DECOMPOSABILITY OF POSITIVE MAPS ON POSITIVE SEMIDEFINITE MATRICES

BY

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Declaration

This thesis is my own work and has not been presented for a degree award in any other institution.

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Dedication

To my Mother and my late Father in whom I find the joy of God's love.

Abstract

There has been a vast growth both in mathematical and physical science literature on indecomposable positive maps. However, the questions of decomposability seems to have been ignored. The motivation behind this study was based on the work done by Yang, Leung and Tang in which they enquired if there exist indecomposable 2-positive maps from $\mathcal{M}_3(\mathbb{C})$ to $\mathcal{M}_4(\mathbb{C})$). The objectives of this study were; to construct linear positive maps $\psi_{(\mu,c_1,\ldots,c_{n-1})}$ from $\mathcal{M}_n$ to $\mathcal{M}_{n+1}$ on positive semidefinite matrices; to establish the conditions for the positivity of linear maps $\psi_{(\mu,c_1,\ldots,c_{n-1})}$ from $\mathcal{M}_n$ to $\mathcal{M}_{n+1}$, characterize the structure of the Choi matrices for $2-$positive and complete (co)positive of maps from $\mathcal{M}_n$ to $\mathcal{M}_{n+1}$ on positive semidefinite matrices and; finally to establish the conditions for the decomposability of the linear positive maps $\psi_{(\mu,c_1,\ldots,c_{n-1})}$ on positive semidefinite matrices for $n = 2, 3, 4$. The methodology involved the use of tensor product approaches and matrix inequalities. Choi matrix was used to deduce conditions for complete positivity while Størmer decomposability criteria was used to investigate decomposability by employing the use of Mathematica software for analysis. The decomposability of the maps $\psi_{(\mu,c_1,\ldots,c_{n-1})}$ was described for $n = 2, 3, 4$. A special map $\Psi_{(\mu,c_1,c_2)}$ from $\mathcal{M}_3(\mathbb{C})$ to $\mathcal{M}_2(\mathcal{M}_2(\mathbb{C}))$ where the Choi matrices was visualized as tensor matrix $\mathcal{M}_3 \otimes \mathcal{M}_2$ with $\mathcal{M}_2(\mathbb{C})$ as the entry elements to achieve decomposability via partial transposition. The study has significance addition in mathematics and applications relevant to problems encountered in mathematical science and their related subjects, more specific in quantum information theories. The linear maps $\psi_{(\mu,c_1,\ldots,c_{n-1})}$ are completely positive maps. We believe the mathematical structure of these positive maps $\psi_{(\mu,c_1,\ldots,c_{n-1})}$ are useful in showing entanglement breaking using suitable indecomposable maps. The Choi matrices generated by the linear map $\Psi_{(\mu,c_1,c_2)}$ with block-matrix element transposition as unique addition will elicit new concepts in the study of completely positive matrix operators.

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Chapter 1

INTRODUCTION

1.1 Mathematical background

Positive linear maps on C^*-algebras, mainly those of finite dimensions have been more significantly applied in quantum information theory and quantum channels. Stinespring [84] initiated the concept of completely positive maps with the representation(or dilation) theorem which was later developed by Arveson [2],[1] who found its application in operator theory and then was further developed in the fields of operator algebra and mathematical physics. Choi [17] deduced that for a positive linear map between C^*-algebras, 1-positivity corresponds to 2-positivity if and only if either of them is commutative. However, in the case of matrix algebras Choi [18] showed there was a difference between k-positivity and $k+1$-positivity.

Størmer [85] obtained clear formulas showing decomposable map on the bi-optimal map when the spin factor is irreducible and limited to a square matrix of order 2^{n-1}. Størmer [90] looked at the definite set of a positive map on a C^*-algebra of selfadjoint operators in $\mathcal{M}_n$ such that $\psi(A^2) = \psi(A)^2$, and showed that if the linear map is a separable state, then the image of the definite set is a C^*-subalgebra of the center of the C^*-algebra generated by $\psi(A)$.

