^*, b_{33} = \frac{\eta_2 u_1^{*2}}{h + u_1^{*2}} - \mu_1.$$

The eigenvalues  $\lambda_i$  of  $J(E_4)$  are obtained by solving the auxiliary equation  $|J(E_4) - \lambda_i I| = 0$ . Therefore,

$$|J(E_4) - \lambda_i I| = \begin{vmatrix} b_{11} - \lambda & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} - \lambda & 0 & 0 \\ b_{31} & 0 & b_{33} - \lambda & b_{34} \\ b_{41} & 0 & b_{43} & b_{44} - \lambda \end{vmatrix} = 0.$$

Using the cofactor method for determinant, the characteristic polynomial for the variational matrix  $J(E_4)$  can be reduced to  $\lambda^4 + e_1 \lambda^3 + e_2 \lambda^2 + e_3 \lambda + e_4 = 0$ , where the coefficients  $e_i (i = 1, 2, 3, 4)$  are defined as  $e_1 = -(b_{11} + b_{22} + b_{33} + b_{44}), e_2 = b_{11} b_{22} + b_{22} b_{33} + b_{22} b_{44} + b_{11} b_{33} + b_{11} b_{44} + b_{33} b_{44} - b_{12} b_{21} - b_{13} b_{31} - b_{14} b_{41} - b_{34} b_{43}, e_3 = b_{12} b_{21} b_{33} + b_{12} b_{21} b_{44} + b_{13} b_{31} b_{22} + b_{14} b_{41} b_{22} +$

$$b_{11}b_{34}b_{43} + b_{13}b_{31}b_{44} + b_{14} + b_{41}b_{33} + b_{22}b_{34}b_{43} - b_{11}b_{22}b_{33} - b_{11}b_{22}b_{44} - b_{22}b_{33}b_{44} - b_{11}b_{33}b_{44} - b_{13}b_{34}b_{41} - b_{14}b_{31}b_{43}, \text{ and } e_4 = b_{12}b_{21}b_{34}b_{43} + b_{11}b_{22}b_{33}b_{44} + b_{13}b_{34}b_{22}b_{41} + b_{14}b_{31}b_{22}b_{43} - b_{12}b_{21}b_{33}b_{44} - b_{11}b_{22}b_{34}b_{43} - b_{13}b_{31}b_{22}b_{44} - b_{14}b_{41}b_{22}b_{33}.$$

According to Routh-Hurwitz criterion [1], the eigenvalues of  $J(E_4)$  will have roots with negative real parts if  $e_1 > 0, e_2 > 0, e_3 > 0, e_4 > 0, e_1e_2 - e_3 > 0$  and  $(e_1e_2 - e_3)e_3 - e_1^2e_4 > 0$ . Therefore, the positive equilibrium point  $E_4$  is locally asymptotically stable if  $e_1 > 0, e_2 > 0, e_3 > 0, e_4 > 0$  and  $(e_1e_2 - e_3)e_3 - e_1^2e_4 > 0$ , otherwise unstable.

### 3.2. Global Stability of $E_1, E_2, E_3$ and $E_4$

In this section, we apply the knowledge of the Lyapunov direct method to investigate the global stability of the System 2. This is done by employing the relevant Lyapunov function as demonstrated in the proof following the theorem below.

**Theorem** Suppose that  $E_1, E_2, E_3$  and  $E_4$  are locally asymptotically stable in  $\mathbb{R}_+^4$ . Then  $E_1, E_2, E_3$  and  $E_4$  are globally asymptotically stable in  $\mathbb{R}_+^4$  whenever the following condition holds

$$\tilde{n}_1\eta_2\xi_2 < \tilde{n}_2\eta_1\xi_1 \tag{32}$$

**Proof:** Let us consider a positive definite function for the coexistence equilibrium point  $E_4$  defined by

$$\begin{aligned} \Phi_4(u_1, u_2, v, w) &= c_1 \left[ (u_1 - u_1^*) - u_1^* \ln \left( \frac{u_1}{u_1^*} \right) \right] + c_2 \left[ (u_2 - u_2^*) - u_2^* \ln \left( \frac{u_2}{u_2^*} \right) \right] \\ &+ c_3 \left[ (v - v^*) - v^* \ln \left( \frac{v}{v^*} \right) \right] + c_4 \left[ (w - w^*) - w^* \ln \left( \frac{w}{w^*} \right) \right], \end{aligned}$$

where  $c_1, c_2, c_3$  and  $c_4$  are positive constants to be suitably chosen.

