JARAMOGI OGINGA ODINGA UNIVERSITY OF SCIENCE AND TECHNOLOGY

SCHOOL OF BIOLOGICAL, PHYSICAL, MATHEMATICS AND ACTURIAL SCIENCES

UNIVERSITY EXAMINATION FOR DEGREE OF MASTER OF SCIENCE IN PURE MATHEMATICS

1st YEAR 1st SEMESTER 2023/2024 ACADEMIC YEAR

MAIN REGULAR

COURSE CODE: SMA 801

COURSE TITLE: ABSTRACT INTEGRATION I

EXAM VENUE: **STREAM: (Msc. Pure Mathematics)**

DATE: **EXAM SESSION:**

TIME: 3.00HRS

Instructions:

1. **Answer any THREE questions only**
2. **Candidates are advised not to write on the question paper.**
3. **Candidates must hand in their answer booklets to the invigilator while in the examination room**

QUESTION ONE [20 MARKS]

a) i) Define a measure and discuss its fundamental properties. [6marks]
ii) Explain how measures generalize the concept of length, area, and volume in mathematics. [6marks]

b) Describe the concept of outer measure and its relationship to measures. Use outer measure to construct the Lebesgue measure on $\mathbb{R}$. [8marks]

QUESTION TWO [20 MARKS]

Show that every bounded Riemann integrable function over $[a, b] \subseteq \mathbb{R}$ is Lebesgue integrable and the two integrals are equal and hence, evaluate both the Riemann integral and Lebesgue integral for the function $f(x)$ on the interval $[0, 5]$ defined by

$$f(x) = \begin{cases} 0, if \quad 0 \leq x < 1 \\ 1, if \ \{1 \leq x < 2\} \cup \{3 \leq x < 4\} \\ 2, if \ \{2 \leq x < 3\} \cup \{4 \leq x \leq 5\} \end{cases}$$ [20marks]

QUESTION THREE [20 MARKS]

a) Explain the significance of the Borel σ-algebra in measure theory. Provide examples of sets that are Borel measurable and sets that are not. [10 marks]

b) Prove that the Borel σ-algebra is the smallest σ-algebra on $\mathbb{R}$ that contains all open intervals. [10marks]

QUESTION FOUR [20 MARKS]

a) Explain Carathéodory's Extension Theorem and how it is used to construct measures. [8marks]

b) i) Describe the process of completing a measure space. When is it necessary to complete a measure space? [7marks]
ii) Give an example of a measure space that cannot be completed. Explain why it cannot be completed. [5marks]

QUESTION FIVE [20 MARKS]

a) Give an example of a sequence of functions that converges pointwise but not almost everywhere. Explain why it does not converge almost everywhere. [5marks]

b) Discuss the limitations of the Monotone Convergence Theorem. State Fatou's Lemma and explain how it complements the Monotone Convergence Theorem. [15marks]