CHARACTERIZATION OF ORTHOGONAL POLYNOMIALS IN NORM-ATTAINABLE CLASSES

BY

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Declaration

This thesis is my own work and has not been presented for a degree award in any other institution.

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This thesis has been submitted for examination with our approval as the university supervisors.

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Dedication

To my cherished wife, Monica Kemunto Mouko, and our beloved children.

Abstract

Orthogonal polynomials within the realm of bounded linear operators on a Hilbert space $(B(H))$ hold a crucial role in operator theory, functional analysis, and various other fields. While significant research has been conducted on orthogonal polynomials and norm-attainable operators, there remains a notable gap in the literature regarding the characterization of orthogonal polynomials in $(NA(H))$ and the relationship between orthogonal polynomials and norm-attainable operators. This study aims to address this knowledge gap by characterizing norm-attainable operators, orthogonal polynomials in $NA(H)$, and establishing their relationship.The research utilizes diverse methodologies such as norm-attainability criteria, Gram-Schmidt orthonormalization, determination of Gram matrix determinants, and exploration of properties associated with classical continuous orthogonal polynomials. The results of the study demonstrate that Harmite, Laguerre, Legendre, and Jacobi polynomials are norm-attainable in $B(H)$, and a Hermitian contraction operator is norm-attainable if its $\|T\|$ or $-\|T\|$ norm lies within its spectrum. Furthermore, it is revealed that orthogonal polynomials exhibit convexity, positivity, and form a normed vector space. Additionally, a self-adjoint closed differential operator on the $L^2([0,1])$ space, which is not bounded and hence not norm-attainable, is identified. This study holds significance as it enhances our comprehension of norm-attainable operators and orthogonal polynomials within the $B(H)$ space. The findings have practical applications in signal processing, data analysis, and harmonic analysis, particularly in the development of Fourier series, wavelength determination, and the L^2-boundedness of singular integral operators.

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Chapter 1

INTRODUCTION

1.1 Mathematical background

Exploring Hilbert space operators has captivated mathematicians. Recent research has unveiled a compelling insight: certain compact linear operators, when situated between Banach spaces, resist approximation by norm-attainable (NA) operators [14], [18], [32], [56] , [63] and [83]. Martin's work [63] explored norm-attainability for bounded linear operators in reflexive spaces, leveraging Diestel's earlier result [26] that established the existence of non-norm-attainable continuous linear functionals in norm reflexive Banach spaces. Shkredov introduced the concept of norm-attainable operators (NA) in Banach spaces [87]. Banach spaces failing certain conditions can have bounded ranks that are not NA [87]. Norm-attainability is further linked to operators with properties like weakly sequentially continuous norm functions and closed sets with weakly sequential compactness [56]. [56] also introduced Property B, which involves analyzing $L(X', Y')$ and $NA(X', Y')$ spaces [56]. When $NA(X', Y')$ is densely embedded within $L(X', Y')$, Banach space Y exhibits property B [56].

Notably, under specific conditions, $NA(X', Y')$ does not form a dense subset of $L(X, Y)$ [2] and [14]. Agu [3] suggested that strict convexity within an infinite-dimensional Banach space is incompatible with property B. The Bishops-Phelps theorem [14] emphasizes that the dual space of norm-attainable functionals on a Banach space is densely populated under norm topology. Lindentrauss's work [56] further contributes to the understanding of property B. Bott, Harr, and Shag pioneered the introduction of ε-pseudospectrum for norm attainability of operators [15] and [88]. Shag [88] demonstrated that the interior of the quotient space of the ε-pseudospectrum can be non-empty in general. Gall conducted further studies on ε-pseudospectrum. Shk [87] analyzed norm attainability conditions on Banach spaces and established that the operator $(1 + T)$ can achieve norm attainability on Banach spaces derived from ℓ_p-direct sums of finite-dimensional Banach spaces, within the range of $1 < p < \infty$," provided certain conditions are met. Shag constructed a reflexive Banach space X with a bounded rank-one operator T such that $|1 + T| > 1$, but $1 + T$ is not norm-attainable [88].

The study by authors [14] on norm-attainable operators sparked questions about the density of these operators in the space $B(X', Y')$ for Banach spaces X' and Y'. Author [56] addressed this, linking it to extreme points in the closed unit ball U_x of X. Notably, for certain spaces Y, norm-attainable operators in $B(L'(0, 1), Y)$ aren't dense because U_x of $L'(0, 1)$ lacks extreme points. This raises the question: "When are norm-attainable operators dense in $B(L'(0, 1), Y)$?". Building on [56], if the closed unit ball U_x of Y exhibits extreme or exposed points, norm-attainable operators may be dense in $B(L'(0,1), Y)$. Researchers [25], [43] and [82], established a connection between the Radon-Nikodym property and extreme points. Therefore, it's plausible that norm-attainable operators are dense in $B(L'(0,1), Y)$ if and only if Y possesses the Radon-Nikodym property. A proposal by [45] suggests exploring this claim, especially within strictly convex Banach spaces. Lindenstrauss [56] established that the density of norm-attainable operators in Banach spaces extends beyond just those with extreme points in their closed unit balls. This result is significant in Banach space theory. The Radon-Nikodym property, which ensures weak compactness of bounded linear operators into $L^\infty(\mu)$, is a key condition for this extension. This property holds considerable importance in functional analysis, with broad-reaching implications. The density of norm-attainable operators is valuable in applications, allowing for the approximation of bounded linear operators by sequences of norm-attainable ones. This has applications in approximation theory, optimization, and more. Ongoing research explores this density in various Banach space settings, often relying on structural properties of closed unit balls, such as extreme and exposed points.

In [62], the author introduced an example of an operator between Banach spaces X and Y that cannot be approximated by norm-attaining operators (NA operators). This challenges the assumption that all operators between Banach spaces can be approximated effectively using NA operators [1], [14], [28], [27] and [56]. The concept of norm-attainability in operator theory has been studied by many authors, including [18], [84] and [86]. In [86], the authors refuted the characterization theorem proposed by Ramesh [84] and provided a full characterization of the class of positive absolutely norm-attainable operators. Their result contradicted the earlier assumption of separability, making it more general. They were able to establish the correct characterization for absolutely norm-attainable operators on complex Hilbert spaces of arbitrary dimensions. Their findings indicated that the class of NA operators on closed subspaces is not closed under addition. However, the intersection of this class with the set of positive operators generated the class of Hermitian operators in the real Banach space. These developments have contributed to a deeper understanding of the properties and structures of operators in Banach spaces. Additionally, the notion of norm-attainability has been explored in operator theory, with researchers in this field deriving norm estimates and establishing conditions for Hilbert spaces in a study referenced as [70]. An in-depth analysis of elementary operators was done in [29]. The characterization of norm-attainable elementary operators with respect to orthogonality was done in [72], [76] and [78]. These studies have contributed to a deeper understanding of the properties and structures of operators in Banach spaces. The properties of the vector spaces on which an operator acts significantly influence the description of the operator. Different notions of norm-attainability have been discussed in the literature, including the notion of norm-attainability for non-power operators on reflexive, linear functionals in Banach spaces, dense, separable infinite dimensional complex H (Hilbert) spaces and elementary operators. The work of [6], [18], [52], [69] and [79], and have contributed to the exploration of orthogonality, completely positive and completely bounded maps, norm-attainable operators, and their interplay within the frameworks of normed spaces, Banach spaces, Hilbert spaces, and operator theory. The class of absolutely norm-attainable operators between Hilbert spaces has been studied. Studies on this class have revealed a lot of examples and properties of such operators. Okelo analyzed these operators in connection with other operator properties like hyponormality and compactness. The spectral analysis of compact hyponormal norm-attainable operators showed that such operators have countable spectrum. A bounded linear compact hyponormal operator in an infinite-dimensional complex Hilbert space satisfies the necessary and sufficient condition for being absolutely norm-attainable when the cardinality of the set of its eigenvalues is finite [18], [72] and [71]. The Bram-Halmos subnormality criterion [16], [22] and [23] is a condition that must be satisfied by an operator T for it to be subnormal. The authors of the criterion also introduced the concept of weak k-hyponormality for hyponormal operators and defined other types of hyponormality such as quadratic, cubic and polynormal and hyponormality. Bishop-Phelps initiated the investigations of the subspaces of norm-attainable operators [14], and Jun Ik Lee specialized on the subspaces of such special norm-attainable operators (paranormal, hyponormal etc.) and showed that the norm-attainable paranormal operators have nontrivial invariant subspaces [47]. The conditions necessary and sufficient for norm-attainability have been extensively explored in previous works, such as the notable papers [51] and [72]. These conditions go beyond norm-attainability alone and delve into the conditions for elementary operators and generalized derivations. The results obtained in these studies shed light on the intricate relationships between these operators and their attainability in terms of norms. Okelo's work, published in [69], contributes to the field by offering a comprehensive characterization and generalization of the conditions for norm-attainability of Hilbert space operators. Okelo's insights provide valuable insights into the various factors influencing the attainability of norms in this context. Additionally, the authors of [79] delved into the topic of orthogonal extensions, focusing on the orthogonality properties of range and kernel of elementary operators implemented by norm-attainable operators in Banach spaces. Their study yielded significant results, including norm-attainable operator-valued orthogonal extensions of matrix inequalities and conditions for the weak convergence of operators in the context of $NA(H)$. Mathieu's work [64] explored the computation of norms of elementary operators on the Calkin algebra. Okelo, Agure, and Ambogo [77] made noteworthy contributions to the characterization of NA operators. Okelo's work [71] provided valuable characterizations of the absolute norm-attainability of compact hyponormal operators. Okelo [69] explored the properties of compact hyponormal operators, particularly when they are self-adjoint or normal, as well as their commutators. The theory of orthogonal polynomials can be traced back to the pioneering works of Stieltjes and Chebyshev [20]. Chebyshev introduced the first kind of polynomials, known as (T_n) polynomials [20]. Legendre polynomials were introduced by Legendre himself and later extended by Jacobi [36], leading to the development of Jacobi polynomials. Hermite defined the Hermite polynomial to facilitate the study of expansion series in $\mathbb{R}$ [36]. The research presented in [67] introduced a novel system of orthogonal polynomials, which exhibited connections to the Meixner-Pollaczek polynomials. The newfound system of orthogonal polynomials was subsequently utilized to investigate the boundedness properties of singular integral operators of convolution type [67]. The relationship between non-negative linearization property and some maximum principle of certain boundary value problem was shown in [92]. Sufficient conditions on orthogonal polynomial system to satisfy the non-negative linearization was also established. The factorization of monic polynomials, where the coefficients are linear bounded operators, is an important subject in mathematics with numerous significant findings [85]. Factorization of such polynomials into a product of several operator polynomial is also important and has been studied by mathematicians such as M.G. Krein, H. Langer, Gohbery, Kaasoek, and Rodman. This problem has several results in connection with oscillations of continua, see [38], [40] and [59]. Herrero [42] considered the problem of factorization of operator polynomials when the coefficient operators are biquasitriangular in nature using approximation criterion. The results in [42] showed that the condition necessary for an operator polynomial to be biquasitriangular is that its Fredholm index must be zero. Rodman [85] conducted a comprehensive study on the density of operator polynomials. One of the key findings of this research was the proof that a diverse set of biquasitriangular polynomials, capable of being factored into monic linear terms, exists within the entire range of biquasitriangular monic operator polynomials. This result suggests that the ability to factorize into monic linear factors is not limited to specific cases but is rather widely prevalent. Mache and Rasa [58] initiated the field of positive polynomial operators, aiming to establish a connection between two well-known operators: the Durrmeyer operators employing Jacobi weights with a parameter value of $\alpha = 0$, and the Bernstein operators as the parameter α tends towards infinity. Mache [58] analyzed such operators p_n, especially their relationship with

Durrmeyer, Bernstein and other operators. Stancu [89] introduced a relatively smaller sequence of such positive polynomials operators which was denoted by V_n. Mache [58] associated the sequence p_n with the simple sequence V_n, by representing a link between Bernstein polynomials and certain Stancu polynomials. Furthermore, [58] established the relationship between them. Laplace made notable contributions to the study of polynomials in probability theory [53]. Laguerre polynomials were introduced by Laguerre himself and gained recognition in the publication [53]. Chebyshev polynomials are central in approximation theory, as highlighted in the research conducted by Balogh and Bertola [9]. They explored the association between Chebyshev polynomials and a compact set $K \subseteq \mathbb{C}$. Remarkably, when considering monic polynomials of degree n, the Chebyshev polynomials emerge as minimizers of the supremum norm over K. This property makes them valuable tools for approximating functions and representing them through polynomial expansions. Balogh and Bertola [9] also considered weighted Chebyshev polynomials with varying weight functions, denoted as w^n. These weight functions were analyzed as factors influencing the supremum norm of weighted polynomials $Q_n w^n$ over a specific subset $\sum \subseteq \mathbb{C}$. The researchers imposed certain standard admissibility conditions on the weight function w, as outlined in the reference [31].

Szegő [91] emphasized the significance of orthogonal polynomials as a valuable tool for analyzing fundamental problems. Various applications such as moment problems, rational and polynomial approximation, interpolation, and numerical quadrature rely on the fundamental properties of orthogonal polynomials. Understanding the theory of general orthogonal and extremal polynomials becomes essential, as it is closely linked to the classical research of Bernstein [13] regarding the asymptotic behavior of L_p extremal polynomials. Benitez [12] conducted an examination of different types of orthogonality in real normed spaces. The authors [38], [39], [59] and [60] investigated the existence of monic operator polynomials with right divisors. They determined the minimum degree required for these multiple polynomials and examined the roles played by various properties of the divisors. They also ventured into exploring the infinite-dimensional case, which posed new technical challenges and yielded significantly different results compared to their previous work. In their analysis, they made use of a generalized Vandermonde operator matrix, which need not be square for it to be of utmost importance. They demonstrated that the invertibility of the Vandermonde matrix, as well as some form of generalized invertibility, played crucial roles in the analysis.

In summary, despite previous attempts to characterize orthogonal polynomials and norm-attainable operators, the characterization of orthogonal polynomials in $NA(H)$ (the space of norm-attainable operators) and the relationship between the two remains an open question. This study aims to bridge that knowledge gap by investigating and shedding light on this unexplored territory.

1.2 Basic concepts

We have included some fundamental definitions and notations that are pertinent to our research in this section.

Definition 1.1. ([51], Definition 1.2) A norm or length function on a vector space V' can be formally defined as a mapping $\|\cdot\| : V' \to \mathbb{R}$ that satisfies the following three key properties:

1. **Non-negativity**: For all vectors x' in V', $\|x'\|$ is a nonnegative real number: $\|x'\| \geq 0$ Moreover, $\|x'\| = 0$ if and only if x' is the zero vector.

2. **Homogeneity**: For all vectors x' in V' and all scalars α in $\mathbb{R}$, the norm scales with the absolute value of the scalar: $\|\alpha x'\| = |\alpha|\|x'\|$

3. **Triangle Inequality**: For all vectors x' and y' in V', the norm of the sum of these vectors is bounded by the sum of their individual norms: $\|x' + y'\| \leq \|x'\| + \|y'\|$

Definition 1.2. ([68], Definition 3.1) An inner product on a vector space V' over the field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$) is a function $\langle \cdot, \cdot \rangle : V' \times V' \to \mathbb{F}$ satisfying the following properties:

1. **Positive-definiteness:** For all vectors $v' \in V'$, $\langle v', v' \rangle \geq 0$, with equality only if $v' = 0$.

2. **Conjugate symmetry:** For all vectors $v', w' \in V'$, $\langle v', w' \rangle = \overline{\langle w', v' \rangle}$.

3. **Linearity in the first argument:** For all vectors $v', w' \in V'$ and scalars $\alpha, \beta \in \mathbb{F}$, $\langle \alpha v' + \beta w', u' \rangle = \alpha \langle v', u' \rangle + \beta \langle w', u' \rangle$.

Example 1.3. Consider complex vectors z' and w' with n' components in the space $\mathbb{C}^n$. The inner product between z' and w' is calculated by taking the sum of the products of their corresponding components. This inner product is denoted as $\langle z', w' \rangle$. Similarly, in the space of continuous functions $C[0,1]$, the inner product between two functions g' and h' defined on the interval $[0,1]$ is obtained by integrating the product of $g'(x)$ and the complex conjugate of $h'(x)$ over the interval $[0,1]$. This inner product is represented as $\langle g', h' \rangle$. In summary, this example outlines the definition of inner products for complex vectors in $\mathbb{C}^n$ and continuous functions in $C[0,1]$.

Definition 1.4. ([51], definition 1.5) An operator T' in $B(H)$ is said to be non-attainable (NA) if there exists a unit vector x_0' in H such that $\|T'x_0'\| = \|T'\|$.

Remark 1.5. In this definition, $\|T'x_0'\|$ represents the norm (or magnitude) of the image of the unit vector x_0' under the operator T', and $\|T'\|$ represents the norm of the operator T' itself. If the norm of $T'x_0'$ is equal to the norm of T', then the operator T' is considered non-attainable.

Definition 1.6. ([4], definition 2.1) Let $T' : H_1' \to H_2'$ be a linear operator. The operator norm of T' is defined as

$$\|T'\| = \inf\{c \in \mathbb{R} : \|T'x'\| \leq c\|x'\| \text{ for all } x' \in H_1'\}.$$

In other words, the operator norm of T' is the smallest real number c such that the norm of $T'x'$ is always less than or equal to c times the norm of x', for all vectors x' in H_1'.

Example 1.7. Taxicab norm (C_1'): For $p = 1$ then $\|T'\|_1 = (\sum_{j=1}^{\infty} s_j'(T'))$ and $\||T'|^2\|_{\frac{1}{2}} = \|T'\|_1^2$. The class of all operators which admit the norm $\|T'\|_1 = (\sum_{j=1}^{\infty} s_j'(T'))$ is called Trace class and is denoted by C_1'.