Differentiating the function  $\Phi_4(u_1, u_2, v, w)$  with respect to time,  $t$  along the solutions of System 2 gives

$$\begin{aligned} \frac{d\Phi_4}{dt} &= \frac{\partial\Phi}{\partial u_1} \cdot \frac{du_1}{dt} + \frac{\partial\Phi}{\partial u_2} \cdot \frac{du_2}{dt} + \frac{\partial\Phi}{\partial v} \cdot \frac{dv}{dt} + \frac{\partial\Phi}{\partial w} \cdot \frac{dw}{dt} \\ &= c_1 \left( \frac{u_1 - u_1^*}{u_1} \right) \frac{du_1}{dt} + c_2 \left( \frac{u_2 - u_2^*}{u_2} \right) \frac{du_2}{dt} + c_3 \left( \frac{v - v^*}{v} \right) \frac{dv}{dt} \\ &+ c_4 \left( \frac{w - w^*}{w} \right) \frac{dw}{dt}. \end{aligned} \tag{33}$$

Now, substituting the values of  $u_1, u_2, v$  and  $w$  from System 2 into Equation 38 leads to

$$\begin{aligned} \dot{\Phi}_4 &= c_1(u_1 - u_1^*) \left[ \tau_1 \left( 1 - \frac{u_1}{\kappa_1} \right) - \gamma_1 + \frac{\gamma_2 u_2}{u_1} - \frac{\eta_1 u_1 v}{h + u_1^2} - \varrho_1 w \right] \\ &+ c_2(u_2 - u_2^*) \left[ \tau_2 \left( 1 - \frac{u_2}{\kappa_2} \right) - \gamma_2 + \frac{\gamma_1 u_1}{u_2} \right] + c_4(w - w^*)(\varrho_2 u_1 + \xi_2 v - \mu_2) \\ &+ c_3(v - v^*) \left[ \frac{\eta_2 u_1^2}{h + u_1^2} - \mu_1 - \xi_1 w \right]. \\ \Phi_4 &= c_1(u_1 - u_1^*) \left[ -\frac{\tau_1}{\kappa_1}(u_1 - u_1^*) + \gamma_2 \left( \frac{u_2}{u_1} - \frac{u_2^*}{u_1^*} \right) - \frac{\eta_1(u_1 - u_1^*)(v - v^*)}{(h + u_1^2)(h + u_1^{*2})} \right] \\ &- c_1 \varrho_1(w - w^*) + c_2(u_2 - u_2^*) \left[ -\frac{\tau_2}{\kappa_2}(u_2 - u_2^*) + \gamma_1 \left( \frac{u_1}{u_2} - \frac{u_1^*}{u_2^*} \right) \right] \\ &+ c_3(v - v^*) \left[ \frac{\eta_2(u_1 - u_1^*)^2}{(h + u_1^2)(h + u_1^{*2})} - \xi_1(w - w^*) \right] + c_4 \varrho_2(u_1 - u_1^*)(w - w^*) \\ &+ c_4 \xi_2(v - v^*)(w - w^*). \end{aligned}$$

Carrying out further mathematical computations result to

$$\begin{aligned} &= -\frac{c_1 \tau_1}{\kappa_1}(u_1 - u_1^*)^2 + c_1 \gamma_2(u_1 - u_1^*) \left( \frac{u_2 u_1^* - u_1 u_2^*}{u_1 u_1^*} \right) - \frac{c_2 \tau_2}{\kappa_2}(u_2 - u_2^*)^2 \\ &- c_1 \left[ \frac{\eta_1(v - v^*)(u_1 - u_1^*)^2}{(h + u_1^2)(h + u_1^{*2})} \right] - c_1 \varrho_1(u_1 - u_1^*)(w - w^*) \\ &+ c_2 \gamma_1(u_2 - u_2^*) \left( \frac{u_1 u_2^* - u_2 u_1^*}{u_2 u_2^*} \right) + c_3 \left( \frac{\eta_2(u_1 - u_1^*)^2(v - v^*)}{(h + u_1^2)(h + u_1^{*2})} \right) \\ &- c_3 \xi_1(v - v^*)(w - w^*) + c_4 \varrho_2(u_1 - u_1^*)(w - w^*) + c_4 \xi_2(v - v^*)(w - w^*). \end{aligned}$$