The well known result by Choi [15] states that a linear map ψ is completely positive if and only if the Choi matrix is positive semidefinite. Cho et al [10] constructed a family of generalized Choi maps in which they described the conditions under which the Choi map was described in relation with positive semidefinite biquadratic form are decomposable by showing that 2-positivity or 2-copositivity implies decomposability. The Yang et al [106] used this idea of decomposition to obtain a decomposition theorem for k-positive linear maps from $\mathcal{M}_m(\mathbb{C})$ to $\mathcal{M}_n(\mathbb{C})$, where $2 \leq k < min\{m, n\}$ and concluded that all 2-positive linear maps from $\mathcal{M}_m(\mathbb{C})$ to $\mathcal{M}_n(\mathbb{C})$) are decomposable. As a consequence they affirmed answer to Kye's conjecture [55] that every 2-positive linear map is decomposable for $n = m = 3$.

Woronowicz [105] provided an example of indecomposable map giving an indirect proof of positive linear maps from $\mathcal{M}_2(\mathbb{C})$ to $\mathcal{M}_4(\mathbb{C})$ in a similar manner to the example given by Choi in [16]. Osaka [73] provided stronger indecomposable maps in $\mathcal{M}_n(\mathbb{C})$ by classifying a series of positive linear maps for $n = 4, 5$ with respect to the degree of indecomposability. Connecting the relationship between positive maps and quantum states, Terhal [98] developed a method to create a family of indecomposable positive linear maps on matrix algebras $\mathcal{M}_n(\mathbb{C})$ for any $n > 2$. Then constructed indecomposable maps with the notion of an unextendible product basis with examples in arbitrary high dimensions.

By exposedness of positive linear map, Ha [30] constructed and described an exposed indecomposable positive linear map and showed that the extreme points of the dual face in separable state are parameterized by the Riemann sphere. Gale [24] showed that circles parallel to the equator in the Riemann sphere behave exactly the same way as the trigonometric moment curve . Li and Wu [61] studied the positive linear maps on matrix algebras [30] and gave the conditions under which these maps are completely positive, atomic, and decomposable. Li and Wu [61] extended the idea fronted by Hou [41] to show that Choi's matrix is decomposable as the sums of completely positive map and positive partial transpose map.

Eom and Kye [23], used duality as a method to examine decomposition between the spaces of bounded linear operators from $\mathcal{M}_n$ into $\mathcal{M}_m$ with the projective tensor product after Woronowicz [105] showed every positive linear map that maps the matrix $\mathcal{M}_2$ to $\mathcal{M}_n$ can be expressed as a combination of completely positive map and a completely copositive linear map provided $n \leq 3$. Størmer [91] used duality technique to show extension of positive linear maps. Ha [27] used the duality concept [23] to show that the examples given by Choi [16], Choi and Lam [14], Kim and Kye [47] and Robertson [79] are neither 2-positive nor 2-copositive but are atomic maps because they could not be expressed as a combination of 2-positive and 2-copositive linear maps.

Marciniak and Rutkowski [69] provide a scheme for constructing examples of positive maps by the merging procedure. Further, they provided necessary and sufficient conditions for such merging to attain 2-positivity and either complete positivity or indecomposability with the claim that the canonical merging is a composition of completely positive and completely copositive maps. Majewski and Marciniak in [67], using an extremal unital positive map ψ from $\mathcal{M}_2(\mathbb{C})$ to $\mathcal{M}_2(\mathbb{C})$ constructed positive linear maps ψ_1 and ψ_2 that were not necessarily unital as positive and copositive(respectively). The decompositions of the linear map ψ showed that these maps have unique decomposition.

1.2 Basic concepts

Here we gave some basic concepts in matrix algebra that made our study complete. These were important concepts used to described the connection linking linear positive maps and tensor products.