Example 1.8. Let us consider the **Hilbert-Schmidt Operators** for the case when $p = 2$. Then $\|T'\|_2 = (\sum_{j=1}^{\infty} s_j^2(T'))^{\frac{1}{2}}$ and $\||T'|^2\|_{\frac{1}{2}} = \|T'\|_1^2$.

The class of all operators which admit the norm $\|T'\|_2 = (\sum_{j=1}^{\infty} s_j^2(T'))^{\frac{1}{2}}$ is called Hilbert-Schmidt norm and is denoted by C_2'.

Definition 1.9. ([80], definition 4.1) For $T' \in B(\mathcal{H}')$

(i). Uniform convergence: $\lim_{n\to\infty} \|T_n' - T'\| = 0$

(ii). Strong convergence: $\lim_{n\to\infty} \|(T_n' - T')x'\| = 0$, $\forall\ x' \in \mathcal{H}'$

(iii). Weak convergence: $\lim_{n\to\infty}\langle (T_n' - T')x', y'\rangle = 0$, $\forall\ x', y' \in \mathcal{H}'$

Moreover, a sequence ${T_n}'$ is considered bounded if there exists a finite number such that the norms of all the operators in the sequence do not exceed that value.

Remark 1.10. It is worth noting that if a sequence converges uniformly to an operator, it automatically implies strong and weak convergence. Additionally, if the norms of the operators in a sequence are bounded, then the sequence itself is also considered bounded.

Definition 1.11. ([5], Definition 4.12) Let $T' \in B(\mathcal{H}')$, then the spectrum of T' denoted as $\sigma(T')$, if defined as $\sigma(T') = \{\lambda \in \mathbb{C} : \lambda' I - T'$is not invertible$\}$. On the other hand, if $(T' - \lambda' I)$ is invertible, then λ' is referred to as the resolvent of T', that is, $\rho(T') = \mathbb{C}\backslash\sigma(T')$.

Definition 1.12. ([68], Definition 1.3) Let $T' : V' \to W'$ be a linear transformation. The image of T' is the set of all vectors in W' that can be written as $T'(v')$ for some vector v' in V'. It is denoted by $\mathrm{im}(T')$:

$$\mathrm{im}(T') = \{w' \in W' \mid w' = T'(v') \text{ for some } v' \in V'\}$$

The kernel of T' is the set of all vectors in V' that are mapped to the zero vector by T'. It is denoted by $\ker(T')$:

$$\ker(T') = \{v' \in V' \mid T'(v') = 0\}$$

The rank of T' is the dimension of the image of T', and the nullity of T' is the dimension of the kernel of T'.

Definition 1.13. ([71], Definition 2.1) Let H_1' and H_2' be Hilbert spaces. A linear operator $T' : H_1' \to H_2'$ is said to be **bounded** if there exists a positive constant M' such that

$$||T'x'||_2 \leq M'||x'||_2$$

for all $x' \in H_1'$. Here, $||x'||_2$ denotes the norm of x' in H_1', and M' is called the **operator norm** of T'.

Definition 1.14. ([71], Definition 2.2) Let H' be a Hilbert space. An operator $T' : H' \to H'$ is said to have an **adjoint**, denoted by $T'^* : H \to H'$, if it satisfies the condition

$$\langle T'x', y' \rangle = \langle x', T'^* y \rangle$$

for all vectors x' and y' in H'. The adjoint operator T'^* is uniquely determined by T'. It can be constructed using the Riesz representation theorem.

- An operator T' is said to be **self-adjoint** if $T' = T'^*$.
- An operator T' is said to be **normal** if $T'T'^* = T'^*T'$.

- An operator T' is said to be **hypernormal** if $\|T'^{*}x'\| \leq \|T'x'\|$ for all $x' \in H'$.

- An operator T' is said to be **positive** if $\langle T'x', x'\rangle \geq 0$ for all $x' \in H'$.

- An operator T' is said to be **symmetric** if $\langle T'x', y'\rangle = \langle x', T'y'\rangle$ for all x' and y' in H'.

Definition 1.15. ([79], Definition 5.1) Let H' be a Hilbert space and let $NA(H')$ be the operator algebra of all bounded linear operators on H'. Two operators T' and P in $NA(H')$ are said to be **orthogonal** if their inner product, denoted by $\langle T', P'\rangle$, is zero. That is,

$$\langle T', P'\rangle = T'r(T'^{*}P') = 0$$

where $T'r$ denotes the trace operator. A collection of operators T'_j and T'_k (where $j, k = 0, 1, \ldots$) are said to **possess orthogonal extensions** if their inner products, denoted by $\langle T'_j, T'_k\rangle$, are all zero. That is,

$$\langle T'_j, T'_k\rangle = 0 \quad \forall j, k$$

Remark 1.16. This definition establishes the notion of orthogonality for operators within the operator algebra $NA(H)$ and extends it to the concept of orthogonal extensions between sets of operators.

Definition 1.17. ([95], definition 3.1) In the context of polynomials, p is considered monic provided its leading coefficient equals to 1. More precisely, if p is a polynomial represented as $p(x) = x^n + b_n x^{n-1} + \ldots$, then p is monic. Moreover, a monic family of polynomials refers to a collection of polynomials in which every member of the collection is monic, meaning each polynomial within the family has a leading coefficient of 1.

Definition 1.18. ([36], Examples 1-5)

(i). The Hermite polynomials denoted as $H_n(x')$ are a sequence of orthogonal polynomials defined on the entire real line $(-\infty, \infty)$, and with a weight function $w'(x^H) = e^{-x^2}$.

(ii). The Laguerre polynomials denoted as $L_n^{(\alpha)}(x'_1)$ are a family of orthogonal polynomials defined on the interval $(0, \infty)$, and they are related with a weight function $w(x'_1) = e^{-x'_2} x'^{(-\alpha)}_1$.

(iii). Legendre polynomials $P_n(x'_1)$ are a family of orthogonal polynomials defined on the interval $(-1, 1)$, and their definition does not involve a weight function; it is simply based on the interval limits.

(iv). Chebyshev polynomials is a class of orthogonal polynomials $\phi_n(x^C)_{n=0}^{\infty}$ defined over the range from -1 to 1 with respect to a weight function $w(x^C) = (1 - x^2)^{-1/2}$.

(v). The Jacobi polynomials $P_n^{(\alpha_1,\alpha_2)}(x)$ are orthogonal polynomials defined on the interval $(-1, 1)$, and they are related to a weight function $w(x'_1) = (1 - x'_1)^{\alpha_1}(1 + x'_1)^{\alpha_2}$.

Definition 1.19. ([88], Definition 3.2) $f(x)$(function) is considered convex when given two points x and y in its domain and any scalar α between 0 and 1, the function value at the convex combination $\alpha x' + y'(1 - \alpha) \leq$ the convex combination of the function values $f'(\alpha x') + (1 - \alpha) f'(y')$. On the other hand, $f'(x')$ is positive definite if, for any non-zero point x', the function value $f'(x')$ is greater than zero. Lastly, $f'(x')$ is termed strictly convex provided for any two distinct points x' and y' in its domain and any scalar α between 0 and 1, the function value at the convex combination $\alpha x' + (1-\alpha)y'$ is strictly less than the convex combination of the function values $f'(\alpha x') + (1-\alpha)f'(y')$.

Definition 1.20. ([44], definition 1) A real polynomial is a mathematical expression involving real numbers and variables, represented as $P'(x') = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$. The coefficients $a_0, a_1, \ldots, a_{n-1}, a_n$ are real numbers, with a_n being nonzero. The degree of the polynomial, denoted by n, is a nonnegative integer that indicates the highest power of the variable.

Definition 1.21. ([7], Definition 2.1) A polynomial with complex coefficients is a function that expressed as:

$$p'(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0, \tag{1.2.1}$$

where $a_i \in \mathbb{C}\ \forall i = 0, 1, \ldots, n$ and z is a complex variable. n is the highest power of z known as the degree of the polynomial and $a_n \neq 0$.

Remark 1.22. In the context of polynomials, the degree of a polynomial P is a nonnegative integer denoted by n. P is expressed by $P(x) = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1} + a_n x^n$. The coefficient a_0 is referred to as the leading coefficient.

Definition 1.23. ([7], Definition 3.1) When we have a polynomial with a degree of $n \geq 1$ written as $4p(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$, we say that a complex number $z_0 \in \mathbb{C}$ is a root of the polynomial if the polynomial equation evaluates to zero when z_0 is substituted into it.

Definition 1.24. ([7], Definition 3.3) In the context of polynomials, the multiplicity of a root z_0 corresponds to the greatest exponent of the term $(z - z_0)$ that can be divided without remainder into the given polynomial $p(z)$. This concept applies to polynomials of degree n or higher, with n being a positive integer. It is crucial to emphasize that the quotient polynomial $q(z)$ should not evaluate to zero when substituting z_0 into it.

Remark 1.25. The term "zero polynomial" refers to a polynomial whose coefficients are all equal to zero, or $(P(x) = 0)$, and whose degree is "undefined."

Definition 1.26. ([24], Definition 1) An orthogonal polynomial associated with a weight function on a given interval (a, b) can be described as a sequence of polynomials, denoted as $q_n(x)$, each having a specific degree. These polynomials possess a unique property: when the product of any two polynomials in the sequence with different degrees is integrated over the interval, the resulting value is zero. However, if the product involves two polynomials with the same degree, the integral yields a non-zero constant.

1.3 Statement of the problem

Orthogonal polynomials are a well-studied class of functions, but their relationship to norm-attainability was not well understood until this study. Norm-attainability is a property of operators that ensures that their norms can be attained by elements of their domains. The goal of this study was to fill the knowledge gap in this area by investigating norm-attainable operators and establishing the norm-attainability conditions for orthogonal polynomials. By doing so, the study provides a comprehensive understanding of the relationship between orthogonal polynomials and norm-attainable operators, offering valuable insights into the properties and behavior of orthogonal polynomials within the context of norm-attainability.

1.4 Objectives of the study

1.4.1 Main objective

The main objective of this study was to characterize orthogonal polynomials in norm-attainable classes.

1.4.2 Specific Objectives

To achieve the main objective, the specific objectives of the study were to:

(i). Characterize norm-attainable operators.

(ii). Establish norm-attainability conditions for orthogonal polynomials.

(iii). Establish the relationship between orthogonal polynomials and norm-attainable operators.

The study achieved these objectives by exploring the characterization of norm-attainable operators and establishing the conditions and connections specific to orthogonal polynomials. The study was able to provide a comprehensive understanding of orthogonal polynomials within the context of norm-attainable classes.

1.5 Significance of the study

This study holds paramount significance in both theoretical and practical domains. The characterization of orthogonal polynomials in norm-attainable classes contributes to the advancement of theoretical knowledge. By delving into the properties and behavior of norm-attainable operators and their connection to orthogonal polynomials, this research enhances our understanding of fundamental mathematical concepts. The findings can potentially inspire further investigations and lay the groundwork for new theoretical frameworks in functional analysis and approximation theory. Moreover, the practical implications of this study are noteworthy. Norm-attainable classes and orthogonal polynomials have numerous applications in diverse fields such as signal processing, numerical analysis, and mathematical physics. Understanding the norm-attainability of orthogonal polynomials enables the development of improved algorithms and methodologies in these domains. It can enhance the efficiency and accuracy of signal reconstruction, data approximation, and solving differential equations. Furthermore, the insights gained from this research can guide the design and optimization of numerical algorithms, contributing to advancements in computational mathematics and scientific computing. Ultimately, the practical significance of this study lies in its potential to impact various industries and scientific disciplines, leading to improved techniques and solutions for real-world problems.

Chapter 2

LITERATURE REVIEW

2.1 Introduction

In this chapter, we examined the existing literature on norm-attainable operators in $NA(H)$, polynomials, and orthogonal polynomials. This comprehensive review was conducted as part of our study to establish norm-attainability conditions for orthogonal polynomials and deepen our understanding of their behavior within the context of norm-attainable classes.

2.2 Norm-attainable operators

Let $T : H \to H$ be a group of linear operators that are bounded with the properties that $\|Tx_0\| = \|T\|$ for a unit vector $x_0 \in H$ are called norm-attainable operators. Such operators have a host of properties and applications which have attracted the interest of many mathematicians. Bounded linear operators which meet different criteria of norm-attainability on Banach spaces form an algebra or class with a host of properties. Some of these properties include, sequential compactness, density among others. After the inquiry by [45] about which X and Y(Banach spaces) have a dense class of norm-attainable operators belonging to $B(X, Y)$, there has been significant progress in the field. In particular, [93] responded to the question when X is isomorphic to the Banach space $[0, 1]$. This means that he investigated which Banach spaces Y have a dense class of norm-attainable operators in $B(L^1[0, 1], Y)$. The paper by [69] explores the essential criteria that must be met for operators and derivations, with a specific focus on self-adjoint and contraction operators, to achieve norm-attainment.

In our study, however, we have applied the spectrum of such operators to the construction of orthogonality in polynomials. Furthermore, we have not considered the characteristics of the range of the norm-attainable operators. Moreover we have considered norm-attainable operators which are also compact in nature.

The findings presented in [62] addressed the question of whether certain compact operators can be effectively approximated by NA operators. In the study cited as [62], counterexamples were utilized to present compelling evidence that refutes the existence of an affirmative solution to the inquiry concerning compact linear operators between Banach spaces. The results of the study revealed that while such operators do exist, they cannot be approximated by operators that achieve their norms. The authors effectively utilized counterexamples to illustrate the infeasibility of approximating linear compact operators between Banach spaces with norm-attainable operators. Hence, their study convincingly demonstrated that although compact linear operators are present in these spaces, they cannot be accurately approximated by operators that attain their norms. Our results however shows a criterion of norm-attainability of compact operators which are self-adjoint in nature. Indeed we have not considered approximations of such compact operators. Furthermore we have confined our study in operators which act strictly on Hilbert spaces and not Banach spaces.

In a previous study in [30], a Banach space X satisfies Dunford-Pettis property if, $\forall$ Banach spaces Y, any weakly compact operator $T \in B(X, Y)$ maps weakly Cauchy sequences to strongly Cauchy sequences. Furthermore, norm-attainability of different compact operators on different Banach spaces was investigated in the same study, and conditions for norm-attainability were established.

(i). A compact operator is norm-attainable if the underlying space is sequentially compact Banach space.

(ii). A weakly compact operator is norm-attainable if it acts on a sequentially compact space and Dunford- Pettis property holds for the space.

(iii). Any finite rank operator is norm-attainable if the Banach space it acts upon has separable quotient space and the superspace of its range has absolutely summable sequences.

Our work has deviated from this matter significantly. Indeed the underlying vector space considered in our study is a Hilbert space (separable) of orthogonal polynomials rather than a separable quotient Banach space.

Moreover, the author conducted an investigation on the density characteristics of a specific class of NA operators in relation to the tensor product. This study primarily focused on the tensor product involving norm-attainable operators and the density characteristics of the domain of such operators. Our study however considered other properties of the norm-attainable operator including normality, self-adjointness and others.

An adjoint of an operator is an involution which can be applied in the definition of other classes of operators like normal, hyponormal, subnormal and paranormal operators. Bram-Halmos criterion for subnormality of an operator is one particular criterion that employs the adjoint operator. The criterion for establishing the subnormality of an operator, as described in references [3] and [6], incorporates the concept of the adjoint. By employing the involution provided by the adjoint, this criterion offers a way to determine whether an operator satisfies the subnormality condition. In a similar fashion, weak k-hyponormality of an operator T was defined in [8]. In 2012, Jun Ik Lee [47] applied norm attainability concept of such special operators as normal, hyponormal, subnormal and paranormal. The subspace of such norm-attainable operators (norm-attainable paranormal operators) were then considered. The study conducted by [47] presents significant findings regarding the characterization of norm-attainable phenomena. Specifically, their research explores the norm-attainability of paranormal entities. Furthermore, within the same study, the authors also provide a characterization of norm-attainable quadratically hyponormal weighted shifts. In a notable study by [47], it was demonstrated that the subnormality property holds for the norm-attainable quadratically

hyponormal weighted shift operator. Although our work has touched on the conditions of norm-attainability of operators, it has not considered quadratically weighted shift operators which are subnormal, hypernormal and more.
The problem of norm attainability on Banach spaces which was done by James in [10] was continued in [87]. However, the main focus of S.Shkran [87] was norm attainability of operators with pseudo spectrum which act on finite dimensional Banach spaces. In the results of [87], the author established a new criterion for establishment of norm attainability of operators which touch on the spectra of the operator. The mentioned result [87] relies on operators that operate on subspaces or quotients which are direct sums of finite-dimensional Banach spaces, specifically the ℓ_p-direct sum. In our scenario, we are working with Hilbert spaces. Moreover it was shown that the new operator $1 + T$ constructed arbitrarily from operator T on condition that the norm of the new operator is greater than unity.

Our study does not consider such operators but instead use norm-attainable operators. A new method of characterization of absolutely norm-attainable operators was established in [71], for compact operators $T \in B(H)$ which also meet the condition $\|Tx\| \geq \|T^*x\|$ for all $x \in H$. In fact, proposition 3.3 in [71], clearly works for absolutely norm-attainable operators but may not work for norm-attainable operators. The author of [71] provided a comprehensive analysis, establishing the precise conditions that both ensure and demonstrate the absolute norm-attainability of bounded compact hyponormal operators in infinite-dimensional complex Hilbert spaces. In the study conducted by [71], they investigated the concept of the algebra of absolutely norm-attainable operators denoted as $AN(H)$. However, their focus primarily revolved around examining the structural properties associated with absolutely norm-attainable operators, which is not directly relevant to our current interest. The study of properties of norm-attainable NA operators, as discussed in the reference [79], reveals an interesting observation: the unit ball of NA operators acting on a Hilbert space, denoted as H, consists solely of isometries and co-isometries. This finding provides a significant result. Moreover, the same reference [79] presents compelling evidence supporting the claim that any arbitrary operator on a Hilbert space, if it is normal, automatically falls into the category of NA operators. Consequently, this provides a positive answer to the question of whether all normal operators are norm-attainable. Contrarily, our study focusses on the application of aspects of norm-attainable operators like spectrum e.t.c. Our work does not touch on the isometries, co-isometries and other operators like normal operators.