Choosing the positive constants  $c_1, c_2, c_3$  and  $c_4$  tactfully as

$$C_1 = 1, \quad c_2 = \frac{\gamma_2 u_2^*}{\gamma_1 u_1^*}, \quad c_3 = \frac{\eta_1}{\eta_2} \text{ and } c_4 = \frac{\tilde{n}_1}{\tilde{n}_2}, \text{ we end up with}$$

$$\begin{aligned} \dot{\Phi}_4 &= -\frac{\tau_1}{\kappa_1}(u_1 - u_1^*)^2 - \left( \frac{\gamma_2}{u_1^*} \right) (u_1 u_2^* - u_2 u_1^*) - \left[ \frac{\eta_1(u_1 - u_1^*)^2(v - v^*)}{(h + u_1^2)(h + u_1^{*2})} \right] \\ &- \varrho_1(u_1 - u_1^*)(w - w^*) - \left( \frac{\gamma_2 u_2^* \tau_2}{\kappa_2 \gamma_1 u_1^*} \right) (u_2 - u_2^*)^2 + \left( \frac{\gamma_2}{u_1^*} \right) (u_1 u_2^* - u_2 u_1^*) \\ &+ \left[ \frac{\eta_1(u_1 - u_1^*)^2(v - v^*)}{(h + u_1^2)(h + u_1^{*2})} \right] - \frac{\eta_1 \xi_1(v - v^*)(w - w^*)}{\eta_2} + \varrho_1(u_1 - u_1^*)(w - w^*) \\ &+ \frac{\varrho_1 \xi_2(v - v^*)(w - w^*)}{\varrho_2}. \end{aligned}$$

Therefore, we finally end up with

$$\begin{aligned} \frac{d\Phi_4}{dt} &= -\frac{\tau_1}{\kappa_1}(u_1 - u_1^*)^2 - \left( \frac{\gamma_2 u_2^* \tau_2}{\kappa_2 \gamma_1 u_1^*} \right) (u_2 - u_2^*)^2 - \frac{\varrho_1 \mu_2 w}{\varrho_2} \\ &- \left( \frac{\varrho_2 \eta_1 \xi_1 - \varrho_1 \eta_2 \xi_2}{\eta_2 \varrho_2} \right) (v - v^*)w. \end{aligned} \tag{34}$$

The time derivative  $\dot{\Phi}_4$  denoted by Equation 34 is negative definite whenever the inequality  $\tilde{n}_1\eta_2\xi_2 < \tilde{n}_2\eta_1\xi_1$  holds. Therefore, the coexistence equilibrium point  $E_4$  is globally asymptotically stable in  $\mathbb{R}_+^4$  whenever Condition 32 holds. The proof for the global stabilities of the other equilibrium points is done in a similar way.

### 4. Numerical Simulations

In this section, numerical simulations are used to verify the obtained analytical results for the System 2. The System 2 is numerically solved using the following set of biologically feasible hypothetical parameters with given initial points

$$\tau_1 = 4, \tau_2 = 3.5, \gamma_1 = 2.5, \gamma_2 = 1.5, \eta_1 = 2, \eta_2 = 1.5, \varrho_1 = 0.25, \kappa_1 = 30, \varrho_2 = 0.025, \kappa_2 = 40, \xi_1 = 0.5, \xi_2 = 0.25, h = 1, \mu_1 = 0.6, \mu_2 = 0.8 \tag{35}$$

With the above parameters, it is noted that all the conditions for existence of the equilibrium points are satisfied. Therefore, the equilibrium points of System 2 exist and given by:  $E_1(28.1, 43.0, 0.0, 0.0)$ ,  $E_2(28.0, 43.8, 0.0, 1.4)$ ,  $E_3(0.8, 23.8, 46.1, 0.0)$  and  $E_4(25.5, 40.7, 0.7, 1.8)$ . Time series of the solutions are then drawn using the in-built ordinary differential equation solver; MATLAB function ode45 software package under the above set of parameters to validate the obtained solutions for each equilibrium point as demonstrated in the subsections that follow.

#### 4.1. The Coexistence Equilibrium Point

The positive equilibrium point's solution gives a threshold level under which all the four species coexist.