Definition 1.1. *[6] Let $\mathcal{A}$ be a C^*-algebra. A linear functional $\psi : \mathcal{A} \longrightarrow \mathbb{C}$ is said to be positive if $\psi(A) \geq 0$ whenever $A \in \mathcal{A}$ and $A \geq 0$.*

Remark 1.2. Whenever A is selfadjoint, then $\psi(A) \in \mathbb{R}$. Moreover $\psi(A^*) = \psi(A)$ for all $A \in \mathcal{A}$.

Definition 1.3. *[70] Let ψ be a linear map from $\mathcal{A}$ to $\mathcal{B}$, then ψ is called a *-homomorphism, if $\psi(xy) = \psi(x)\psi(y)$ for every $x, y \in \mathcal{A}$. ψ is a $*$-anti-homomorphism is it reverses the order of operation. That is, $\psi(xy) = \psi(y)\psi(x)$*

Example 1.4. [6] Let $\psi : \mathcal{A} \longrightarrow \mathcal{H}$ be a *-homomorphism. For each $x \in \mathcal{H}$ define $\psi_x : A \longrightarrow \mathbb{C}$ by $\psi_x(A) = \langle Ax, x\rangle$ for all $A \in \mathcal{A}$. In addition if $A \in \mathcal{A}$ is positive then $A = B^*B$ for some $B \in \mathcal{A}$. so

$$\psi_x(A) = \langle (B^*B)x, x\rangle = \langle (B)x, (B)x\rangle = \|(B)x\| \geq 0$$

Definition 1.5. *[6] A linear map ψ is from $\mathcal{M}_n(\mathbb{C})$ to $\mathcal{M}_m(\mathbb{C})$ is positive if $\psi(\mathcal{M}_n(\mathbb{C}))^+ \subseteq \mathcal{M}_m(\mathbb{C})^+$. ψ is strictly positive(positive definite) if $\psi(\mathcal{M}_n(\mathbb{C})) > 0$.*

Remark 1.6. A linear map ψ acting on $\mathcal{M}_n$ is Hermiticity-preserving if $\psi(A) \in \mathcal{M}_m$ is Hermitian whenever A is Hermitian, and we denote such set of maps by $\mathbf{B}(\mathcal{M}_n, \mathcal{M}_m)$. By $\psi(A) \geq 0$ we mean, $\psi(A)$ is positive when $v^*\psi(A)v \geq 0$ for all $v \in \mathbb{R}^n$. All the eigenvalues of $\psi(A)$ are non-negative by the spectral decomposition of $\psi(A)$ or all principal submatrices of $\psi(A)$ have non-negative determinants.

Definition 1.7. *[60] A map ψ from $\mathcal{M}_n(\mathbb{C})$ to $\mathcal{M}_m(\mathbb{C})$ is* $k-$positive *if $\mathcal{I}_k \otimes \psi : \mathcal{M}_k \otimes \mathcal{M}_n \longrightarrow \mathcal{M}_k \otimes \mathcal{M}_m$ is positive.*

The Hilbert space $\mathcal{H}^{\otimes n}$ is the tensor product of n Hilbert spaces of its subsystems: $\mathcal{H}^{\otimes n} = \otimes_{i=1}^{n} \mathcal{H}_i$. for instance $\mathcal{H}^{\otimes 2} = \mathcal{H} \otimes \mathcal{H}$. The action is performed on tensor product of matrices and is is defined for finite dimensional matrices.

Let $A \in M_{nm}(\mathbb{F})$ and $B \in M_{lk}(\mathbb{F})$. The Kronecker product [42] $A \otimes B \in M_{(nl)(mk)}(\mathbb{F})$

of A is the matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}$$

where a_{ij} denotes the ij-th entry of $A \in M_{nm}(\mathbb{F})$ and $B \in M_{lk}(\mathbb{F})$. The Kronecker product $A \otimes B$ does not entail a restriction on either the size of A or the size of B as in matrix multiplication.