Results of [70] further exposed the intricate relationship between the notion of norm-attainability and positive normal and normaloid operators. Precisely, it was shown in [70] the conditions that should be made for positive normal and normaloid operators to be NA.

Theorem 2.1. *([70], Theorem 4.1) Suppose $S \in B(H)$ i a positive normal and normaloid operator. If $\alpha \in \mathbb{C}\backslash\{0\}$ is such that $\|S - \alpha x\| < m$ for $m > 0$. Then $(0 \leq)\|S\| - r(S) \leq \frac{1}{2}\frac{m^2}{\|\alpha\|}$. Moreover, S is NA.*

Our work does not consider the construction of NA operators or the operators applied in their construction. We have only given example of NA

operators, calculated their norms and applied their eigenvalues to construct orthogonal polynomials system.

The idea of norm-attainability of elementary operators [72], was investigated with respect to contractions and unitary operators as the inducing operators. In the document [72], it was established that existence of a unitary operator is a necessity for elementary operator to be norm-attainable. Our work does not involve elementary operator, isometry or co-isometry. Furthermore results of [70] and [72] did not consider the applications of any aspect of elementary operators. Topological considerations of properties of norm-attainable operators were investigated by [80]. Precisely, relationships between different convergences of norm-attainable operators and those of norm-attainable norms in projective norms were shown in [80]. Our study has touched on power bounds but it is notable that we considered polynomial bounds for norm-attainable operators. Furthermore, we have not investigated convergence of the sequences in the projective norm induced by tensor product.

Other properties of norm-attainable operators were investigated by [73]. Such properties as separability and norm density of range, supraposinormality, coposinormality, posinormality and α-hyponormality of positive invertible norm-attainable operators. The author in [73] utilizes the Fuglede-Putnam theorem to establish the necessary conditions for positive invertible norm-attainable operators to possess certain properties. Lemma 3.1 in the paper presents these conditions. If the range of the positive invertible operator Q is both separable and dense, then the operator A is shown to be supraposinormal, and the kernel of A^{α} is contained within the kernel of $A^{\alpha *}$. Similarly, if the positive invertible operator P has a dense separable range, then A is proven to be supraposinormal and dominant, and again the kernel of A^{α} is included in the kernel of $A^{\alpha*}$. Furthermore, if Q is positive invertible and norm-attainable, then the α-supraposinormal operator A is also α-posinormal and therefore α-hyponormal. On the other hand, if P is positive invertible and norm-attainable, A is established as an α-supraposinormal operator, which implies it is also an α-coposinormal operator. When both P and Q are positive invertible and norm-attainable, A exhibits both posinormal and coposinormal properties, with the kernels and ranges of A^{α} and $A^{\alpha*}$ being equal. Lastly, if either P or Q is dominant while both are positive invertible, A is characterized as both an α-coposinormal and an NA operator, and the intersection of the kernels of A^{α} and $A^{\alpha*}$ is equivalent to the intersection of the ranges of A^{α} and $A^{\alpha*}$.

As in the case of [73], we generated a spectrum for norm-attainable operators, but we have not used positive invertible operators. Furthermore, we have not involved Fugledge-Putman theorem in our study. Finally, it is notable that we have only used the real subset of the spectrum of norm-attainable operators to induce orthogonality in polynomials.

The concept of approximating operators was extensively discussed by Martin in their notable work [62]. They demonstrated that within the framework of Banach spaces, the existence of a compact linear operator is guaranteed, which defies any attempts at approximation through NA operators. This finding was proved by [62] using Lindenstrauss theorem and Efflo's example. Other than this [62] also investigated the properties of subspaces of the class of norm-attainable operators which act on convex subspaces of Banach spaces.

Theorem 2.2. *([62], Theorem 1) Compact linear operators between Banach spaces do exist, but they cannot be approximated by operators that are NA.*

In contrast, our work on the other hand has not touched on approximation of compact operators by norm-attainable operators. However, we have applied the idea of approximation of polynomials by functions. We have also considered only the real subspace of the spectrum of compact NA operators.

The paper by [10], investigates the norm-attainment property of random operators. The authors examined the norm-attainment property of random operators and focussed on operators whose coefficients are random variables. They also explored the conditions under which such random operators attain their norms with high probability. The analysis in the paper involved the concentration of measure phenomenon and provided probabilistic bounds for the probability of norm attainment. In [55] the authors focussed on the uniform norm-attaining property of operators, which relates to the uniform convergence of the norms of operators over a given set. The authors examined the conditions under which an operator uniformly attains its norm on a set and provide characterizations of operators with this property. They also investigated the connections between uniform norm attainment and other operator properties, such as compactness and uniform convexity. This is different from our work where we have offered specific conditions under which certain types of operators, such as compact and self-adjoint operators, normal operators with certain properties, or functional calculus operators, belong to the norm-attainable class.

2.3 Orthogonal Polynomials

In [44], the author analyzed the decomposition of polynomials into sums of other unique polynomials. Theorem 3.4 in [44], involve the application of unique factorization theorem and it is shown that such algorithm depend on the degree of the polynomial. Furthermore, such decomposition depends on whether the remainder $r(x) = 0$ or not. In our study, we have focussed on the norm-attainability of classical orthogonal polynomials using Rodrigues algorithm and integration by parts. We have also considered positivity, convexity and strict convexity of polynomials.

Just like polynomials, a lot of results exist on orthogonal polynomials. These results range from their construction, analysis of their moment of generating functions, analysis of their zeros, types of polynomials which are orthogonal (Jacobi, Laguerre, Hermite, Besel, Legendre) and their applications among others.

The construction of orthogonal polynomials is addressed squarely in [75] by using arbitrary moment generating functions $\mu(x)$ and a sequence of positive numbers. In our study, we have used a sequence of eigenvalues of defined operators to generate such orthogonal polynomials. Additionally, we have investigated various properties such as positivity, norm, and others, for orthogonal polynomials defined on multiple variables using non-orthonormal bases. We have also generated a normed vector space of orthogonal polynomials upon which differential operators act.

In [67] Proposition 1.2.1, the author constructed the result on the use of weight functions to generate a system of monic orthogonal polynomials. Contrary to this, our work has considered properties of operators which act on a space of such monic orthogonal polynomials. The work of

[67], clearly considers the construction of orthogonal polynomials system which form a Hilbert space $L^2(w_2)$ with norm $\sqrt{n+1}$. However, our work considered norms for different classical orthogonal polynomials. We have also used the norms due to such orthogonal polynomials to determine whether each type is NA. Furthermore we have applied the constructed orthogonal polynomials to construct a normed vector space.

In our study, we have utilized orthogonal polynomials in conjunction with various weight functions to investigate the norm-attainability of classical orthogonal polynomials. Our research builds upon the work of previous researchers [24], who established the presence of a three-term recurrence relationship in a system of orthogonal polynomials. Moreover, they demonstrated that the zeros of consecutive orthogonal polynomials on any given interval are distinct. Additionally, they provided evidence that a sequence of orthogonal polynomials is always linearly independent when the degree of the initial polynomial is finite.

The application of matrices to orthogonal polynomials was pioneered by [90]. In their work, they utilized non-square matrices as coefficients in the three-term relation of orthogonal polynomials, resulting in a collection of distinct coefficient matrices. While their study focused on the three-term relation of orthogonal polynomials, our research diverges by exploring the properties of convexity and positivity associated with these orthogonal polynomials. Moreover, we have not specifically investigated the coefficient matrices generated by this particular family of orthogonal polynomials.

A lot of literature exists on the application of linear functionals on polynomials to generate orthogonal polynomial family. The work of [91] involved the use of linear functionals to generate a system of orthogonal polynomials. After that, the three term relation for the orthogonal polynomial system was examined under the restriction of the theorem 3.2. Our study also involved linear functionals on the space of orthogonality of polynomials. Indeed we have considered the properties of linear differential operators which include normality, self-adjoint etc. However, we have not involved the use of matrices as coefficients of the three-term recurrence formula.

Using linear functional and moment of finite order, [21] investigated the conditions necessary for a system of polynomials to be orthogonal. Still in [21], the properties of the functionals used together with moments of finite order to generate orthogonal polynomials were considered using Favards theorem for one variable. Furthermore, we have not applied Favards theorem although we have considered different classical orthogonal polynomials of one variable.

Our research primarily concentrated on linear differential operators, similar to the ones examined in [21] (specifically, theorem 2.1). It is important to note that the operators we employed in our study were not in matrix form. Consequently, we did not explore the determinants of these operators. Moreover, we have not considered the properties of the functionals which induce the orthogonality as in the case in [21]. However, we have analyzed the linear functionals which act on the space of such orthogonal polynomials. Furthermore, we have not considered the moments of such functionals.

Author [34] demonstrated the application of a real number sequence to the orthogonality of polynomials, particularly focusing on the three-term recurrence relation. Their study utilized this approach to establish the existence of a moment function associated with an indeterminate of bounded variation. By employing this methodology, they were able to provide a robust proof for the existence of such a moment function. Although our study has not touched on bounded variations, we have used the definite integral $\int_a^b q_n(x)q_m(x)\delta\alpha(x)$ to prove norm-attainability for different classical orthogonal polynomials.

The work of [11], also considered the use of regular linear functionals on the orthogonality of polynomials. Indeed it was shown that if such a functional exists, then the system of orthogonal polynomials exists such that the orthogonality is not induced by moment generating functions. Such functionals were also used to generate monic orthogonal polynomial with characteristic matrix (Gram), whose determinant determine the orthogonality. Similar to the approach employed by [11], our study also incorporates the use of linear functionals. However, we extend beyond the scope of regular functionals and consider a broader range of functionals in our research. Additionally, in our study, the orthogonality of the system is exclusively determined by the inner product and the moment generating function. Similarly, in the work of [21], the construction of orthogonal polynomials was facilitated by the application of Favard's theorem. However, it is important to note that the functionals utilized in [21] are moment functionals.

The researcher in [61], investigated the relationship between orthogonal polynomials of infinite order and those of finite order and generated by regular linear functionals. Additionally, we have examined the derivatives of these orthogonal polynomials in relation to the orders of the monic polynomials Contrarily, our study does not consider relationship between monic orthogonal polynomials of different order. Furthermore, no considerations about our work have been made on the derivatives of the orthogonal polynomials. Although we have used the sum of the orthogonal polynomials derivatives to generate a linear functional with a variety of properties.

It was further shown in [21] that the system of orthogonal polynomials can be modified to be strictly monic and that a Hankel determinant exists and that if it is manipulated analytically, it generates the leading coefficient of the parent polynomial. This is a contradiction to our work which involves construction of a functional (linear differential operator) which is not dependent on the moments at all.

The paper by Mourad [66], focussed on the norm attainment properties of orthogonal polynomials in the L^2 space, which is a space of square-integrable functions. It provides conditions under which the norms of orthogonal polynomials can be attained, meaning there exist functions that achieve the maximum or minimum value of the norm. The paper also characterizes the extremal functions that achieve these norms, providing insights into the behavior and properties of the orthogonal polynomials. Further, in [94], the author investigated the norm attainment properties of orthogonal polynomials in various contexts, including classical and non-classical orthogonal polynomials. The paper presents necessary and sufficient conditions for the norm attainment of orthogonal polynomials. These conditions indicate when it is possible for orthogonal polynomials to attain their maximum or minimum norm.The paper further explores the connection between norm attainment and extremal problems, which involve finding functions that optimize certain properties related to the orthogonal polynomials.

It is clear that the authors discussed norm attainment properties of orthogonal polynomials in a more general context different from our work, where we have focussed on specific cases and families of orthogonal polynomials with different weight functions. We have specifically addressed the concept of norm-attainability conditions for Hermite polynomials with a normal distribution weight function, Laguerre polynomials with a gamma distribution weight function, Legendre polynomials with a constant weight function, and Jacobi polynomials with a Beta distribution weight function. We have highlighted the norm-attainability properties of these specific families of orthogonal polynomials on specific intervals with different weight functions.

2.4 Link between OP and NAO in NA(H)

A considerations of polynomials whose coefficients are operators which are numbers of Schatten-Von Neumann ideals for the compact operators in Hilbert spaces was considered extensively in [65]. The author [65] applied the regular points and eigenvalues of such bounded invertible polynomials to determine how close the spectra of two such polynomials are if one is induced by the regular points and another by the eigenvalues of the compact operator. Indeed the polynomials discussed in [65] clearly did not involve the OP as in the case of our study. Furthermore, our study did not consider the polynomials involving regular points of the operator in question. Lastly the results of [65] never considered the polynomials which involve norm-attainable operators as coefficients. It is also notable that the eigenstructure of members of Schatten-Von Neumann ideals of compact operators may be significantly different from those of norm-attainable operators used in our work.

The concept of orthogonality in the real normed space have a variety of meanings: Roberts, Birkhoff's orthogonality among others. In [69] the author studied orthogonality for elementary operators which are implemented by norm-attainable operators. This is a classical of application of NA operators in characterization of other operators. Indeed the proposition 2.2 in [69], shows a condition necessary for an inner derivation as an elementary operator to be norm-attainable. Furthermore, for other elementary operators, a criterion for norm attainability follows from the proposition.

In our work, we have shown that the orthogonal polynomials form a normed vector space. We have also shown that an operator which act on such a space is self-adjoint, closed but not compact. Furthermore, it is notable that our study on the other hand does not consider elementary operators.

In the analysis of linear positive operator sequences denoted as p_n, a study was conducted in [69] on operators of the form $[q_{n,k}(x)] = \binom{n}{k} x^k (1-x)^{n-k}$, where $x \in [0,1]$, $n \geq 1$, and $k = 0, 1, \ldots, n$. The authors established a connection between these operator sequences p_n and a simpler sequence denoted as v_n, which was originally introduced by Stancu [89]. This link provided insights into the relationship between Bernstein polynomials and Stancu polynomials as discussed in [13, 89]. The properties such as convexity and spectra of such sequences were investigated. Indeed the theorems in [89], the eigen polynomial q_r corresponding to λ_r can be chosen as a monic polynomial of degree r to show the connection between such sequences and also give the criterion of determining the eigenstructure of such operators. In our study though, such operators have not been considered. Instead, we have considered linear differential operators.

Investigations on polynomials whose coefficients are linear bounded operators between fixed separable complex Hilbert spaces have always touched on the problem of factorization. As such problem of factorization of such polynomials is important, and has been proved by [85] to be strongly influenced by the density of a set of diagonalizable linear operators in a finite dimensional vector spaces.

The author of [85] investigated this observation on a set of all (BQT) biquasitriangular operator polynomials of finite degree with induced topology. This has proved to vital in our study thus

Theorem 2.3. *([85], Theorem 1) The set of all degree 1 factorable biquasitriangular mononomials is dense in BQT.*

This theorem applies direct sum decomposition of the separable Hilbert space into invariant subspaces. It also involves biquasitriangular operators which may or may not be norm-attainable. The monic nature of the polynomials in question does not apply to our study. Although our work similarly apply the idea of this functional calculus, the set of polynomials used are those that are mutually orthogonal. Moreover, the operator used as the coefficients are the companion operators with a host of properties which are different from those of norm-attainable operators as it can be seen in [85].

The authors in [19], investigated the connection between orthogonal polynomials and norm-attaining operators in a Hilbert space. They explored the properties of orthogonal polynomials and their role in the characterization of norm-attainable operators and established conditions under which a bounded linear operator on a Hilbert space is norm-attainable. They further demonstrated that the existence of a certain type of orthogonal polynomials associated with the operator's spectrum is closely related to the operator's norm-attainability. In our study however, we have provided the properties of the operator $T(u)$, such as its adjoint, closure, eigenvalue problem, and the norm-attainability of $T(u)$ on the function space $C^0([0,1])$. These results are specific to the operator $T(u)$ and its behavior within the function space setting. This is a contrast from [19] where the authors explored the general connection between orthogonal polynomials and norm-attainability of bounded linear operators on a Hilbert space.

In [41], the author focused on the interplay between orthogonal polynomials and norm-attaining operators in the context of weighted Bergman spaces. They examined the connection between the asymptotic behavior of orthogonal polynomials and the norm-attainability of certain integral operators where they established the necessary and sufficient conditions for the norm-attainability of integral operators in weighted Bergman spaces using the properties of orthogonal polynomials. They further provided explicit examples illustrating the relationship between orthogonal polynomials and norm-attaining operators. The authors established general conditions for the norm-attainability of integral operators in weighted Bergman spaces using orthogonal polynomials. In contrast, we have provided specific results and formulas for the adjoint operator, norm, and eigenvalue problem of the given differential operator where we focussed on analyzing the properties of this particular operator $T(u)$ in the context of $C^0([0,1])$.

The paper by [54], investigated the norm-attainability of Toeplitz operators and its connection to orthogonal polynomials. It explored the relationship between the properties of orthogonal polynomials and the norm attainment of Toeplitz operators on Hardy spaces and the authors found the conditions for the norm-attainability of Toeplitz operators in terms of the zero distribution of orthogonal polynomials. They also gave examples illustrating the role of orthogonal polynomials in determining the norm attainment of Toeplitz operators. In our study we have discussed a specific second-order differential operator $T(u)$ and its properties, different from [54] where the authors focusessed on the norm-attainability of Toeplitz operators on Hardy spaces in relation to orthogonal polynomials.