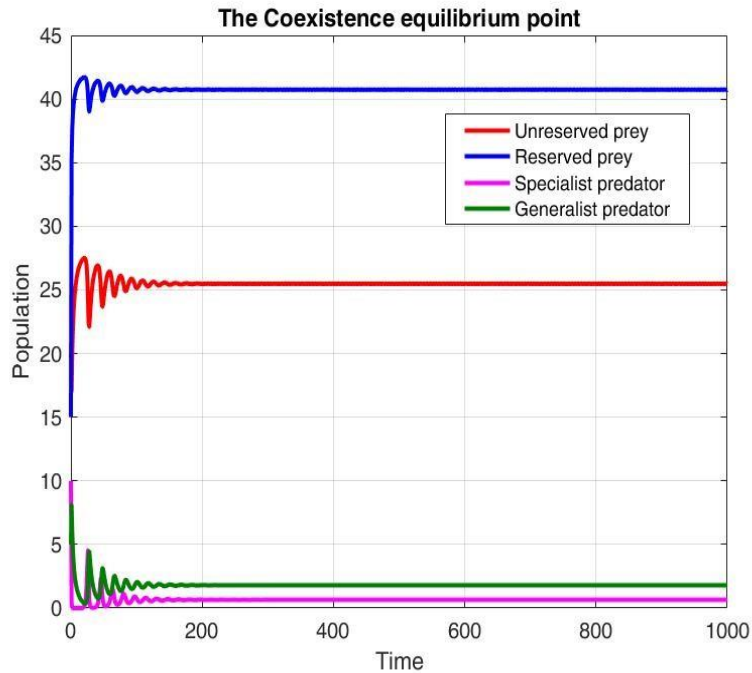


Fig. 1 Graphical representation of  $E_4$  with the data given by Equation 40

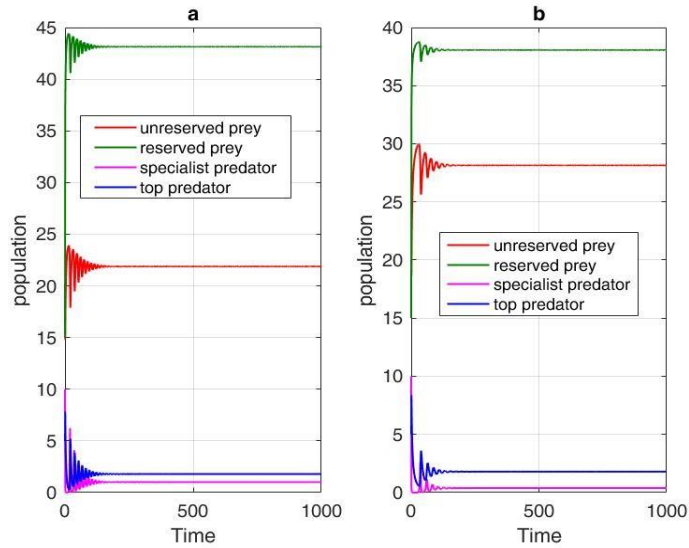


Figure 1 shows that as  $t$  tends to infinity, the solutions  $u_1(t)$ ,  $u_2(t)$ ,  $v(t)$  and  $w(t)$  oscillates for a small period of time before asymptotically tending to the positive equilibrium point  $E_4(25.5, 40.7, 0.7, 1.8)$ , in the long-time limit obtained by solving the algebraic equations for equilibrium points, which agree with the simulation results. Thus, the respective populations can coexist.

At this point, we study the effects of varying certain parameters of the System 2 on the dynamical behavior of its solutions. We demonstrate the effects of varying key parameters, and the corresponding effects on the dynamical behavior of the System 2 illustrated graphically.

### 4.2. Effect of varying migration rate coefficient $\gamma_1$

For various values of  $\gamma_1$ , the solution oscillates briefly before asymptotically approaching the equilibrium points  $(21.9, 43.1, 1.0, 1.8)$  when  $\gamma_1 = 3.5$  and  $(28.1, 38.1, 0.4, 1.8)$  when  $\gamma_1 = 1.8$ .