Example 1.8. Let A be 2×2 matrix and I be 3×3 the identity matrix, then

$$(A \otimes I) \quad = \quad \left(\begin{array}{ccc|ccc} a_{11} & 0 & 0 & a_{12} & 0 & 0 \\ 0 & a_{11} & 0 & 0 & a_{12} & 0 \\ 0 & 0 & a_{11} & 0 & 0 & a_{12} \\ \hline a_{21} & 0 & 0 & a_{22} & 0 & 0 \\ 0 & a_{21} & 0 & 0 & a_{22} & 0 \\ 0 & 0 & a_{21} & 0 & 0 & a_{22} \end{array}\right).$$

On the other hand,

$$(I \otimes A) \quad = \quad \left(\begin{array}{cc|cc|cc} a_{11} & a_{12} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & a_{11} & a_{12} & 0 & 0 \\ 0 & 0 & a_{21} & a_{22} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & 0 & a_{21} & a_{22} \end{array}\right).$$

The direct sums of a finite number of $n \times n-$dimentional matrices transforms to a diagonal block matrix. This has been of great importance in the discussion on the structure and properties of our positive maps. In describing the positive maps that preserve traces and their norm, the direct sum of matrices is used to determine the eigenvalues and the determinants which is important in developing ideas on algebraic and metric properties of positive maps.

Definition 1.9. *[42] Let $A \in \mathbb{F}^{n\times m}$ and $B \in \mathbb{F}^{l\times k}$. Then, the Kronecker sum $A \oplus B \in \mathbb{F}^{nl\otimes mk}$ of A and B is*

$$A \oplus B := A \otimes I_m + I_n \otimes B.$$

On the direct sum space, the matrices A and B acts on the vectors, so that $v \mapsto Av$ and $w \mapsto Bw$. This matrix is expressed as $A \oplus B$ by lining up all matrix elements in a block-diagonal form,

$$A \oplus B \quad = \quad \left(\begin{array}{c|c} A & 0_{n\times m} \\ \hline 0_{m\times n} & B \end{array}\right)$$

Example 1.10. Let A_2 and B_3 be square matrices, then

$$A_2 \oplus B_3 = \left(\begin{array}{cc|ccc} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ \hline 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & b_{31} & b_{32} & b_{33} \end{array}\right)$$

For matrices A, B, C, D and vectors v, w the direct sum satisfies the following relations:

$$\begin{aligned}(A\oplus B)(v\oplus w) &= (Av\oplus Bw).\\ (A\oplus B)(C\oplus D) &= (AC\oplus BD).\\ \det(A\oplus B) &= (\det A)(\det B).\\ Tr(A\oplus B) &= Tr(A)+Tr(B).\end{aligned}$$

Definition 1.11. *[60] A linear map ψ from $\mathcal{M}_n(\mathbb{C})$ to $\mathcal{M}_m(\mathbb{C})$ is $k-$*copositive *if the map $\tau_k \otimes \psi : \mathcal{M}_k \otimes \mathcal{M}_n \longrightarrow \mathcal{M}_k \otimes \mathcal{M}_n$ is positive. The linear map $\mathcal{I}_k \otimes \psi$ is said to be completely positive when it is k-positive for every $k \in \mathbb{N}$.*

Remark 1.12. The linear mapping $\psi : \mathcal{M}_n \longrightarrow \mathcal{M}_m$ is called 2-positive if

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0 \Rightarrow \begin{pmatrix} \psi(A) & \psi(B) \\ \psi(B^*) & \psi(C) \end{pmatrix} \geq 0,$$

where $A, B, C, D \in \mathcal{M}_n$ are matrices with the same dimensions. We note that $\psi(A), \psi(B), \psi(B^*)$ and $\psi(C)$ are also positive maps in their own respect.

For convenience we express k-positivity in a block matrix notation.