Chatzikonstantinou and Nestoridis [19] provided a comprehensive overview of the relationship between orthogonal polynomials and norm-attaining operators. They covered a wide range of topics, including the characterization of norm-attaining operators using orthogonal polynomials, the computation of the norm of norm-attaining operators using orthogonal polynomials, and applications of the relationship between orthogonal polynomials and norm-attaining operators in quantum mechanics, signal processing, and linear algebra.

Li and Zheng [54] studied the relationship between orthogonal polynomials and norm-attaining Toeplitz operators. They showed that the norm attainment condition for Toeplitz operators could be expressed in terms of the orthogonal polynomials associated with the symbol function. They also showed that the norm of a norm attainable Toeplitz operator could be expressed in terms of the coefficients of the associated orthogonal polynomials.

Zhu and Zhu [97] studied the relationship between orthogonal polynomials and norm-attaining operators on Fock spaces. They showed that the norm attainability of an operator on a Fock space could be characterized using properties of the orthogonal polynomials associated with the operator. They also showed that the norm of a norm attainable operator on a Fock space could be expressed in terms of the coefficients of the associated orthogonal polynomials.

In this study, we established properties and relationships involving the inner product between functions, the adjoint operator, and the eigenvalue problem associated with T(u) in a Hilbert space H. Our results complemented the existing literature on orthogonal polynomials and norm-attaining operators by providing specific and explicit results for a particular differential operator.

Despite all these attempts to characterize orthogonal polynomials and norm-attainable operators, the relationship between the two remained an open question until this study bridged that knowledge gap by investigating and shedding light on the characterization of orthogonal polynomials in the space of norm-attainable operators.

Chapter 3

RESEARCH METHODOLOGY

3.1 Introduction

This chapter provides an overview of the fundamental principles, essential results, and key equalities and inequalities that played a significant role in proving our main results regarding the characterization of orthogonal polynomials and norm-attainable operators in $NA(H)$.

3.2 Fundamental Principles

The study of norm-attainable operators (NAO) and their connections to orthogonal polynomials (OP) is a fascinating and important area of research in operator theory. Understanding the behavior and properties of these operators and their associated eigenfunctions not only deepens our understanding of mathematical analysis but also has significant implications in various fields, including physics, engineering, and data science. In this thesis, we have investigated the relationship between OP and NAO within the framework of the norm-attainable class $NA(H)$. In achieving our goal, we adopted a comprehensive methodology that incorporated fundamental principles and concepts from operator theory. Our methodology encompasses differential operators, adjoint operators, inner products, integration by parts, closed operators, eigenvalue problems, self-adjointness, normality, and null spaces. By employing these principles, we analyzed and characterized the behavior of the operator $T(u)$ and its associated eigenfunctions in different function spaces.

Differential operators also played a pivotal role in our analysis. By studying second-order differential operators and their adjoint operators, we gained insights into their self-adjointness and the properties of their eigenvalue problems. This enabled us to explore the behavior of the eigenfunctions, which formed the basis of our investigation. The utilization of inner products and integration by parts allowed us to measure the similarity or orthogonality between functions and manipulate integrals involving derivatives. These tools were essential in analyzing the relationships between functions and operators and formed a fundamental part of our methodology. The concept of closed operators was another key aspect of our study. By examining the closedness of operators, we were able to determine their stability and completeness, shedding light on their inclusion in the norm-attainable class $NA(H)$. We also explored the properties of self-adjoint and normal operators. Self-adjoint operators possess a special symmetry property that plays a significant role in their analysis, while normal operators satisfy a commutation relationship with their adjoints.

These properties provide valuable insights into the behavior and classification of operators within $NA(H)$. Furthermore, the study of null spaces, or kernels, of operators is a crucial component of our analysis.
By investigating the properties and relationships between null spaces of different powers of an operator, we gain a deeper understanding of their behavior and their connection to norm-attainability. By employing these fundamental principles and concepts from operator theory, our research aims to contribute to the understanding of the relationship between OP and NAO within $NA(H)$. We believe that our comprehensive methodology will provide valuable insights into the behavior and classification of operators and their associated eigenfunctions, paving the way for advancements in mathematical analysis and its applications in various fields.

3.3 Known Useful Results

Our research delves into the realm of orthogonal polynomials (OP) and norm-attainable operators within the norm-attainable class NA(H), employing a suite of sophisticated methodologies and theoretical underpinnings. A fundamental pillar of our analysis lies on Spectral Theorem for Compact SA Operators, which serves as a cornerstone for understanding the spectral properties of these operators.

Theorem 3.1. *[86] Suppose A' is a compact self-adjoint operator on a Hilbert space H. Then there exists an orthonormal basis $\{e'_n\}$ of H and* *a sequence of real numbers* $\{\lambda'_n\}$ *such that*

$$A'e'_n = \lambda'_n e'_n \quad \textit{for all } n,$$
$$\lambda'_n \to 0 \ \textit{as } n \to \infty,$$
$$H = \bigoplus_{n=1}^{\infty} span(e'_n).$$

The eigenvalues λ'_n *known as eigenvalues of* A' *and* e'_n *known as eigenvectors of* A'.

This theorem establishes that every compact SA operator within a Hilbert space is characterized by a complete set of orthonormal eigenvalues. It serves as the foundational cornerstone for comprehending the spectral properties inherent in compact self-adjoint operators.
To further broaden our theoretical framework, we employ Spectral Theorem for Normal Operators, which finds its application within finite-dimensional complex inner product spaces V. This theorem is pivotal in elucidating the spectral traits of normal operators, offering a thorough comprehension of how they behave.

Theorem 3.2. *[70] Given* T' *as a normal operator on a finite-dimensional complex inner product space* V'. *Then there exists a unitary operator* U' *so that* $U^*T'U'$ *is a diagonal matrix. Moreover, there exists a unique set of projection operators* $\{E'_\lambda : \lambda' \in \sigma(T')\}$ *such that:*

1. $E'_\lambda E'_\mu = 0$ *for* $\lambda' \neq \mu$.
2. $\sum_{\lambda' \in \sigma(T')} E'_\lambda = I$.
3. $T' = \sum_{\lambda' \in \sigma(T')} \lambda E'_\lambda$.

The set of projection operators $\{E'_\lambda : \lambda' \in \sigma(T')\}$ *becomes the spectral family of* T'*, and the decomposition* $T' = \sum_{\lambda' \in \sigma(T')} \lambda E'_\lambda$ *is called the spectral decomposition of* T'*.*

This theorem establishes that any normal operator upon a Hilbert space can be transformed into a multiplication operator on the $L^2(\Omega, \mu)$ space through a unitary equivalence, with Ω representing the spectrum of the operator.

3.4 Technical Approaches

The technical methodologies employed in our research outcomes were based on a set of key techniques. First, spectral theory played a fundamental role in our analysis. By exploring the characteristics of eigenvalues and eigenvectors of specific operators, spectral theory provided valuable tools, including the spectral decomposition of self-adjoint operators and the notion of an operator's spectrum. Additionally, norm-attainability was a crucial concept in several propositions we presented. We investigated norm-attainable functions or polynomials, which are functions for which the norm is achieved by an element within the function space. This exploration of norm-attainable functions allowed us to establish important conditions and relationships. Moreover, compact operators played a significant role in our work. These operators possess the property of mapping bounded sets to relatively compact sets. Leveraging the compactness property, we derived various results, including the normality of operators and the properties of their eigenspaces.

Functional calculus was another technique we employed, enabling us to extend the concept of a function to operators. Through functional calculus, we defined functions of operators and analyzed their properties, thereby establishing connections between operators and their spectral properties. In propositions related to second-order differential equations, we utilized techniques from differential equations and calculus, specifically focusing on differential operators. These operators were analyzed to determine properties such as self-adjointness and closedness.Finally, we leveraged the concepts of convexity and positive definiteness, which are fundamental in functional analysis. These properties played a crucial role in establishing conditions for norm-attainability and proving the equivalence of different statements.
In summary, our research incorporated a combination of these technical approaches: spectral theory, norm-attainability, compact operators, functional calculus, differential operators, and the utilization of convexity and positive definiteness. Through their application, we derived significant results and contributed to the understanding of the subject matter.

3.5 Fundamental Equalities and Inequalities

In our research findings, we have employed several fundamental equalities and inequalities [4], that serve as crucial tools in our analysis. Firstly, we have utilized the triangular inequality, which states that for any vectors x' and y' in a normed vector space, the absolute difference between the magnitudes of their components, denoted as $\|x'\| - \|y'\|$, is always less than or equal to the magnitude of their vector difference, $\|x' - y'\|$. That is,

$$\|x'\| - \|y'\| \leq \|x' - y'\|.$$

Another important inequality we have employed is the Cauchy-Schwarz inequality.

$$\|\langle x', y'\rangle\| \leq \|x'\|\|y'\|.$$

This inequality applies to vectors in an inner product space and asserts that the absolute value of the inner product of two vectors, denoted as $\langle x', y'\rangle$, is bounded by the product of their magnitudes, $\|x'\|$ and $\|y'\|$. In other words, $\|\langle x', y'\rangle\| \leq \|x'\|\|y'\|$. Furthermore, we have utilized Holder's inequality

$$\|x'y'\| \leq \|x'\|_p\|y'\|_q,$$

which holds for vectors x' and y' in a normed vector space and positive real numbers p and q satisfying $\frac{1}{p} + \frac{1}{q} = 1$. This inequality states that the absolute value of the product of the components of x' and y', denoted as $\|x'y'\|$, is less than or equal to the product of their ℓ^p and ℓ^q norms, denoted as $\|x'\|_p$ and $\|y'\|_q$, respectively.

Additionally, we have applied Jensen's inequality, which relates to convex functions. If f' is a convex function defined on an interval I, and $x'_1, x'_2, \ldots, x'_n$ are points within this interval, while $a'_1, a'_2, \ldots, a'_n$ are non-negative real numbers that sum up to 1, then Jensen's inequality states that the value of the convex function evaluated at the weighted average of the x' values, namely $f'(a'_1x'_1 + a'_2x'_2 + \ldots + a'_nx'_n)$, is less than or equal to the weighted sum of the function values at each x' value, i.e.,

$a'_1 f'(x'_1) + a'_2 f'(x'_2) + \ldots + a'_n f'(x'_n)$. Moreover, we have made use of Parseval's identity, which applies to vectors x' and y' in a Hilbert space. This identity states that the square of the magnitude of x', denoted as $\|x'\|^2$, is equal to the sum of the squared absolute values of the inner products between x' and an orthonormal basis y'_n of the Hilbert space. Mathematically, it can be expressed as $\|x'\|^2 = \sum_{n=0}^{\infty} \|\langle x', y'_n \rangle\|^2$. Lastly, we have employed Bessel's inequality, which also applies to vectors x' and y' in a Hilbert space with an orthonormal basis y'_n. Bessel's inequality states that the sum of the squared absolute values of the inner products between x' and the orthonormal basis elements, $\sum_{n=0}^{\infty} \|\langle x', y'_n \rangle\|^2$, is always less than or equal to the square of the magnitude of x', i.e., $\|x'\|^2$.

By incorporating these fundamental equalities and inequalities into our analysis, we have been able to establish and support our results in a rigorous manner, ensuring proper credit is given to these mathematical principles.

Chapter 4

RESULTS AND DISCUSSION

4.1 Introduction

The prerequisites for norm-attainable operators, the characteristics of orthogonal polynomials, and the connection between orthogonal polynomials and $NA(H)$ have all been covered in this chapter.

4.2 Norm-attainability conditions

We examine norm-attainability requirements and operator characteristics in both Banach and Hilbert spaces. Our findings demonstrate the attainability of norms for compact operators, self-adjoint operators, and self-adjoint contractions.

Proposition 4.1. *Suppose a function T' from a vector space H' to itself is both compact and self-adjoint then $T' \in NA(H')$.*

Proof. Since T' is SA by the proposition, then $\exists$ a vector x' of H' such that $\|T'\| = \sup\{|\langle T'x', x'\rangle| \ : \|x'\| = 1\}$. This implies $\exists$ $\{x'_n\}$ of unit vectors such that $|\langle T'x'_n, x'_n\rangle|$ converges to $\|T'\|$. But T' is SA implying $|\langle T'x'_n, x'_n\rangle|$ is real. Hence there $\exists$ a subsequence $\{y'_n\}$ in H such that $y'_n \to \|T'\|$ or $y'_n \to -\|T'\|$. Thus if the norm y'_n equals to 1 $\forall$ $n \in \mathbb{N}$ we have $\langle T'y'_n, y'_n\rangle \to \beta$, the real value of either $\|T'\|$ or $-\|T'\|$. This implies;

$$\begin{aligned}
\|T'y'_n - \beta y'_n\|^2 &= \|T'y'_n\|^2 - 2\beta\langle T'y'_n, y'_n\rangle + \beta^2\|y'_n\|^2 \\
&\leq \|T'\|^2\|y'_n\|^2 - 2\beta\langle T'y'_n, y'_n\rangle + \beta^2\|y'_n\|^2 \\
&= \beta^2 - 2\beta\langle T'y'_n, y'_n\rangle + \beta^2 \\
&= \beta^2 - 2\beta^2 + \beta^2 = 0
\end{aligned}$$

This implies for $\beta \neq 0$, $\|T'y'_n - \beta y'_n\|^2 \to 0$. Since T' is compact, $\{T'y'_n\}$ has a subsequence $\{T'y'_{n_k}\}$ such that $T'y'_{n_k} \to y' \in H'$. Implying;

$$\begin{aligned}
y'_{n_k} &= \frac{(\beta y'_{n_k} - T'y'_{n_k}) + T'y'_{n_k}}{\beta} \\
&\Rightarrow \ y'_{n_k} = \frac{0 + T'y'_{n_k}}{\beta} \\
&\Rightarrow \ y'_{n_k} = \frac{y'}{\beta}
\end{aligned}$$

as $T'y'_{n_k} \to y$. Here, $\{y'_{n_k}\}$ are unit vectors, implying $y' \neq 0$. Hence $y'_{n_k} = \frac{y'}{\beta}$ and due to continuity of T', we have $T'y'_{n_k} = T'\frac{y'}{\beta}$ which yields to $\frac{T'y'}{\beta} = y'$ (since $T'y'_{n_k} \to y'$). Hence $T'y' = \beta y' \Rightarrow (T' - \beta I)y' = 0$. Hence, an eigenvalue of T' is implied by β, where β is either $-\|T'\|$ or $\|T'\|$. Hence by [86] T' is NA. □

Proposition 4.2. *Let $T_1 = T_2 + iT_3$ be normal with T_2 Hermitian and T_3 the operator with diagonal α_n given a positive integer n. Then T_1 is $NA(H)$.*

Proof. Since T_1 is normal, we have $\|T_1 f\|^2 = \langle T_1 f, T_1 f\rangle = \langle T_1^* T_1 f, f\rangle = \langle T_1 T_1^* f, f\rangle$, $\forall f \in H$. Therefore, $\|T_1 f\| = \|T_1^* f\|$ for all $f \in H$. Let f be an eigenvector of T_3 with eigenvalue α. Then $T_1 f = T_2 f + iT_3 f = T_2 f + i\alpha f$. Since T_2 is Hermitian, we have $\|T_2 f\|^2 = \langle T_2 f, T_2 f\rangle = \langle f, T_2 T_2^* f\rangle = \langle f, T_1 T_1^* f\rangle$. Therefore, $\|T_2 f\|^2 < \|T_1 f\|^2$ if $\alpha \neq 0$. This means that the only eigenvectors of T_3 that are also eigenvectors of T_1 are those with eigenvalue $\alpha = 0$. Since the eigenvectors of T_3 with eigenvalue $\alpha = 0$ span the entire space H, we can conclude that T_1 is diagonalizable. Moreover, the diagonal entries of T_1 are all real, since T_1 can be expressed as summation of a Hermitian operator and a purely imaginary operator. Therefore, T_1 is a normal, diagonalizable operator with real diagonal entries. This means that T_1 is self-adjoint, and hence norm-attainable. □

Proposition 4.3. *Let $T : H_1 \to H_2$ be normal and T^p where $p > 0$ be compact then T is NA.*

Proof. Let a multiplication operator T be induced by linear and bounded measurable $m \times n$ function (ψ) on a suitable measurable space. Implying T^p is also a multiplication operator induced by ψ^p. Then T^p becomes a diagonal operator and a direct sum of $m \times n$ zero entry matrices where the diagonal entries are convergent to zero. Consequently, we can express T as the direct sum of T with a diagonal operator whose diagonal elements tend to zero. This decomposition implies that T is compact in nature.

Let M be the closed unit ball of a Hilbert space H_1, defined as

$$M = \{x \in H_1 : \|x\|_{H_1} \leq 1\}$$

Due to the compact nature of $T : H_1 \to H_2$, therefore, under the norm topology, $T(M) \subset H_1$ which is also compact and $\|.\|_{H_1} : T(H_1) \to (0, \infty)$ is a continuous function on $T(H_1)$. This implies that

$$\sup_{x \in M} \|Tx\|_{H_1} = \max\{\|Tx\|_{H_1} : x \in M\}$$

Therefore, there exists $x_0 \in H_1$ such that $\|Tx_0\|_{H_1} = \|T\|$. □

In the following proposition, we demonstrate the attainability of norms by contractions.

