JARAMOGI OGINGA ODINGA UNIVERSITY OF SCIENCE AND TECHNOLOGY

SCHOOL OF BIOLOGICAL, PHYSICAL, MATHEMATICS AND ACTUARIAL SCIENCES

UNIVERSITY EXAMINATION FOR MASTERS OF BACHELOR OF ACTUARIAL SCIENCE

1ST YEAR 2ND SEMESTER, 2023/2024 ACADEMIC YEAR

REGULAR (MAIN)

COURSE CODE: SMA 814

COURSE TITLE: FUNCTIONAL ANALYSIS II

EXAM VENUE: **STREAM: (PURE MATHEMATICS)**

DATE: **EXAM SESSION:**

TIME: 3.00 HOURS

Instructions:

1. **Chose ANY THREE questions.**
2. **Candidates are advised not to write on the question paper.**
3. **Candidates must hand in their answer booklets to the invigilator while in the examination room.**

QUESTION ONE (20 marks)

a) Let X be an inner product space over $\mathbb{K}$. If $x_i, y_j \in X$ and $\alpha_i, \beta_j \in \mathbb{K}$ where $i = 1, \dots, n$ and $j = 1, \dots, m$. Show that

$\langle \sum_{i=1}^{n} \alpha_i x_i , \sum_{j=1}^{m} \beta_j y_j \rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \overline{\beta_j} \langle x_i, y_j \rangle$. (8 marks)

b) Let X be a normed linear space over $\mathbb{K}$, X^* its dual and X^{**},the dual of X^*. Consider the map J on X defined by $(Jx)(f) = f(x) \forall f \in X^*$. Show that J and Jx are both bounded and linear. (7 marks)

c) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a normed linear space X. Then show that if weak limit of $(x_n)_{n\in\mathbb{N}}$ exists then it is unique. (5 marks)

QUESTION TWO (20 marks)

a) Let $(X, \langle , \rangle)$ be a normed linear space over $\mathbb{K}$ and $(x_n), (y_n)$ be sequences of X, then show that

i) If $x_n \to x$ and $y_n \to y$ strongly then $\langle x_n, y_n \rangle \to \langle x, y \rangle$ in $(\mathbb{K}, d)$. (5 marks)

ii) If $(x_n), (y_n)$ are strongly Cauchy, then the sequence $(\langle x_n, y_n \rangle)_{n=1}^{\infty}$ of scalars converges in $(\mathbb{K}, d)$. (5 marks)

b) Let Y be an inner product space. If E is a nonvoid subset of Y and $x \perp E$ then show that $x \perp \bar{E}$. Also if E is dense in Y, then $x = \bar{0}$. (5 marks)

c) Let Y be an inner product space and $\{x_\alpha : \alpha \in \Lambda\}$be a summable family elements of Y with sum x. Let $y \in Y$, then show that the family $\{\langle x_\alpha, y \rangle : \alpha \in \Lambda\}$ of scalars are summable to $\langle x, y \rangle$. (5 marks)

QUESTION THREE (20 marks)

a) Let G, H and K be Hilbert spaces over $\mathbb{C}$ and $T \in B(G, H)$ and $S \in B(H, K)$, then show that

i) $(S+T)^* = S^* + T^*$. (4 marks)

ii) $(ST)^* = T^*S^*$. (4 marks)

b) Let X be a Banach space, Y a normed linear space and $(T_n)_{n=1}^{\infty} \in B(X,Y)$ converging strongly to T, then show that $T \in B(X,Y)$. (6 marks)

c) Let X, Y a normed linear space, $D \subseteq X$ and $T: D \to Y$ be a linear transformation bounded from below. Then show that $T^{-1}: R_T \to D$ is bounded. (6 marks)

QUESTION FOUR (20 marks)

a) State and prove the polarization identity. (10 marks)

b) Let X be a Banach space, Y a normed linear space, $D \subseteq X$ and $T: D \longrightarrow Y$ bounded from below be a closed linear transformation then, show that R_T is a closed linear subspace of Y. (10 marks)

QUESTION FIVE (20 marks)

a) State and prove Cauchy-Bunyakowskii-Schwarz inequality. (8 marks)

b) Let Y be an inner product space, N a linear subspace of Y and M a closed linear subspace of Y such that $N \supset M$ properly. Show that there exists a non-zero vector $z \in N$ such that $z \perp M$. (4 marks)

c) Let Y be an inner product space and $M \subseteq Y$. Then show that $M^{\perp}$ is a closed linear subspace of Y and $M \cap M^{\perp} \subseteq \{\bar{0}\}$. (8 marks)