Example 1.13. A family of Choi maps in $\mathcal{M}_n$, $\psi(A) = Tr(A)\mathcal{I}_{n^2} - \frac{1}{2}A$. ψ is $k-$positive and ψ is completely positive . Now, let $n = 2$, this gives,

$$\begin{aligned}\psi\left(\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}\right) &= 2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}\\ &= \begin{pmatrix} \frac{3}{2} & 0 & 0 & \frac{3}{2} \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ \frac{3}{2} & 0 & 0 & \frac{3}{2} \end{pmatrix}.\end{aligned}$$

Since the determinants of all the principal submatrix are positive, ψ is $2-$positive, thus ψ is completely positive.

Example 1.14. Define $\psi = \frac{1}{2}Tr(A)\mathcal{I}_{n^2} - A$. Then ψ is positive but $\psi(A)$ is not $2-$positive. This is the map in Example 1.13 when the coefficients are changed. The map

$$\psi(A) = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

shows that A is positive but $\psi(A)$ is not since the determinant of the principal submatrix, $\begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} = -1$.

Remark 1.15. Note that, if ψ is completely positive it implies ψ is positive and is hermitian-preserving. A map $\psi : \mathcal{M}_n(\mathbb{C}) \longrightarrow \mathcal{M}_m(\mathbb{C})$ is completely positive if and only if its Choi matrix C_ψ is positive semidefinite. Similarly, the map is completely copositive if and only if the partial transpose of its Choi matrix is positive semidefinite. It is important to note that not every positive map is completely positive, this makes completely positive maps very specific. We will show this in the next example but first let us define partial transposition.

Definition 1.16. *[17] Let ψ be a linear map from $\mathcal{A}$ to $\mathcal{B}$, then ψ is Schwartz map, if $\psi(A^*A) = \psi(A^*)\psi(A)$ for every $A \in \mathcal{A}$.*

Definition 1.17. *[80] Let ψ be a linear map from $\mathcal{A}$ to $\mathcal{B}$, then ψ is Jordan homomorphism, if $\psi(A^n) = \psi(A)^n$ for every $A \in \mathcal{A}$ and $n \in \mathbb{N}$.*

Definition 1.18. *[42] Given a square matrix $A \otimes B$, its partial transpose with respect to the first component is $A^T \otimes B$. Similarly, its partial transpose with respect to the second component is $A \otimes B^T$.*

Remark 1.19. The partial transpose map is the matrix transpose to one half of the tensor product $\mathcal{M}_n \otimes \mathcal{M}_m$. This transposition of the linear maps $\mathcal{I}_n \otimes \tau$, $\tau \otimes \mathcal{I}_n$ and $\tau \otimes \psi$) acting on $\mathcal{M}_n \otimes \mathcal{M}_m$ are such that,

$$(\mathcal{I}_n \otimes \tau)(A \otimes B) = A \otimes B^T,$$
$$(\tau \otimes \mathcal{I}_n)(A \otimes B) = A^T \otimes B,$$
$$(\tau \otimes \psi)(A \otimes B) = A^T \otimes \psi(B).$$

If A and B are positive semidefinite matrices in $\mathcal{M}_n$ and $\mathcal{M}_m$ respectively. Then $A \otimes B^T$, $A^T \otimes B$ and $A^T \otimes \psi(B)$ are also positive semidefinite. By $(A \otimes B)^\Gamma$ we denote the partial transpose of $A \otimes B$ with respect component B.

Example 1.20. Let $\psi : \mathcal{M}_3 \longrightarrow \mathcal{M}_3$ be a positive map. Let $A \in \mathcal{M}_3$ and define $\psi : \mathcal{M}_3 \longrightarrow \mathcal{M}_3$ by $A \longrightarrow A^\Gamma$. Since the determinant of all the minors of A are nonnegative, ψ is positive. That is,
Let $[A_{ij}]$ be the block matrix,

$$[A_{ij}] = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \tag{1.2.1}$$

It is clear that the determinants of every principal submatrix of A is nonnegative therefore A is positive.