Lemma 4.4. *Given that $T : H \to H$ as a SA contraction, then T is NA iff $-\|T\|$ or $\|T\|$ belongs to the spectrum of T $(\sigma(T))$.*

Proof. Suppose that $-\|T\|$ or $\|T\|$ is in $\sigma(T)$, then for a corresponding eigenvector $x \in H$, we obtain a new eigenvector $\frac{x}{\|x\|} = x_0$ by orthonormalization of x so that $1 \geq \|T\| = \|T(\frac{x}{\|x\|})\| = \|Tx_0\| \leq 1$ say. Conversely, suppose that a normalized vector x_0 exists in D so that $\|T(x_0)\| = \|T\|$. Then $\langle (1 - T^2)x_0, x_0 \rangle = \|x_0\|^2 > \|Tx_0\|^2$ and $(1 - T^2) > 0$ is strictly positive. Then $(1 + T)(x_0 - Tx_0) > 0$ since $x_0 - Tx_0 > 0$. Let there be $x' \in H$ in such a way $x' = (x_0 - Tx_0)$ and $y' = \frac{x'}{\|x'\|}$ therefore $Ty' = -y'$ i.e y' is unitary eigenvector corresponding to eigenvalue $-\|T\| = -1$. □

Proposition 4.5. *Let $H \neq L^1(0, 1)$. For a SA contraction $T \in NA(H)$ which is also p'-normal. Then $\alpha T^{p'} \in NA(H)$ for some $p' \leq 1$ and*

$0 < \alpha < 1$ *iff* $\|T^{p'}\|$ *or* -$\|T^{p'}\|$ *is the farthest point of the numerical range* $W(T^{p'})$.

Proof. Given M and m as a positive operator defined as $M = (I+T)\|T^{p'}\|$ and
$m = (I-T)\|T^{p'}\|$, so that

$$\langle T^{p'}x_o, x_o\rangle \leq \langle T^{p'}x_o, x_o\rangle + \langle Mx_o, x_o\rangle = \|T^{p'}\| \quad (4.2.1)$$

or

$$\langle T^{p'}x_o, x_o\rangle \geq \langle T^{p}x_o, x_o\rangle - \langle mx_o, x_o\rangle = -\|T^{p'}\|. \quad (4.2.2)$$

By supposition that $T^{p'} \in NA(H)$, then $\|T^{p'}\|$ or $-\|T^{p'}\|$ becomes extreme point of $W(T^{p'})$, because
$\langle T^{p'}x_o, x_o\rangle \leq \|T^{p'}\|$, $\langle T^{p'}x_o, x_o\rangle \geq -\|T^{p'}\|$, $\forall x_0 \in D$. Conversely, if -$\|T^{p'}\|$ or $\|T^{p'}\|$ extreme point of $W(T^{p'})$, then we can obtain $x_0' \in D$ such that $\|T^{p'}\| = \langle T^{p'}x_0', x_0'\rangle$ or $-\|T^{p'}\| = -\langle T^{p'}x_0', x_0'\rangle$. From the inequalities 4.2.1 and 4.2.2, it is clear that $\langle Mx_0', x_0'\rangle = 0$ and because M or m is positive, $(M \text{ or } m)x_0' = 0$. So $T^{p'}x_0' = \|T^{p'}\|x_0'$ or $T^{p'}x_0' = -\|T^{p'}\|x_0$. Let $0 < \alpha < 1$ be given then from the definition of m,

$$\langle \alpha T^{p'}x_o, x_o\rangle \leq \langle \alpha T^{p'}x_o, x_o\rangle + \langle Mx_o, x_o\rangle \;=\; \|T^{p'}\|$$

or

$$\langle \alpha T^{p'}x_o, x_o\rangle \leq \langle \alpha T^{p'}x_o, x_o\rangle - \langle Mx_o, x\rangle \;=\; -\|T^{p'}\|$$

□

Theorem 4.6. *Let T be $NA(H)$, p-normal SA compact operator with $D \supset \sigma(T)$ with a positive measure $d\mu_x$. Then $w(T)$ is NA with $|f(T)| \leq \|f\|$ and f is in $R(D)$.*

Proof. If $D \supset \sigma(T)$, then $Re(1 - zT)^{-1} \geq 0$ is equivalent to $w(T) \leq 1$, for every $\{z \in \mathbb{C} : |z| < 1\}$ with series expansions giving

$$(1 - zT)^{-1} = 1 + \sum_{n=1}^{\infty} z^n T^n.$$

Thus

$$\begin{aligned} \langle Re(1 - zT)^{-1}x, x_0\rangle &= \langle 1 + \sum_{n=1}^{\infty} z^p T^p x_0, x_0\rangle \\ &= \|x_0\| + \sum_{n=1}^{\infty} z^p \langle T^p x_0, x_0\rangle. \end{aligned}$$

By p-normality of T, then $\langle T^n x_0, x_0\rangle$ is norm-attainable for $n \equiv p$. Therefore, a measure μ_x which is also positive exists on $[0, 2\pi]$ such that $\|x_0\| + \sum_{n=1}^{\infty} z^p \langle T^p x_0, x_0\rangle\|$ takes the integrand form of $\int \frac{1}{1-ze^{i\theta}} d\mu_{x_0}\theta$ for $\theta \in [0, 2\pi]$ and $|z| < 1$ which can be expanded to obtain

$$\langle T^p x_0, x_0\rangle = 2 \int e^{in\theta} d\mu_{x_0}(\theta) n = 1, 2, ... \qquad (4.2.3)$$

Application of equation 4.2.3 to the polynomial $f(z) = \sum_{k=1}^{n} \alpha_k z^k$ generates $\langle f(T)^p x_0, x_0\rangle = 2 \int f^n(e^{in\theta}) d\mu_{x_0(\theta)}, n = 1, 2, ..$ Given that $\|f\| \leq 1$, then $\|f(T)^p\|$ is bounded and thus

$$\begin{aligned} \langle (1+\sum_{m=1}^{\infty} z^p f(T)^p)x_0, x_0\rangle &= \|x_0\|^2 + 2\sum_{m=1}^{\infty} z^p \int f(e^{i\theta})^n d\mu_{x_0}(\theta) \\ &= \int \frac{1}{1-zf(e^{i\theta})} d\mu_{x_0}(\theta). \end{aligned}$$

Since the part of the integrand is positive, the above proposition takes over. □

Banach spaces have a variety of properties which determines the behavior of operators which act upon them. We consider such properties as density, reflex, e.t.c in this section. For the sake of clarity, the subset of $B(X, Y)$ will be denoted by $NA(X, Y)$ consisting the NAO. The following two theorems due to James and Bourgain are imperative. We have show the condition for norm-attainability of compact operators acting on L^p.

Lemma 4.7. *Suppose H_1 and H_2 are two L^2 spaces with weak topologies H_1^w and H_2^w respectively. Let T be a linear operator from H_1 to H_2 Consequently, T is continuous from H_1^w to H_2^w and vice versa.*

Proof. Let $T : H_1 \to H_2$ be a linear operator. We want to show that T is continuous from H_1^w to H_2^w, and vice versa. First, assume that T is continuous from H_1^w to H_2^w. Let $l \in H_2^*$ be an arbitrary linear functional on H_2. Then, the function $l \circ T : H_1 \to K$ is linear, and it is continuous with respect to the weak topology on H_1, since for any net $(x_\alpha)_{\alpha\in A}$ in H_1 that converges weakly to some $x \in H_1$, we have
$l(T(x_\alpha)) = l(T(x_\alpha)-T(x)+T(x)) \to l(0)+l(T(x)) = l(T(x))$, as $\alpha \to \infty$, where we used the linearity of l, the continuity of T from H_1^w to H_2^w, and

the fact that $l(0) = 0$. Therefore, by the definition of the weak topology on H_2, we conclude that T is continuous from H_1 to H_2. Conversely, assume that T is continuous from H_1 to H_2. Let $(x_\alpha, y_\alpha)_{\alpha \in A}$ be a net in the graph of T, i.e., a net in $H_1 \times H_2$ such that $y_\alpha = T(x_\alpha)$ for all $\alpha \in A$. Suppose that this net converges weakly to some $(x, y) \in H_1 \times H_2$, i.e., for any linear functionals $f \in H_1^*$ and $g \in H_2^*$, we have
$f(x_\alpha) \to f(x) \quad \text{and} \quad g(y_\alpha) \to g(y)$, as $\alpha \to \infty$. Then, since T is continuous from H_1 to H_2, we have $g(y_\alpha) = g(T(x_\alpha)) \to g(T(x))$, as $\alpha \to \infty$. By the uniqueness of limits in K, we must have $g(y) = g(T(x))$, for all linear functionals $g \in H_2^*$. This implies that $y - T(x) = 0$, since the linear functionals on H_2 separate points. Therefore, we have shown that $(x, y) \in G(T)$, where $G(T)$ denotes the graph of T. This means that the graph of T is closed with respect to the weak topology on $H_1 \times H_2$, which we denote by $(H_1 \times H_2)^w$. This topology is induced by the product of the dual spaces of H_1 and H_2, i.e., $(H_1 \times H_2)^w = (H_1^* \times H_2^*)^*$. By the closed graph theorem, we conclude that T is continuous from H_1^w to H_2^w. □

In the following proof, we establish the James Theorem, a well-known result, as a prerequisite for our forthcoming result

Theorem 4.8 (Known James Theorem)**.** *The James theorem states that a Banach space X is reflexive if and only if every continuous linear functional f on X attains its supremum on the closed unit ball in X, i.e., there exists $x \in X$ with $\|x\| \leq 1$ such that $f(x) = \|f\|$.*

Proposition 4.9. *Let T be an element of $B(H_1, H_2)$. Then $T \in NA(H)$ iff a L^2 space H_2 is a finite dimensional and H_1 is reflexive.*

Proof. Let H_2 be a space of finite dimensions and that $T \in B(H_1, H_2)$.

Since H_1 is a reflexive space, the unit ball U_x in H_1 becomes compact(weakly). If the mapping T is continuous when considering the topologies induced by the norms on H_1 and H_2, respectively, and the transformation from H_1 to H_2 is done continuously, then T will also be continuous when considering the weak topologies on H_1 and H_2. So T maps compact(and weakly) sets of H_1 compact(and weakly) sets in H_2. So $T(U_x)$ becomes weakly and compact in H_2. Norm topology and weak topology of H_2 coincide because H_2 is finite dimensional, therefore $T(U_x)$ is compact, hence T is norm-attainable. Conversely suppose that all $T \in B(H_1, H_2)$ are norm-attainable in finite dimensional T, then by James theorem, H_1 is reflexive. □

Corollary 4.10. *If H_1 is reflexive and H_2 a finite dimensional Banach space, then any compact $T \in B(H_1, H_2)$ is NA.*

Proof. Assume that T in $B(H_1, H_2)$ is bounded and linear, then by proposition 4.9 , when H_1 is reflexive, then $T(U_x)$ becomes closed. Furthermore, provided T is compact, then $T(U_x)$ is also compact hence T is NA. On the other hand, if $T(U_x)$ is arbitrary and T has upper and lower bounds and is also linear, then $T(U_x)$ is bounded. So the closure of $T(U_x)$ is also compact since H_2 is finite dimensional. □

Proposition 4.11. *Let T be a compact operator on a Banach space X. If X is reflexive, then T is norm-attainable.*

Proof. Assume that X is reflexive and T is a compact operator on X. We want to show that T is norm-attainable. Let f be a norm-attaining linear functional on X, which means there exists a unit vector $x \in X$ such that

$f(x) = \|f\|$. Since T is compact, it is also continuous. Therefore, we can calculate the norm of T as follows:

$$\|T\| = \sup\{\|Tx\| : x \in X, \|x\| = 1\} \geq \|Tf\| = |f(Tx)| = |T^*(fx)|.$$

Equality occurs when $x = \frac{T^*(f)}{\|T^*(f)\|}$. Therefore, T is norm-attainable. $\square$

4.3 OP in $NA(H)$

In this section, we present the proofs for the norm-attainability of classical orthogonal polynomials. We note that for $m \neq n$, the zero property for the norm is immediate and, therefore, omitted.

Proposition 4.12. *Let $\phi_n(x^C)_{n=0}^{\infty}$ be the sequence of Chebyshev orthogonal polynomials defined on the interval $[-1, 1]$ with respect to the weight function $w(x^C) = (1 - x^2)^{-1/2}$. Then, for any non-negative integer n, the polynomial $\phi_n(x^C)$ is norm-attainale in H, i.e.,$\|\phi_n(x^C)\|_H = K$.*

Proof. Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. We want to show that for any non-negative integer n, there exists a constant $K > 0$ such that $\|\phi_n(x)\|_H = K$. Consider the Chebyshev orthogonal polynomial $\phi_n(x^C)$. By definition, $\phi_n(x^C)$ is orthogonal to all lower degree Chebyshev polynomials, that is,

$$\langle \phi_n(x^C), \phi_m(x^C) \rangle = 0 \quad \text{for all } m < n.$$

Now, let's consider the norm of $\phi_n(x^C)$ in H. We have:

$$\begin{aligned}\|\phi_n(x^C)\|_H^2 &= \langle \phi_n(x^C), \phi_n(x^C)\rangle + \sum_{m<n} \langle \phi_n(x^C), \phi_m(x^C)\rangle \\ &= \langle \phi_n(x^C), \phi_n(x^C)\rangle + 0 \quad \text{(by orthogonality)} \\ &= \langle \phi_n(x^C), \phi_n(x^C)\rangle.\end{aligned}$$

Since $\phi_n(x^C)$ is a non-zero polynomial, then $\langle \phi_n(x^C), \phi_n(x^C)\rangle > 0$. Let $K = \langle \phi_n(x^C), \phi_n(x^C)\rangle$. Then, we have:

$$\|\phi_n(x^C)\|_H^2 = \langle \phi_n(x^C), \phi_n(x^C)\rangle = K^2.$$

Hence, we can conclude that $\|\phi_n(x^C)\|_H = K$, where $K > 0$. □

Proposition 4.13. *Let* $w(x) = e^{-x^2}$ *be a normal distribution and* $H_n(x)$ *be Hermite polynomials defined on the interval* $(-\infty, \infty)$*. Then* $H_n(x)$ *is* NAP_n *for some* $n \in \mathbb{N}^+$*.*

Proof. Consider a space that can be measured $L^2(X, \mu)$ where some measure μ is specified on the S of the X support. For Hermite polynomials, Rodriguez's formula takes the form:

$$H_n(x) = \frac{(-1)^n}{w(x)} D^n w(x) = (-1)^n e^{x^2} D^n e^{-x^2}, n = 0, 1, 2, ...$$

$\forall x \in X$. Suppose $x_0 \in X$ so that $x_0 \in U_x$, that is, $\|x_0\| = 1$. Then $\|H_n(x_0)\|^2$ takes the formula

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x_0) H_n(x_0) dx_0 = (-1)^n \int_{-\infty}^{\infty} H_m(x_0) D^n e^{-x_0^2} d(x_0)$$

for $m < n$. If we perform n integrations on the right-hand side of the equation, it will eventually vanish. When we consider the case where $m = n$, and apply n successive integration by parts to the right-hand side of the equation, it can be deduced that the right-hand side leads to the following result.

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-x_0^2} H_n(x_0) H_n(x_0) dx_0 &= (-1)^n \int_{-\infty}^{\infty} H_n(x_0) D^n e^{-x_0^2} dx_0 \\ &= \int_{-\infty}^{\infty} D^n H_n(x_0) e^{-x_0^2} dx_0 \\ &= \alpha_n n! \int_{-\infty}^{\infty} e^{-x_0^2} dx_0 = 2^n n! \sqrt{\pi}.\end{aligned}$$

Thus for $x_0 \in U_x$, $\|H_n\| = \sup\{2^n n!\sqrt{\pi} : \|H_n(x_0)\| \leq 2^n n!\sqrt{\pi}\|x_0\|\}$, that is, $\|H_n\| = \|H_n(x_0)\|$. □

Before we proceed with the next proof, we introduce a key definition that will be essential for our upcoming result

Definition 4.14. A gamma distribution is a type of continuous probability distribution that is defined by two parameters: a shape parameter (α) and an inverse scale parameter (β). That is, $f(x) = \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}$. Here, x is the random variable, $\alpha$is the shape parameter, β is the inverse scale parameter and $\Gamma(\alpha)$ is the gamma function.

Proposition 4.15. *Let* $w(x'_1) = e^{-x'^2} x'^{(-\alpha)}$ *be gamma distribution function for some* $x'_1 \in X$ *and* $\alpha > -1$. *Then the Laguerre polynomials* $L_n^{(\alpha)}(x'_1) \in NAP_n$ *in some interval* $(0, \infty)$ *and for somen* $\in \mathbb{N}, n > 0$.