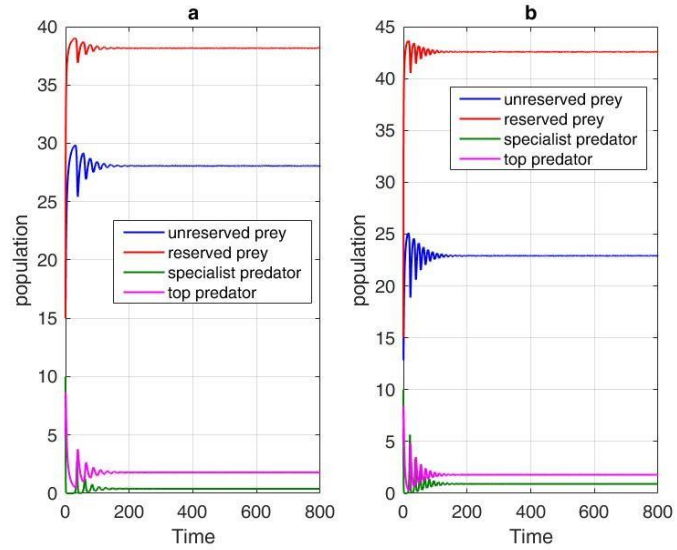


**Fig. 2** Convergence of the solution with (a)  $\gamma_1 = 3.5$  and (b)  $\gamma_1 = 1.8$ .

The population density of prey in both the reserved and unreserved zones grows with increase in  $\gamma_1$ . On the other hand, decreasing the value of  $\gamma_1$  results in an increase in the density of unprotected prey and a slight decrease in the density of protected prey. However, the top predator is not affected by these variations.

### 4.3. Effect of varying migration rate coefficient $\gamma_2$

Figure 3 below illustrates the dynamics of System 4.2.4 in which the solutions approach the stable points  $(28.07, 38.16, 0.39, 1.80)$  when  $\gamma_2 = 2$  and  $(22.90, 42.57, 0.91, 1.80)$  when  $\gamma_2 = 1.12$ .

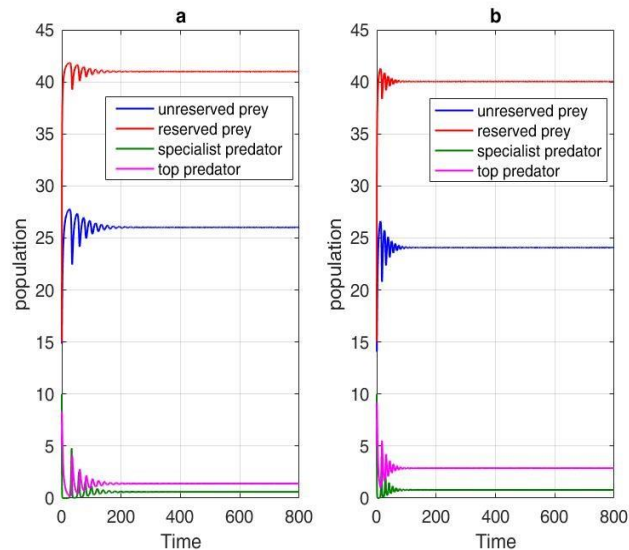


**Fig. 3** Convergence of the solution (a)  $\gamma_2 = 2.0$  and (b)  $\gamma_2 = 1.12$ .

It can be seen from Figure 3(a) that as  $\gamma_2$  increases, the number of unreserved prey increases while the population density of the reserved prey decreases noticeably. This indicates a surplus of prey in unprotected region and as a result, the overall predator’s rate of prey conversion to food rises. The population density of prey in the unreserved region and the specialist predators are also severely affected by a decline in  $\gamma_2$ .

**4.4. Dynamics of System 4.2.4 with different values of  $\mu_1$**

The solutions approaches the stable points (26.0, 41.0, 0.6, 1.4) for  $\mu_1 = 0.8$  and (24.1, 40.0, 0.8, 2.9) for  $\mu_1 = 0.06$  as confirmed in the Figure 4 below.



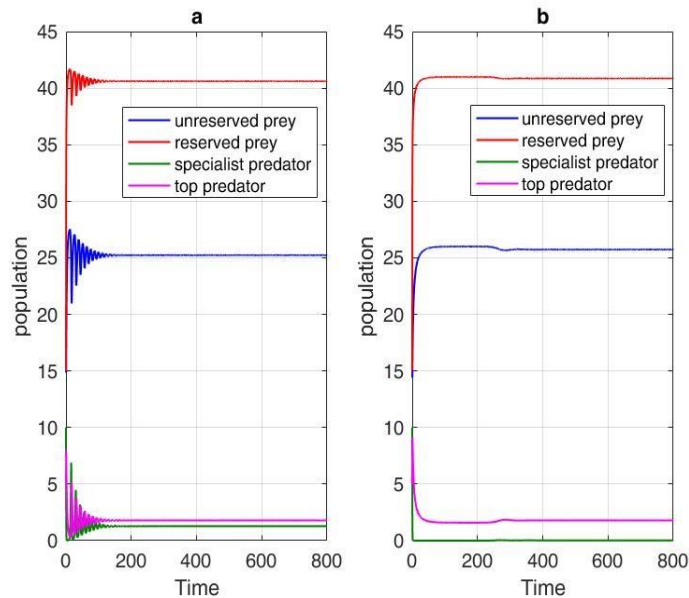
**Fig. 4** Convergence of the solution with (a)  $\mu_1 = 0.8$  and (b)  $\mu_1 = 0.06$ .