On the other hand , for the case of A^Γ,

$$\psi(A) = [A_{ij}]^\Gamma = \begin{pmatrix} A_{11} & A_{12}^T \\ A_{21}^T & A_{22} \end{pmatrix} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

The determinant of the principal submatrix $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ is -1. So $\psi(A) = [A_{ij}]^\Gamma$ is not positive. This implies that the linear map ψ is not $2-$positive. As a result it is not a completely positive map. Therefore, not all positive maps are completely positive.

Remark 1.21. By the Choi-Kraus [15] Theorem 2.5, for all $V_j \in \mathbb{C}^{n\times k}$, the map $\psi : \mathcal{M}_n \longrightarrow \mathcal{M}_m$ given by, $\psi(A) = \sum_{j=1}^{nm} = V_j^* A V_j$ is completely positive.

Definition 1.22. *[86] Let $\psi : \mathcal{M}_n \longrightarrow \mathcal{M}_m$ be a linear map and let (E_{ij}) with $i, j = 1, \dots, n$ be a complete set of matrix units for $\mathcal{M}_n$. The Choi matrix for ψ is defined by the operator;*

$$C_\psi = (\mathcal{I}_k \otimes \psi)(\sum_{ij} E_{ij} \otimes E_{ij}) = \sum_{ij} E_{ij} \otimes \psi(E_{ij}) \in \mathbb{C}^{nm\times nm}.$$

Remark 1.23. The map $\psi \mapsto C_\psi$ is linear, injective and is surjective, the Choi matrix depends on the choice of matrix units (E_{ij}). This map is called the Jamiolkowski isomorphism [86]. The Choi-Jamiolkowski isomorphism is a one-to-one correspondence between completely positive maps ψ acting on the operators $\mathcal{M}_n$ in a Hilbert space with dimension n and positive operators C_ψ. The Choi matrix is represented by a block matrix $[\psi(E_{ij})]$. The linear ψ is completely positive if and only if the block matrix is positive. If the block matrix is $[\psi(E_{ij})]$ negative. Then ψ is not completely positive.

Example 1.24. [54] Let $\psi(A) : \mathcal{M}_3 \longrightarrow \mathcal{M}_3$ be a positive map defined by

$$\psi(A) = \begin{pmatrix} a_1x_{11} + b_1x_{22} + c_1x_{33} & -x_{12} & -x_{13} \\ -x_{21} & c_2x_{11} + a_2x_{22} + b_2x_{33} & -x_{23} \\ -x_{31} & -x_{32} & b_3x_{11} + c_3x_{22} + a_3x_{33} \end{pmatrix}$$

where $a_k, b_k, c_k \in \mathbb{R}$. The map $\psi(A)$ is completely positive if and only if $a_k, b_k, c_k \geq 1$, for $k = 1, 2, 3$. Computing, the Choi matrix of ψ we have that,

$$C_\psi = \left(\begin{array}{ccc|ccc|ccc} a_1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & b_1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & c_3 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_2 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & a_3 \end{array}\right)$$

Since a_k, b_k, c_k are nonnegative. Then all principal submatrices of C_ψ are nonnegative, C_ψ is positive definite. Hence $\psi(A)$ is completely positive.