Proof. For some $n \in \mathbb{N}$, $L_n^{(\alpha)}(x)$ is defined by Rodriguez's formula as

$$\begin{aligned} L_n^{\alpha}(x'_1) &= w^{-1}\frac{1}{n!}(x)D^n[w(x'_1)x^n] \\ &= \frac{1}{n!}e^{-x}x^{-\alpha}D^n[e^{-x}x^{n+\alpha}], n = (0, 1, 2, ...) \end{aligned}$$

Application of the rule due to Leibniz on the above formula generates

$$L_n^{\alpha}(x') = \sum_{k=0}^{n}(-1)^k \begin{pmatrix} n+\alpha \\ n-k \end{pmatrix} \frac{x'^k}{k!}, n = 0, 1, 2, ...$$

in some arbitrary $x'_1 \in X$. Taking $X = \mathbb{R}$ and a positive Borel measure μ. Then $P_n^{(\alpha)}(x'_1) : L^2(x', \mu) \to \mathbb{R}$ is defined and has a norm given by

$$h_n = \|P_n^{(\alpha)}(x')\|^2 = \int_0^{\infty} \frac{x'^{\alpha}}{e^{x'}} L_m^{\alpha}(x')L_n^{\alpha}(x')dx$$

in some $x' \in X$ and $m, n \in \{0, 1, 2, ...\}$. Suppose $x_0 \in U_{x_0}$, that is, $\|x_0\| = 1$ and $\mu_n = \int_0^{\infty} e^{-x_0}x_0^{n+\alpha}dx_0$. Then $\lim_{n\to\infty} h_n = t$, $n = 0, 1, 2, ...$ exists for some t. Thus for $\alpha > -1$ and $\mu = \Gamma(n + \alpha + 1) > 0$, and so the Rodrigues formula changes thus:

$$\int_0^{\infty} e^{-x_0}x_0^{\alpha}L_m^{(\alpha)}(x_0)L_n^{(\alpha)}(x_0)dx_0 = \frac{1}{n!}\int_0^{\infty} L_m^{(\alpha)}(x_0)D^n[e^{-x_0}x_0^{n+\alpha}]dx.$$

By performing integration by parts n times on the right-hand side of the given integral equation, it is observed that the resulting expression becomes zero. This holds true when n is less than m. When n is equal to m, integrating n times using the same method yields the following outcome

$$\int_0^\infty D^n L_n^{(\alpha)}(x_0)e^{-x_0}x_0^{n+\alpha}dx_0 = \lambda_n n! \int_0^\infty e^{-x_0}x_0^{n+\alpha}dx_0$$
$$= (-1)^n\Gamma(n+\alpha+1).$$

Thus;

$$\|L_n^\alpha\| = \sup_{\|x_0\|=1}\left\{\frac{\Gamma(n+\alpha+1)}{n!} : \frac{\Gamma(n+\alpha+1)}{n!}\|x_0\| \geq \|L_n^\alpha(x_0)\|\right\}$$

for some natural number n. □

Proposition 4.16. *Given be an arbitrary weight function $w(x'^1) = 1$, for some $x'^1 \in X$. Then Legendre polynomials $P_n(x'^1) \in NAP_n$ for some element x'^1 of X, and $n = 0, 1, 2, \ldots$*

Proof. There exists some $n \in \mathbb{N}$, $P_n(x'^1)$ is given by Rodrigues formula as

$$P_n(x'^1) = 2^{-n}\frac{(-1)^n}{n!}w^{-1}(x'^1)D^n[w(x'^1)(1-x^2)^n] = \frac{(-1)^n}{2^n n!}D^n[(1-x^2)^n],$$

$\forall n = 0, 1, 2, \ldots$ which is Jacobi polynomial's special case for $\alpha = \beta = 0$ and D^n defined by Leibniz's rule. Let $X = \mathbb{R}$ also μ to represent a Borel measure supported on X. Then $P_n(x'^1) : L^2(X, \mu) \to \mathbb{R}$ is defined on X and has a norm defined by

$$h_n = \|P_n(x'^1)\|^2 = \int_{-1}^1 P_m(x'^1)P_n(x'^1)dx$$

in some x'^1 in X, $m, n = \{0, 1, 2, \ldots\}$. Suppose that $x_0 \in X$, exists with $\|x_0\| = 1$. Then integrating by parts n times, the Rodrigues formula above gives

$$\begin{aligned}\int_{-1}^{1} P_m(x_0)P_n(x_0)dx &= \frac{(-1)^n}{2^n n!}\int_{-1}^{1} P_m(x_0)D^n[(1-x_0^2)^n]dx_0 \\ &= \frac{1}{2^n n!}\int_{-1}^{1} D^n P_m(x_0)(1-x_0^2)^n dx_0\end{aligned}$$

and vanishes for $m < n$. When $m = n$, with substitution $\frac{1-x_0}{2} = t_0$, $n = 0, 1, 2, ...$ and integrating $(1-x_0^2)^n dx_0$ from -1 to 1, we get

$$\begin{aligned}\int (1-x_0^2)^n dx_0 &= \int_{-1}^{1} (1+x_0)^n(1-x_0)^n dx_0 \\ &= \int_{0}^{1} (2t_0)^n(2-2t_0)^n 2dx_0 = 2^{2n+1}A(n+1, n+1) \\ &= 2^{2n+1}[\Gamma(n+1)\Gamma(n+1)][\Gamma(2n+2)]^{-1} \\ &= [2^{n+1}(n!)^2][(2n+1)!]^{-1}\end{aligned}$$

Now,

$$\begin{aligned}\|P_n\| &= \sup\,[2^{n+1}(n!)^2][(2n+1)!]^{-1} : [2^{n+1}(n!)^2][(2n+1)!]^{-1}\|x_0\| \\ &\geq \|P_n(x_0)\|, \|x_n\| = 1\end{aligned}$$

□

We will now introduce a crucial definition before proceeding with the proof of our next result.

Definition 4.17. The beta distribution is a two-parameter continuous probability distribution. The two parameters of the beta distribution are the shape parameter α and the scale parameter β. The beta distribution can be written in the following form:

$$f(x;\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}x^{\alpha-1}(1-x)^{\beta-1}$$

Where: $x>0$, $\alpha>0$, $\beta>0$, $\Gamma(\alpha)$ is the gamma function, which is a special function that is defined for all positive values of α.

Proposition 4.18. *Assume that* $w(x'^{1})=(1-x'^{1})^{\alpha_1}(1+x'^{1})^{\alpha_2}$ *to be a Beta distribution function for* $P_n^{(\alpha_1,\alpha_2)}$*(Jacobi polynomials),* $n=0,1,...$ *Then* $P_n^{(\alpha_1,\alpha_2)}(x)$ *is* NAP_n *for the interval* $(-1,1)$ *in some* x *in* X*.*

Proof. The polynomial $P_n^{(\alpha_1,\alpha_2)}(x'^{1})$ defined by Rodrigue's formula

$$P_n^{(\alpha_1,\alpha_2)}(x'^{1}) \;=\; 2^{-n}\frac{(-1)^n}{n!}w^{-1}(x'^{1})D^n[w(x'^{1})(1-x'^{2}_{1})^n]$$

which equals to
$\frac{2^{-n}(-1)^n}{n!}(1-x'^{1})^{-\alpha_1}(1+x'^{1})^{-\alpha_2}D^n[(1-x'^{1})^{n+\alpha_1}(1+\alpha_1)^{n+\alpha_2}]$
and take the form $P_n^{(\alpha_1,\alpha_2}(x'^{1})$ equals to

$$(-1)^n2^{-n}\sum_{k=0}^{m}(-1)^k\begin{pmatrix} m+\alpha_1 \\ k \end{pmatrix}\begin{pmatrix} m+\alpha_2 \\ m-k \end{pmatrix}(1+k)^k(1-x'^{1})^{m-k},$$

$n=0,1,2,...$

We consider $X=\mathbb{R}$ with μ as a positive Borel measure supported on X. Then $P_n^{(\alpha_1,\alpha_2)}(x'^{1}):L^2(X,\mu)\to\mathbb{R}$ is defined and has a norm defined by

$$\begin{aligned} h_n &= \|P_n^{(\alpha_1,\alpha_2)}(x'^{1})\|^2 \\ &= \int_{-1}^{1}(1+x'^{1})^{\alpha_2}(1-x'^{1})^{\alpha_1}P_m^{(\alpha_1,\alpha_2)}(x'^{1})P_n^{(\alpha_1,\alpha_2)}(x'^{1})dx \end{aligned}$$

for some $x'^{1} \in X$. Suppose that $x_0 \in X$, exists with $\|x_0\| = 1$, and $\alpha_1, \alpha_2 > -1$, $\forall m, n \in \{0, 1, 2, ...\}$. Then integrating by parts the Rodrigues formula above n times gives for m equals to n, $\int_{-1}^{1}(1+x_0)^{\alpha_2}(1-x_0)^{\alpha_1}\{P_n^{(\alpha_1,\alpha_2)}(x_0)\}^2 dx_0$ to be

$$\begin{aligned}
&= \frac{2^{-n}(-1)^n}{n!}\int_{-1}^{1} P_n^{(\alpha_1,\alpha_2)}(x_0) D^n[(1+x_0)^{n+\alpha_2}(1-x_0)^{n+\alpha_1}]dx_0 \\
&= \frac{2^{-n}}{n!}\int_{-1}^{1} D^n P_n^{(\alpha_1,\alpha_2)}(x_0)[(1+x_0)^{n+\alpha_2}(1-x_0)^{n+\alpha_1}]dx_0 \\
&= \frac{2^{-n}(n+\alpha_1+\alpha_2+1)n}{n!}\int_{-1}^{1}(1+x_0)^{n+\alpha_2}(1-x_0)^{n+\alpha_1}dx_0 \\
&= \frac{2^{-n}\Gamma(2n+\alpha_1+\alpha_2+1)}{\Gamma(n+\alpha_1+\alpha_2+1)n!}\int_{-1}^{1}(1+x_0)^{n+\alpha_2}(1-x_0)^{n+\alpha_1}dx_0 \\
&= [2^{(2n+\alpha_1)}2^{(\alpha_2+1)}]\frac{\Gamma(n+\alpha_1+1)\Gamma(n+\alpha_2+1)}{(2n+\alpha_1+\alpha_2+1)\Gamma(2n+\alpha_1+\alpha_2+1)}
\end{aligned}$$

for $(n = 0, 1, 2, ...)$. Thus

$$\|L_n^{(\alpha_1,\alpha_2)}\| = \sup\left\{\frac{\Gamma(n+\alpha_1+1)\Gamma(n+\alpha_2+1)}{(2n+\alpha_1+\alpha_2+1)\Gamma(2n+\alpha_1+\alpha_2+1)}\right.$$

such that
$\left.\|L_n^{(\alpha_1,\alpha_2)}(x_0)\| \leq \frac{\Gamma(n+\alpha_1+1)\Gamma(n+\alpha_2+1)}{(2n+\alpha_1+\alpha_2+1)\Gamma(2n+\alpha_1+\alpha_2+1)}\|x_0\| = 1\right\}$, for $\|x_0\| = 1$. □

Theorem 4.19. *The claims below are both true and equivalent with respect to the norm-attainability of the function $p_n(x)$ on the interval $[-1, 1]$:*

(i). $(p_n(x))^{\frac{1}{t}}$ is a norm in $\mathbb{R}^n$ for some $t \in \mathbb{R}$, $t \geq 2$, $n \in \mathbb{R}$.

(ii). $p_n(x)$ is convex and positive definite

(iii). for $\alpha_1, \alpha_2 \in \mathbb{K}$ and x, y in $[-1, 1]$ with $x \neq y$, then

$p_n(\alpha_1 x + \alpha_2 y) \leq \alpha_1 p_n(x) + \alpha_2 p_n(y)$.

Proof. (i.) $(i) \Rightarrow (ii)$. Let $t \in \mathbb{R}$, $t \geq 2$, $\|p_n(x)\| = (p_n(x))^{\frac{1}{t}}$. By the preceding propositions 4.13, 4.15, 4.16 and 4.18 above, $\|p_n(x)\|$ exists and it is positive. Thus $p_n(x)$ is also positive. Furthermore given $\alpha_1, \alpha_2 \in \mathbb{K}$ and $p_n(x)$, then

$$\begin{aligned} \|\alpha_1 p_n(x) + \alpha_2 p_m(x)\| &= \left(\sqrt{\frac{2\alpha_1}{2n+1}} + \sqrt{\frac{2\alpha_2}{2m+1}}\right) \\ &\leq \alpha_1^{\frac{1}{2}}\sqrt{\frac{2}{2n+1}} + \alpha_2^{\frac{1}{2}}\sqrt{\frac{2}{2m+1}} \end{aligned}$$

and from Cauchy-Schwarz inequality we get $\leq \alpha_1\|p_n(x)\| + \alpha_2\|p_m(x)\|$. Thus $(p_n(x))^{\frac{1}{t}}$ is also convex.

(ii.) $(ii) \Rightarrow (iii)$. Suppose $p_n(x)$ is convex and positive definite. Let also $p_n(x)$ not be strictly convex. Then for some $x_0, y_0 \in [-1, 1]$ with $x_0 \neq y_0$ for α_1 and α_2 in $\mathbb{K}$, $\alpha_1 + \alpha_2 = 1$ such that
$p_n(x)(\alpha_1 x_0 + \alpha_2 y_0) = \alpha_1 p_n(x_0) + \alpha_2 p_n(y_0)$. Defining $f(\alpha_1)$ by
$f(\alpha_1) = p_n(x_0 + \alpha_1(y_0 - x_0))$, then it is noteworthy that f restricts p_n to the line which further shows that p_n is convex and positive definite in α_1 (given that $\alpha_2 = 1 - \alpha_1$ or $\alpha_1 = 1 - \alpha_2$). Let
$g(\alpha_1) + (f(1) - f(0))\alpha_1 - f(0) = f(\alpha_1)$. Because $g(\alpha_1)$ is the sum of two convex orthogonal polynomials, it is also convex. Furthermore for $\alpha_2, \alpha_1 \in \mathbb{K}$, $\alpha_1 + \alpha_2 = 1$, we get $g(x) \geq 0$. Indeed since $f(\alpha_1)$ is convex $f(\alpha_1 x_0 + \alpha_2 y_0) \geq \alpha_1 f(x_0) + \alpha_2 f(y_0), x_0, y_0 \in [-1, 1]$. Clearly $g(0) = g(1) = 0$. The convexity and non-negativity of f on [-1,1] means that $g(\alpha_1) = 0$ which implies that $g = 0$.
Therefore f has finite values and is positive and thus f is constant, which is a contradiction, because $\lim_{\alpha_1 \to \infty} f(\alpha_1) = \infty$. Because $\lim_{\alpha_1 \to \infty} \|p_n(x_0 + \alpha_1(y_0 - x_0))\| = \infty$ and $f(\alpha_1) = p_n(x_0 + \alpha_1(y_0 - x_0))$

therefore $\lim_{\alpha_1 \to \infty} f(\alpha_1) = \infty$. $p_n(x)$ is positive definite, so for some $\overline{x}$ in $[-1,1]$ $p_n(\overline{x}) > 0$. Now let $\overline{x} = argmin_{\|x\|=1} p_n(x)$ and λ be a positive scalar, so $T = \left(\frac{\lambda}{p_n(\overline{x})}\right)^{\frac{1}{t}}$. For any $x \in [-1,1]$ with $\|x\| = T$, it can be established that $p_n(x) \geq min_{\|x\|=T} p_n(x) \geq T^* p_n(\overline{x}) = \lambda$. Thus $\lim_{\|x\| \to \infty} p_n(x) = \infty$.

(iii.) $(iii) \Rightarrow (i)$. For some positive $\lambda \in \mathbb{R}$, then $(p_n(x))^{\frac{1}{t}}$ is homogeneous because $\|\lambda p_n(x)\| = |\lambda| \|p_n(x)\|$. For any (x, y) within the range of $[-1,1]$ with $x \neq y$ then $p_n(y) > p_n(x) + \nabla p_n(x)^T (y - x)$. Clearly $p_n(x) > 0$ for $x = 0$ since $p_n(0) = 0 = \nabla p_n(0)$.

Thus $p_n(x)$ is a positive definite polynomial and so is $\|p_n(x)\|$. Suppose $f = (p_n(x))^{\frac{1}{t}}$ with $M_f = \{x : (p_n(x))^{\frac{1}{t}} \leq 1\} = N_{p_n}$ and $M_f = \{x : p_n(x) \leq 1\} = N_{p_n}$. Now because $p_n(x)$ is strictly convex, N_{p_n} is also convex and so is M_f. For some x, y in $[-1,1]$, then $\frac{x}{p_n(x)} \in M_f$ and $\frac{y}{p_n(y)} \in M_f$. By convexity of M_f, as a result, we obtain that $f\left(\frac{f(x)}{f(x)+f(y)} \cdot \frac{x}{f(x)} + \frac{f(y)}{f(x)+f(y)} \cdot \frac{y}{f(y)}\right) \leq 1$ and by homogeneity of f we get $\frac{1}{f(x)-f(y)} \cdot f(x+y) \leq 1$. Hence $\|p_n(x)\|$ meets the triangle inequality criterion.

□

Properties of univariate orthogonal polynomials touching on their zeros, three recurrence formula and others make them useful in the analysis of differential equations. These properties can be extended to multivariate orthogonal polynomials with some modifications [96]. Given a monomial $x \in \mathbb{R}$ of several variables, $x_1^{\alpha_1}$, $x_2^{\alpha_2} ... \, x_d^{\alpha_d}$ we denote by $|d| = \alpha_1 + ... + \alpha_n$ the monomial's overall degree. For such monomial let Borel positive measure μ on $\mathbb{R}^d$ generate finite moments given by $\mu_\alpha = \int_{\mathbb{R}^d} x^\alpha d\mu(x)$

on which application of Gram-schmidt process involving the monomials with respect to inner product gives multivariate orthogonal polynomials $\int_{\mathbb{R}^d} f(x)g(x)d\mu(x)$ in $L^2(\mu)$. The major problem with multivariate orthogonal polynomials is that they are not unique. Furthermore different total orders give different sequences of orthogonal polynomials. We therefore consider the following spaces instead of fixing total order.

$$\begin{aligned} NAP^0 &= \{P : P \in \Pi_n^d, and, \exists \|Pw(x^d)\| < \infty, \forall x^d \in \mathbb{R}^d, with, \|x^d\| = 1\} \\ \Pi_n^d &= \{P : \langle P, Q\rangle = 0, \forall Q \in \Pi^d, degP > degQ\} \end{aligned}$$

Specifically, this refers to collection of orthogonal polynomials of degree n with regard to μ.

$$\begin{aligned} V^d_{n(\forall Q \in \Pi^d_{n-1})} &= \{P \in \Pi_n^d : \langle P, Q\rangle = 0\} \\ NA\Pi_n^d &= \{P \in \Pi_n^d : PinNAP^0\}. \end{aligned}$$

A multivariate sequence of polynomials $P_j \in \Pi_n^d$, $j \in \mathbb{N}$ is called orthogonal if $\langle P_i, P_j\rangle = \delta_{ij}$. The space V_n^d has a variety of bases which need not be orthonormal.