An increase in  $\mu_1$  leads to an abundance of prey in the unprotected zone and a small decline in the generalist predator’s ability to convert prey into food and hence a slight decrease in the density of top predator as depicted in Figure 4(a). However, decreasing the value of  $\mu_1$  only results in a noticeable increase in the top predator while the other species are only slightly affected as illustrated by Figure 4(b).

#### 4.5. Effect of varying the natural death rate $\mu_2$

Figure 5 shows that a change in  $\mu_2$  initiates oscillations for a short time before approaching the equilibrium points (25.23, 40.61, 1.28, 1.80) when  $\mu_2 = 0.95$  and (25.7, 40.8, 0.2, 1.8) for  $\mu_2 = 0.7$ .

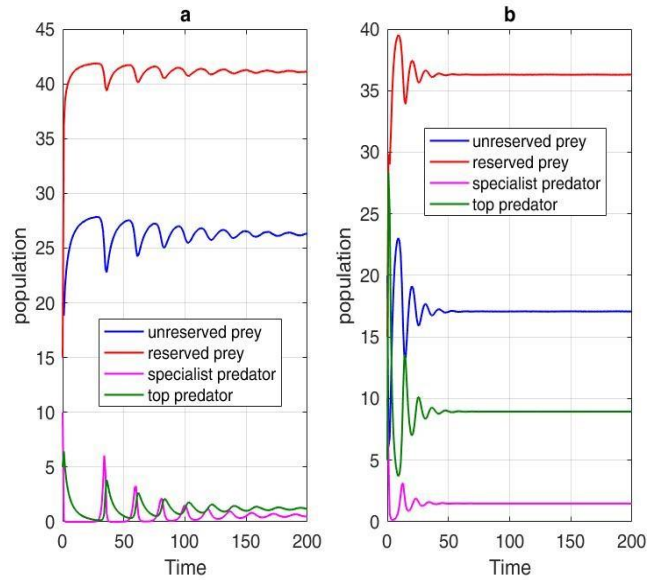
The densities of reserved prey, unreserved prey, as well as generalist predator are insignificantly affected by changes in  $\mu_2$ . However, an increase in  $\mu_2$  leads to an increase in the density of specialist predators as shown in Figure 5(a) and vice versa as illustrated in Figure 5(b). This indicates an abundance of prey in exposed areas, increasing the rate at which prey becomes food for specialist predators.



**Fig. 5** Convergence of the solution with (a)  $\mu_2 = 0.95$  and (b)  $\mu_2 = 0.70$ .

#### 4.6. Effect of varying the attack rate $\xi_1$

The System 2 loses persistence as a result of the numerous oscillations caused by an increase in the value of  $\zeta_1$ , and the chain is broken as depicted in Figure 6(a). However, the solutions asymptotically approach the stable points (26.3, 41.1, 0.6, 1.2) when  $\zeta_1 = 0.75$  and (17.1, 36.3, 1.5, 9.0) when  $\zeta_1 = 0.1$



**Fig. 6** Convergence of the solution with (a)  $\zeta_1 = 0.75$  and (b)  $\zeta_1 = 0.1$ .

## 5. Conclusions

In this paper, a total of five equilibrium points were found to exist and the conditions for their existence derived. Using the eigenvalue approach, Routh-Hurwitz criterion and the Lyapunov technique, all the equilibrium points of the System 2 were investigated for their stability and the conditions for asymptotic stability of each derived. To verify the analytical results, numerical simulations have been performed using MATLAB function ode45 software package and time series graphs of the solutions of System 2 drawn using the in-built ordinary differential equation solver. According to the simulation results, the solutions of the System 2 have been found to undergo a few oscillations before settling down to the equilibrium points in the long run. Moreover, a small variation in the migratory parameters  $\gamma_1$  and  $\gamma_2$  caused the species to significantly change. Hence, this confirms the significance of the reserved zone in affecting the dynamical stability of the System 2.

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## Declaration

**Conflict of interest:** The authors declare no conflict of interest. The authors have no relevant financial interest to disclose.

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