Proposition 4.20. *If $[\{p_n\}_{m=0}^{\infty}]_{m\in\mathbb{N}_0} = \{p_\alpha^d : |\alpha| = n\}$, where p_0 equals to 1, is a family of multivariate polynomials Π_n^d. Then the following are equivalent:*

(i). p_n is $NA\Pi_n^d$ for each $m \in \mathbb{N}_0$.

(ii). $p_n(x_d)$ is both convex and positive function for $x_d \in \mathbb{R}^n$, $m \in \mathbb{N}_0$ $(d \leq n)$.

(iii). p_n is strictly convex for all $m \in \mathbb{N}_0$.

Proof. (i.) $(i) \Rightarrow (ii)$. Consider the function for product weight $W(x) = w_1(x_1), ...w_d(x_d)$ and Gegenbauer polynomials $C_k^{\lambda}(x)$ with monomials, $p_k^n \in \Pi_n^2$ defined by $p_k^n(x,y) = h_{k,n} C_{n-k}^{k+\mu+\frac{1}{2}}(x)(1-x^2)^{\frac{k}{2}} C_k^{\mu}\left(\frac{y}{\sqrt{1-x^2}}\right), 0 \le k \le n$ on $B^2 = \{(x,y) : x^2 + y^2 \le 1\}$. The orthogonality of these polynomials are the existence of the functional hn can be verified by the formula

$$\int_{B^2} p_n^2(xy) p_m^2(xy) W_\mu(xy) \quad = \quad \int_{-1}^{1} \int_{-(1-x^2)^{\frac{1}{2}}}^{(1-x^2)^{\frac{1}{2}}} p_n^2(xy) p_m^2(xy) W_\mu(xy)$$

which equals to $\int_{-1}^{1} \int_{-1}^{1} p_n^2(x,(1-x^2t)^{\frac{1}{2}}) p_m^2(x,(1-x^2t)^{\frac{1}{2}})(1-x^2)dxdt$ for $y = t \Rightarrow dy = dt$. Application of $y = r\sin\theta$, $x = r\cos\theta$ (as polar coordinates) and Chebyshev polynomials T_m and U_m of the first and the second kinds the families of monomials

$$\begin{array}{lll} h_{j,1} p_j^{(\mu-\frac{1}{2}, n-2j+\frac{d-2}{2})}(2r^2-1) r^{n-2j} \cos(n-2j)\theta & , & 0 \le 2j \le n, \\ h_{j,2} p_j^{(\mu-\frac{1}{2}, n-2j+\frac{d-2}{2})}(2r^2-1) r^{n-2j} \sin(n-2j)\theta & , & 0 \le 2j \le n-1 \end{array}$$

with normalization constants $h_{j,i}^n$. For each $n \in \mathbb{N}_0$, these monomials generate $n+1$ polynomials on B^2 of degree verified by

$$\int_{B^2} p_n^2(B^2) p_m^2(B^2) W_\mu^{B^2}(B^2) \quad = \quad \int_0^1 \int_0^{2\pi} p_n^2(B^2) p_m^2(B^2) d\theta r dr$$

where the relations $r = \|x\|$, $T_m\left(\frac{x}{\|x\|}\right) = \cos m\theta$ and

$U_{m-1}\left(\frac{x}{\|x\|}\right) = \frac{\sin m\theta}{\sin\theta}$ hold. A set

$$p_{k^n}(B^2) \;=\; C_n^{\mu+\frac{1}{2}}\left(a\cos\frac{k\pi}{n+1} + b\sin\frac{k\pi}{n+1}\right), 0 \le k \le n$$

of monomials in particular, for $a, b \in B^2$

$$p_{k^n}(B^2) \;=\; \frac{1}{\sqrt{\pi}} U_n\left(a\cos\frac{k\pi}{n+1} + b\sin\frac{k\pi}{n+1}\right), \mu = \frac{1}{2}, 0 \le k \le n.$$

also establish an orthogonal basis with regard to the Lebesgue measure on B^2[22]. The collection of polynomials
$V_k^n(B^2) = x^k y^{n-k} + q(B^2)$ generated

$$(1 - 2(t_1 a + t_2 b) + \|t\|^2)^{-\mu-\frac{1}{2}} \;=\; \sum_{n=0}^{\infty}\sum_{k=0}^{n} t_1^k t_2^{n-k} V_k^n(B^2), t = (t_1, t_2)$$

where $q \in \Pi_{n-1}^2$ and

$$U_k^n(B^2) \;=\; (1 - a^2 - b^2)^{-\mu+\frac{1}{2}} \frac{\partial^k}{\partial a^k}\frac{\partial^{n-k}}{\partial b^{n-k}}(1 - a^2 - b^2)^{n+\mu-\frac{1}{2}}$$

are orthogonal since computation of h_n by integration by parts gives

$$\int_{B^2} V_k^n(a,b) U_j^n(B^2) W_\mu^{B^2}(B^2) \;=\; \begin{cases} h_n\delta_{nm} & k = j \\ 0 & k \ne j. \end{cases}$$

Given $\lambda_1, \lambda_2 \in \mathbb{K}$, $\lambda_1 + \lambda_2 = 1$ and two sets of monomials $\overline{p}_k^n(B^2)$ and $p_k^n(B^2)$. From theorem 4.19 above, the rest follows. Indeed by

Cauchy-Schwarz inequality

$$p_n(B^2)(\lambda_1\overline{p}_k^n(B^2)+\lambda_2 p_k^n(B^2)) \leq \lambda_1 p_n(B^2)(\overrightarrow{p}_k^n(B^2))+\lambda_2 p_n(B^2)(p_k^n(B^2)) \quad (4.3.1)$$

for $(a, b) \in B^2$.

(ii.) $(ii) \Rightarrow (iii)$. With the following substitutions, the supposition follows from theorem 4.19 above. $p_n(x) \equiv p_n(B^2)$, $a_0 \equiv \overline{p}_k^n(B^2)$ and $b_0 \equiv p_k^n(B^2)$. Finally, $(iii) \Rightarrow (i)$ (See the substitutions above in 4.3.1).

□

4.4 Relationship between OP and NAO in $NA(H)$

Consider a non-decreasing function $\alpha(x)$ defined on the interval $[a, b]$, and let $g(x)$ be a measurable function in the Lebesgue space $L^p(X, \xi_1, \mu)$, where $x \in X$. We assume that $\int_a^b |g(x)|^p d\alpha(x)$ exists. When $\alpha(x) = x$ and $p = 2$, we denote this space as $L^2(a, b)$. Exploiting the Hilbert space structure, we can define the inner product $\langle g_1, g_2 \rangle = \int_a^b g_1(x)\overline{g_2(x)} d\alpha(x)$ for monotonic $\alpha(x)$. This definition can be extended to cases where $\alpha(x)$ has bounded variation. If the limits a and b of the interval $[a, b]$ are finite, then $\alpha(x)$ is bounded, and $g(x) \in L^p(x)$ is continuous. In such cases, the integration formula $\int_a^b \alpha(x)dg(x) + \int_a^b g(x)d\alpha(x) = g(b)\alpha(b) - g(a)\alpha(a)$ is applicable. Here, the term $d\alpha(x)$ can be interpreted as a continuous or discontinuous mass distribution within the interval $[a, b]$, and the limit

$[x_1, x_2] \subset [a, b]$ contributes a mass defined by $[\alpha(x_2) - \alpha(x_1)]$. When $\alpha(x)$ is absolutely continuous, the expression $\int_a^b g_1(x)\overline{g_2(x)}d\alpha(x)$ can be written as $\int_a^b g_1(x)g_2(x)w(x)d(x)$ if the integral exists according to Lebesgue's theory. In this case, $w(x)$ is a positive and Lebesgue measurable function, and $\int_a^b w(x) > 0$. The function $w(x)$ is referred to as the weight function on the interval $[a, b]$. The total mass on the interval $[x_1, x_2]$ is given by $\int_{x_1}^{x_2} w(x)dx$ for a distribution $w(x)dx$. Suppose $[a, b]$ is finite with $d\alpha(x)$ or $w(x)$ as fixed distribution and vector functions in $L^2_\alpha(a, b)$. In the sequel, we consider linear differential operators which act on a space of continuous and differentiable functions $C^0([0, 1])$ and $C^\infty([0, 1])$.

Proposition 4.21. *Consider a second order differential operator $T(u)$ expressed by $T(u) = c_1D^2(u)+c_2(D(u)+c_3(u)$. Given $u_1, u_2 \in C^\infty([0, 1])$, then*

$$\langle u_2, Tu_1\rangle - \langle T^*u_2, u_1\rangle = [(\overline{u_2}u_1' - \overline{u_2'}u_1) + (c_2 - c_1)\overline{u_2}u_1]_0^1$$

for an adjoint of T defined by

$$T^*u_2 = (u_2)'' - (c_2u_2)' + c_3u_2, c_1, c_2, c_3 \in C^0([0, 1]).$$

Proof. Applying $\langle , \rangle$ for the usual $L^2([0,1])$, and integrating by parts give:

$$\begin{aligned}
\langle u_2, Tu_1 \rangle &= \int_0^1 \overline{u_2}(c_1 D^2 u_1 + c_2 D u_1 + c_3 u_1)dx \\
&= \int_0^1 (c_1 u_2 D^2 u_1 + c_2 u_2 D u_1 + c_3 u_2 u_1)dx = \overline{u_2} D u_1 \\
&+ c_2 \overline{u}_2 u_1 + \int_0^1 \{-(D\overline{u}_2 D u_1 + c_2 D\overline{u}_2 + c_3 \overline{u}_2 + c_3 \overline{u}_2 u_1\} dx \\
&= [\overline{u_2} D u_1 - \overline{u_2}) u_1 + c_3 \overline{u}_2 u_1]_0^1 + \int_0^1 (D^2 u_2 + c_2 \overline{D u_2} + c_3 u) dx
\end{aligned}$$

□

Example 4.22. If $c_1 = 1$, $c_2 = -2x$, $c_3 = 2n$, $n = 0, 1, 2....$, then Tu is a differential operator acting on a space of Hermite orthogonal polynomials $H_n(x)$ in $C^\infty([0,1])$. So $T(H_n(x^H)) = D^2 H_n(x^H) - 2xDH_n(x) + 2nH_n(x^H)$ has defined adjoint expressed as

$$T^*(H_n(x^H)) \ = \ D^2 H_n(x^H) + \overline{2x} D H_n(x^H) + 2nH_n(x^H), n = 0, 1, 2, ...$$

On integration by parts, we have:

$$\begin{aligned}
\langle u_2, Tu_1 \rangle &= \int_0^1 \overline{u_2}(u_1'' - 2xu_1' + 2nu_1)dx \\
&= \int_0^1 (\overline{u_2} u_1'' - 2x\overline{u_2} u_1' + 2n\overline{u_2} u_1)dx \\
&= \overline{u_2} u_1' - 2x\overline{u_2} u_1 + \int_0^1 \{-(\overline{u_2})' u_1' + 2x(\overline{v})' + 2n\overline{u_2} u_1\} dx \\
&= [\overline{u_2} u_1' - (\overline{u_2}) u_1 + 2n\overline{u_2} u_1]_0^1 + \int_0^1 ((u_2)'' + \overline{2x}(u_2)' + 2nu_1)dx
\end{aligned}$$

Example 4.23. Let $c_1 = x$, $c_2 = (1 - x + \alpha)$, $c_3 = n$ for $n = 0, 1, 2....$ then Tu is a differential operator defined on Laguerre orthogonal polynomials

$L_n^{(\alpha)}(x) \in C^\infty([0,1])$. Thus

$$T(L_n^{(\alpha)})(x^L) = xD^2L_n^{(\alpha)}(x^L) + (1-x+\alpha)DL_n^{(\alpha)}(x^L) + nL_n^{(\alpha)}(x^L)$$

has an adjoint given by

$$T^*(L_n^{(\alpha)}(x^L) \;=\; \overline{x}D^2L_n^{(\alpha)}(x^L) - (\overline{1-x+\alpha})DL_n^{(\alpha)}(x^L) + nL_n^{(\alpha)}(x^L)$$

On integrating by parts we have:

$$\begin{aligned}\langle u_2, Tu_1\rangle &= \int_0^1 \overline{u_2}(xu_1'' + (\alpha+1-x)u_1' + nu_1)dx \\ &= \int_0^1 (x\overline{u_2}u'' + (1-x+\alpha)\overline{u_2}u_1' + n\overline{u_2}u_1)dx.\end{aligned}$$

This equals to

$\overline{u_2}u_1' + (1-x+\alpha)\overline{u_2}u_1 + \int_0^1\{-(\overline{u_2})'u_1' + \overline{(1-x+\alpha)}(u_2)' + n\overline{u_2}u_1\}dx$

and finally equals to

$[\overline{u_2}u_1' - (\overline{u_2})u_1 + n\overline{u_2}u_1]_0^1 + \int_0^1(x(u_2)'' + \overline{(1-x+\alpha)}(u_2)' + nu_1)dx$

Example 4.24. Let $c_1 = (-x^2+1)$, $c_2 = [\beta - \alpha(2+\alpha+\beta)]$, $c_3 = n(n+1+\alpha+\beta)$ for $n = (0, 1, 2...)$. $T(u)$ is then a differential operator defined on Jacobi orthogonal polynomials $P_n^{(\alpha,\beta)}(x^J) \in C^\infty([0,1])$, $(n = 0, 1, 2...)$. Thus

$T(P_n^{(\alpha,\beta)}(x^J)) = (-x^2+1)D^2P_n^{(\alpha,\beta)}(x^J) + \beta - \alpha(2+\alpha+\beta)x^J)DP_n^{(\alpha,\beta)}(x^J)$
$+ n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x^J)$

has an adjoint,

$T^*(P_n^{(\alpha,\beta)}(x^J) = \overline{(-x^2+1)}D^2(P_n^{(\alpha,\beta)}(x^J)$
$-(\overline{\beta - \alpha(2+\alpha+\beta)x^J})D(P_n^{(\alpha,\beta)}(x^J) + n(n+\alpha+\beta+1)(P_n^{(\alpha,\beta)}(x^J).$

Therefore integrating by parts gives:

$$
\begin{aligned}
\langle u_2, Tv\rangle &= \int_0^1 \overline{u_2}((-x^2+1)u_1'' + (\beta - \alpha(2+\alpha+\beta)x^J)u_1' + P_n^{(\alpha,\beta)}(x^J))dx \\
&= \int_0^1 ((-x^2+1)\overline{u_2}u'' \\
&+ (\beta - \alpha(2+\alpha+\beta)x^J)\overline{u_2}u_1' + n(n+1+\alpha+\beta)\overline{u_2}u_1)dx \\
&= \overline{u_2}u_1' + ([\beta - \alpha(2+\alpha+\beta)x])\overline{u_2}u_1 + \int_0^1 \{-(\overline{u_2})'u_1' \\
&+ \overline{(1-x+\alpha)}(u_2)' + n(n+1+\alpha+\beta)\overline{u_2}u_1\}dx \\
&= [\overline{u_2}u_1' - (\overline{u_2})u_1 + n\overline{u_2}u_1]_0^1 + \int_0^1 ((-x^2+1)(u_2)'' \\
&+ \overline{(\beta - \alpha(2+\alpha+\beta)x^J)}(u_2)' + n(n+1+\alpha+\beta)P_n^{(\alpha,\beta)}(x^J))dx
\end{aligned}
$$

Proposition 4.25. *Let $T(u)$ be defined as in 4.21 with $c_1, c_2, c_3 \in C^0[0,1]$ and $u_1, u_2 \in C^\infty[0,1]$. Then, $T(u)$ is SA, that is,*
$\langle u_1, Tu_2\rangle - \langle T^* u_1, u_2\rangle = [u_1(\overline{u}_2 u_1' - \overline{u'}_2 u_1) + (c_2 - c_1')u_2\overline{u}_1]_0^1$ holds.

Proof. Consider orthogonal polynomials $u_1, u_2 \in C^\infty([0,1])$ for which there exist sequences $u_1, u_2 \in C^\infty([0,1])$ such that $u_{1n} \longrightarrow u_1$, $u_{2n} \longrightarrow u_2$. Thus we have
$\langle Tu_{1n}, u_{2n}\rangle - \langle u_{1n}, T^* u_{2n}\rangle = [c_1(\overline{u'}_{1n}u_{2n} - \overline{u}_{1n}u_{2n}') + (c_3 - c_1')\overline{u}_{1n}u_2]_0^1$.
Because $\lim_{u\longrightarrow\infty} Tu_{1n} \longrightarrow Tu_1$ and $\lim_{u\longrightarrow\infty} T^* u_{2n} \longrightarrow T^* u_2$ with the limits in $L^2((0,1))$, the boundary terms converge point wise. □

Proposition 4.26. *If $T : L^2([0,1]) \longrightarrow L^2([0,1])$ is an operator defined as $Tu = c_1 D^2(u) + c_2 D(u) + c_3(u)$, then T is a closed operator.*

Proof. Let (u_n) be a sequence in $D(T)$ such that $u_n \to u$ in $L^2([0,1])$ and $Tu_n \to v$ in $L^2([0,1])$. We want to show that $u \in D(T)$ and $Tu = v$. Since $u_n \in D(T)$, we have $Tu_n = c_1 D^2(u_n) + c_2 D(u_n) + c_3(u_n)$. Taking the limit of both sides as $n \to \infty$, we get

$$\lim_{n\to\infty} Tu_n = c_1 D^2(u) + c_2 D(u) + c_3(u)$$

by the continuity of the differential operators D^2 and D. Since $Tu_n \to v$ in $L^2([0,1])$, we have $\lim_{n\to\infty} Tu_n = v$. Therefore,

$$v = c_1 D^2(u) + c_2 D(u) + c_3(u)$$

which means that $u \in D(T)$ and $Tu = v$. Therefore, T is a closed operator. □

Proposition 4.27. *Let $c_1, c_2, c_3 \in C^0[0,1]$ be real-valued function with $c_1(x) > 0$ for all $x, y \in [0,1]$. The eigenvalue problem*
$c_1 D^2(u) + c_2 D(u) = -c_3(u)$, with $u(0) = u(1) = 0$ has eigenfunctions which form orthonormal basis of $L^2([0,1])$ and therefore normal.

Proof. Taking the inner products of $c_1 D^2(u) + c_2 D(u) = -c_3(u)$ with u and integrating by parts, gives
$\int_0^1 \{c_1 D^2|u|^2 + c_2|u|^2\} du = -c_3 \int_0^1 |u|^2 du$. Let $\lambda_1 = \min_{0\le x\le 1} c_1(x)$,
$\lambda_2 = \min_{0\le x\le 1} c_2(x)$ and because $c_1 > 0$, then $\lambda_1 > 0$ and suppose $c_2 > 0$ we get $\lambda_2 > 0$ which contradicts the possibility of $\lambda_2 \le 0$.

$$\int_0^1 \{\lambda_1 D^2|u|^2 + \lambda_2 D|u|^2\} du + c_3 \int_0^1 |u|^2 du \le 0$$

So we get

$$\lambda_1 \int_0^1 D^2|u|^2 du + \lambda_2 \int_0^1 D|u|^2 du + c_3 \int_0^1 |u|^2 du \le 0.$$

This shows that $T - \lambda I$ has real values if $\lambda_2^2 \geq 4\lambda_i c_3$, hence λ_1, λ_2 forms σT. So the SA resolvent operator $R_{\lambda_i} : i = 1, 2$ is given by $R_{\lambda_i} = (\lambda_i I - T) : L^2([0,1]) \longrightarrow L^2([0,1])$ for which $R_{\lambda_i} f(x) = -\int_0^1 [g_{\lambda_i}(x,y)] f(y) dy$, $i = 1, 2$ where g_{λ_i} is the Greens function for $T - \lambda_i I$, $i = 1, 2$. Now $\int_0^1 \int_0^1 [g_{\lambda_i}(x,y)]^2 dx dy < \infty, i = 1, 2,$ since λ_i is continuous for $i = 1, 2$. Therefore R_{λ_i} is compact and Hilbert-Schmidt. An orthonormal basis $L^2([0,1])$ exists consisting of $\{u_n : n \in \mathbb{N}\}$ of R_{λ_i} whose eigenvalues $\lambda_{in} : n \in \mathbb{N}$ such that $\lim_{n \longrightarrow \infty} \lambda_{in} = 0$. Since $(\lambda_i I - T) R_{\lambda_i} = I$ we have $u_n \in D(T)$ and $T u_n = \lambda_{in} u_n$ where $\lambda_{2n}^2 \geq 4\lambda_{in} c_3$. So $\lim_{n \longrightarrow \infty} \lambda_{in} = \infty$ and therefore T has complete orthonormal set of eigenvalues which form a basis of $L^2([0,1])$. □

Theorem 4.28. *An operator $T(u)$ defined as in 4.21 is not NA on $C^0([0,1])$.*

Proof. Suppose that $u \in C^0([0,1])$, $u : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is a function on which a derivative $D(u(x))$ exists for every single point $x \in [0,1]$, that is, $Du(x) : \mathbb{R}^m \longrightarrow \mathbb{R}^n (D^2 u(x) : \mathbb{R}^m \longrightarrow \mathbb{R}^n)$. Then by Lipschitz principle, $Du(x)$ is characterized as $ux + uh - u(x) = Du(x).h + th$, $h \longrightarrow 0$. Thus $Du(x)$ is defined such that its norm estimates is provided by

$$\|D^m u(x)\| \leq \left(\sum_{i=1}^{m} \left| \frac{d^i u(x)}{dx^i} \right|^2 \right)^{\frac{1}{2}}.$$

Now for each $x \in [0,1]$,

$$\|u(x)\| = \langle u(x), u(x) \rangle = \int_0^1 \overline{u(x)} u(x) dx = 1$$

thus for a differential operator $T(u)$, given by $T(u) = c_1 D^2(u) + c_2 D(u) + u$, the norm estimates for operator takes the form

$$\|T(u)(x)\| \leq |c_1| \left(\sum_{i=1}^{2} \left| \frac{du(x)}{dx} \right|^2 \right)^{\frac{1}{2}} + |c_2| \left(\sum_{i=1}^{1} \left| \frac{du(x)}{dx} \right| \right) + |c_3|.$$

Thus the operator $T(u)$ is SA but unbounded hence not NA. □

Proposition 4.29. *Let $T(u)$ be as defined in proposition 4.21. Then T is normal and has orthogonal eigenspaces.*

Proof. We show that T is diagonalizable by demonstrating that it has orthogonal eigenspaces. Suppose T has an eigenvalue λ and let u represent the corresponding eigenvector. Then we have:

$$\begin{aligned} T(u) &= \lambda u c_1 D^2(u) + c_2 D(u) + c_3 u \\ &= \lambda u c_1 D^2(u) + c_2 D(u) + (c_3 - \lambda)u = 0. \end{aligned}$$

We can rewrite this equation as a homogeneous second-order linear differential equation $c_1 u'' + c_2 u' + (c_3 - \lambda)u = 0$. Now, let's assume both u_1 and u_2 are linearly independent solutions of this differential equation. Since T is a second-order operator, we expect to find two linearly independent eigenvectors corresponding to each eigenvalue. Therefore, we seek a second solution u_2. By employing the technique of variation of parameters, we can determine the second solution u_2 in the following manner: $u_2(x) = u_1(x) \int \frac{e^{-\int \frac{c_2}{c_1} dx}}{u_1^2(x)} dx$. The solution u_2 is linearly independent of u_1 as long as the integral above is not identically zero. The linear independence of solutions u_1 and u_2 implies that they can be used as a basis for the eigenspace corresponding to the eigenvalue λ. Moreover, these solutions are orthogonal with respect to the inner product induced by the differential operator T. Hence, we have shown that for each eigenvalue λ of T, $\exists$ two linearly independent eigenvectors u_1 and u_2, forming an orthogonal basis for the eigenspace associated with λ. Therefore, the operator T is diagonalizable. Finally, suppose λ_i are eigenvalues of T for $i \in \mathbb{N}$. Then T takes the Jordan form. Let $L = T - \lambda_i I$ with the condition that $ker(L) \subset ker(L^2) \subset ... \subset ker(L^n)$. Then the least value of n for $ker(L^n = ker(L^{n+1})$, is the greatest Jordan block. Thus for $n = 1$, the diagonalizability of T holds because all of the Jordan blocks will be 1×1. Therefore if $ker(B) = ker(B^n)$ for $n \geq 1$, the result is as required. □

Proposition 4.30. *Let $T(u)$ be as defined in proposition 4.21 above, and consider u_1 and u_2 belonging to the space $C^\infty([0,1])$, which represents infinitely differentiable functions defined on the interval $[0,1]$. If the following conditions are satisfied:*
$\langle T(u_1), T(u_2)\rangle = \langle T^*(u_1), T^*(u_2)\rangle$ and $Ker(T) = ker(T^*)$ then the operator T becomes normal.

Proof. We need to show that if the two conditions are satisfied, then the operator T is normal. First, let's consider the inner product of $T(u_1)$ and $T(u_2)$. Using the definition of T, we have:

$$\langle T(u_1), T(u_2)\rangle = \langle c_1 D^2(u_1) + c_2 D(u_1) + c_3 u_1, c_1 D^2(u_2) + c_2 D(u_2) + c_3 u_2\rangle.$$

Expanding this inner product, we get:

$$\begin{aligned}\langle T(u_1), T(u_2)\rangle &= c_1c_1^*\langle D^2(u_1), D^2(u_2)\rangle + c_1c_2^*\langle D^2(u_1), D(u_2)\rangle \\ &+ c_1c_3^*\langle D^2(u_1), u_2\rangle + c_2c_1^*\langle D(u_1), D^2(u_2)\rangle \\ &+ c_2c_2^*\langle D(u_1), D(u_2)\rangle + c_2c_3^*\langle D(u_1), u_2\rangle \\ &+ c_3c_1^*\langle u_1, D^2(u_2)\rangle + c_3c_2^*\langle u_1, D(u_2)\rangle + c_3c_3^*\langle u_1, u_2\rangle,\end{aligned}$$

where c_1^*, c_2^*, c_3^* represent the complex conjugates of c_1, c_2, c_3 respectively. Now, let's consider the adjoint of T, denoted as T^*. The adjoint of an operator is obtained by taking the complex conjugate of its coefficients and reversing the order of the derivatives. In our case, T^* is given by $T^*(u) = c_1^*D^2(u) + c_2^*D(u) + c_3^*u$. To satisfy the first condition of the proposition, we need to show that $\langle T^*(u_1), T^*(u_2)\rangle$ is equal to the inner product of $T(u_1)$ and $T(u_2)$. Substituting $T^*(u_1)$ and $T^*(u_2)$ into the inner product, we have
$\langle T^*(u_1), T^*(u_2)\rangle = \langle c_1^*D^2(u_1)+c_2^*D(u_1)+c_3^*u_1, c_1^*D^2(u_2)+c_2^*D(u_2)+c_3^*u_2\rangle$.
Expanding this inner product, we get:

$$\begin{aligned}\langle T^(u_1), T^(u_2)\rangle &= c_1c_1^*\langle D^2(u_1), D^2(u_2)\rangle + c_1c_2^*\langle D^2(u_1), D(u_2)\rangle \\ &+ c_1c_3^*\langle D^2(u_1), u_2\rangle + c_2c_1^*\langle D(u_1), D^2(u_2)\rangle \\ &+ c_2c_2^*\langle D(u_1), D(u_2)\rangle + c_2c_3^*\langle D(u_1), u_2\rangle \\ &+ c_3c_1^*\langle u_1, D^2(u_2)\rangle + c_3c_2^*\langle u_1, D(u_2)\rangle + c_3c_3^*\langle u_1, u_2\rangle.\end{aligned}$$

Comparing this with the previous expression for $\langle T(u_1), T(u_2)\rangle$, we observe that the two are equal. Hence, the first condition of the proposition is satisfied.

Next, we need to show that the kernel of T is equal to the kernel of its adjoint, T^*. The kernel of an operator consists of all the functions u for which $T(u) = 0$. Therefore, we need to show that if u belongs to the kernel of T, then it also belongs to the kernel of T^*, and vice versa. Let's assume that u is in the kernel of T. This means $T(u) = 0$. Substituting the expression for $T(u)$, we have $c_1 D^2(u) + c_2 D(u) + c_3 u = 0$. Similarly, assuming v is in the kernel of T^*, we have $T^*(v) = 0$. Substituting the expression for $T^*(v)$, we get $c_1^* D^2(v) + c_2^* D(v) + c_3^* v = 0$. Since D represents the derivative operator, $D^2(u)$ and $D^2(v)$ are second derivatives of u and v, respectively. By applying integration by parts, we can equate the coefficients of the derivatives and show that u being in the kernel of T implies v is in the kernel of T^*, and vice versa. Therefore, the two conditions are satisfied, which implies that T is a normal operator. □

Proposition 4.31. *Let $T(u)$ be as defined in proposition 4.21 above and let $u_1, u_2 \in C^\infty([0, 1])$. Suppose T is normal operator and $n > 1$, then the null space (kernel) of T is equal to the null space of T^n.*

Proof. Let's consider a function u that belongs to the null space (kernel) of T, denoted as $\text{Ker}(T)$. This means that $T(u) = 0$. We want to show that u also belongs to the null space of T^n, denoted as $\text{Ker}(T^n)$, where $n > 1$. In other words, we need to prove that $T^n(u) = 0$. We can start by using mathematical induction to prove this statement. The base case is $n = 2$. In this case, we have $T^2(u) = T(T(u)) = T(0) = 0$. Since $T^2(u) = 0$, u satisfies the condition for being in $\text{Ker}(T^2)$. Now, let's assume that the proposition holds for some positive integer k, i.e., if $T^k(u) = 0$, then u is in $\text{Ker}(T^k)$. We want to show that the proposition also holds for $n = k + 1$, i.e., if $T^{k+1}(u) = 0$, then u is in $\text{Ker}(T^{k+1})$. Using the assumption, we know that $T^k(u) = 0$, and since T is a normal operator, we have $\text{Ker}(T) = \text{Ker}(T^k)$. This means that u is in $\text{Ker}(T)$. Now, applying T to both sides of $T^k(u) = 0$, we get $T(T^k(u)) = T(0) = 0$. Using the definition of T, we can expand this as $T^{k+1}(u) = 0$. Therefore, u satisfies the condition for being in $\text{Ker}(T^{k+1})$. By induction, we have shown that if $T^k(u) = 0$ for some positive integer k, then u is in $\text{Ker}(T^k)$. In particular, when $k = n - 1$, we have $T^{n-1}(u) = 0$, which implies that u belongs to $\text{Ker}(T^{n-1})$. But since $T^{n-1}(u) = 0$, applying T one more time gives us $T^n(u) = T(T^{n-1}(u)) = T(0) = 0$. Therefore, u satisfies the condition for being in $\text{Ker}(T^n)$, which completes the proof. We have shown that if T is a normal operator and $n > 1$, then $\text{Ker}(T) = \text{Ker}(T^n)$. □

Chapter 5

CONCLUSION AND RECOMMENDATIONS

5.1 Introduction

On the basis of the study's stated aims and its findings, we come to conclusions and provide recommendations.

5.2 Conclusion

This study aimed to characterize norm-attainable operators and orthogonal polynomials in the context of $B(H)$, and establish the relationship between them. The research was structured into four chapters, each contributing to the overall objectives of the study. Chapter 1 provided essential definitions and concepts necessary for understanding the subsequent chapters. Chapter 2 encompassed a comprehensive review of the literature on polynomials and norm-attainable operators, setting the foundation for the research. In Chapter 3, the study presented the methodologies, procedures, and tools employed to solve and prove the main results regarding the characterization of orthogonal polynomials and operators in $NA(H)$. In addressing the initial objective, Proposition 4.9 conclusively demonstrated that an operator achieves its norm when applied to weakly reflexive L^2 spaces. Furthermore, the investigation in Lemma 4.4 unveiled a noteworthy result: a self-adjoint contraction operator T attains its norm if and only if $\|T\|$ or $-\|T\|$ resides within its spectrum. Transitioning to the second objective, the study has yielded noteworthy findings as detailed in Theorem 4.19 and Proposition 4.20. Specifically, the investigation has successfully demonstrated that several classical orthogonal polynomials exhibit characteristics of convexity, strict convexity, and norm-attainability within a singular variable framework. Furthermore, Proposition 4.20 extends these insights to the realm of multivariate orthogonal polynomials, specifically addressing the case of $P_n(x, y)$ in $\mathbb{R}^2$. The proposition establishes and elucidates the convexity properties inherent in these multivariate polynomials. In addressing the third objective, a notable discovery, as evidenced by Proposition 4.28, pertains to the identification of a self-adjoint closed differential operator within the space $L^2([0, 1])$. Importantly, this operator is found to be unbounded, thereby establishing its non-norm-attainability as a normal operator.

In summary, this study successfully achieved its objectives by providing insights into the characterization of norm-attainable operators and orthogonal polynomials in $B(H)$. The research outcomes contribute to the theoretical understanding of these topics and have practical implications in various fields, including signal processing and harmonic analysis. Future research can build upon these results to explore further properties and applications of norm-attainable operators and their relationship with orthogonal polynomials.

5.3 Recommendations

The study of orthogonal polynomials is important in both operator theory especially in approximation of polynomial functions in $L^2(X, \mu)$. In this study we analyzed properties of orthogonal polynomials in such $L^2(X, \mu)$ in connection to properties of norm-attainable operators like convexity and continuity. There are several areas that could be further explored in the study on characterizing orthogonal polynomials in norm-attainable classes. Some potential avenues for future research include:

(i). Generalization to different function spaces: The current study focuses on norm-attainability in specific function spaces. Future research could extend the investigation to other function spaces, such as Sobolev spaces or function spaces defined on manifolds. Exploring the norm-attainability of OP in these broader contexts can provide a deeper understanding of the interplay between different function spaces and NAO.

(ii). Analysis of specific norm-attainable operators: The study can delve into specific classes of NAO and examine their relationship with OP. For example, investigating the norm-attainability properties of integral operators, differential operators, or specific classes of operators arising in mathematical physics could shed light on their connection with orthogonal polynomials and reveal additional insights into the underlying mathematical structures.

(iii). Application-driven research: While the study acknowledges the practical significance of norm-attainability and OP, further research could focus on specific applications in different fields. For instance, exploring the use of norm-attainable OP in image processing, data compression, or solving partial differential equations can provide practical methodologies and algorithms that harness the properties of norm-attainable operators and orthogonal polynomials.

(iv). Numerical methods and algorithms: Investigating efficient computational methods and algorithms for computing with NAO and OP could be an interesting direction for future research. This could involve developing robust numerical techniques for approximation, interpolation, or solving inverse problems using norm-attainable orthogonal polynomials, thereby bridging the theoretical results with practical computational tools.

(v). Connections with other mathematical concepts: Exploring the connections between NAO, OP, and other mathematical concepts can open up new research directions. For example, investigating the relationship between norm-attainable operators and orthogonal polynomials with topics like function approximation, harmonic analysis, or operator theory can lead to novel insights and applications in these areas.

These directions can deepen NAO/OP understanding, broaden applications, and spark interdisciplinary collaborations.

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