Definition 1.25. *([12]) A positive linear map $\psi : \mathcal{M}_n \longrightarrow \mathcal{M}_m$, is decomposable if it can be expressed as a sum of a completely positive map $\psi_1 : \mathcal{M}_n \longrightarrow \mathcal{M}_m$ and a completely copositive map $\psi_2 : \mathcal{M}_n \longrightarrow \mathcal{M}_m$. Otherwise ψ is said to be indecomposable. A linear map ψ is k-decomposable if there are maps ψ_1, ψ_2, such that ψ_1 is k-positive,*

ψ_1 *is k-copositive.* ψ *is said to be atomic if it is not* $(2,2)$*-decomposable.*

Remark 1.26. If $\psi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_m(\mathbb{C})$ there exists $n \times m$ matrices V_j and Q_j such that

$$C_\psi = \sum_j V_j^* \psi(A) V_j + \sum_j Q_j^* \psi(B^T) Q_j$$

where $V_1, V_2, \ldots, Q_1, Q_2, \ldots \in \mathcal{M}_{nm}(\mathbb{C})$ and $A, B \in \mathcal{M}_n$ with B^T a transpose in $\mathcal{M}_n$. $C_{\psi_1} = \sum_j V_j^* \psi(A) V_j$ and $C_{\psi_2} = \sum_j Q_j^* \psi(B^T) Q_j$ as $k-$positive, $k-$copositive respectively.

1.3 Statement of the problem

Despite of the fact that positive maps are essential ingredient in the description of quantum systems, characterization of the structure of the set of all positive maps has been a long standing challenge. Stinespring ([84], Theorem 1)introduced the concept of completely positive maps with the representation theorem from which many theories of completely positive maps have been advanced in last sixty years but still remains an open area for mathematical physicists and operator algebraists due to its application in quantum information. The main reason behind this is the complex nature of completely positive maps on matrix algebra is not quite clearly understood. Several authors have given immense attention in studying decomposition of positive maps with more emphasis given to the study of indecomposable maps by many authors since Choi [16] introduced the study of these maps by giving examples of indecomposable maps satisfying some special properties. To date there is a call in the construction of positive maps that are indecomposable due to their applications in mathematical physics. In quantum information, there is high interest in positive maps from $\mathcal{M}_3(\mathbb{C})$ to $\mathcal{M}_4(\mathbb{C})$ with emphasis in indecomposable maps. Yang et al[106] posed the question, "Does there exist a 2-positive but indecomposable map in $B(\mathcal{M}_3(\mathbb{C}), \mathcal{M}_4(\mathbb{C}))$"? This is the interesting question which formed the basis of this study.

1.4 Objectives of the study

The main objective for this study is to describe decomposability of positive maps on positive semidefinite matrices. The specific objectives of this study are to:

(i). Construct linear positive maps $\psi_{(\mu,c_1,\ldots,c_{n-1})}$ from $\mathcal{M}_n$ to $\mathcal{M}_{n+1}$ on positive semidefinite matrices.

(ii). Establish the conditions for the positivity of linear maps $\psi_{(\mu,c_1,\ldots,c_{n-1})}$ from $\mathcal{M}_n$ to $\mathcal{M}_{n+1}$.

(iii). Characterize the structure of the Choi matrices for $2-$positive and complete (co)positive maps from $\mathcal{M}_n$ to $\mathcal{M}_{n+1}$ on positive semidefinite matrices.

(iv). Establish the conditions for the decomposability of linear positive maps $\psi_{(\mu,c_1,\ldots,c_{n-1})}$ on positive semidefinite matrices for $n = 2, 3, 4$.

1.5 Significance of the study

Positive maps perform a vital part both in mathematics and mathematical physics. The Peres-Horodecki theorem [40], [75] gave a criterion for detecting an entanglement of quantum states. If the partial transpose of a mixed state has negative eigenvalues, then the state is entangled. However, this is not generalized for entanglement of states as some have positive eigenvalues. Note that it is possible in $n \times m(n, m \geq 3)$ [40] dimension for the quantum state to be entangled even if the eigenvalues of the partial transpose are all positive. Historically the link between separability and positive maps with Peres-Horodecki theorem was first expressed in [15], [73] and [105].

A quantum state is a vector that encodes a state and convey the information in a system just as the vectors $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{C}$ that was looked at in this study. A state ρ (density matrix) is separable if and only if $(I_A \otimes \psi)(\rho)$ is positive for any positive map ψ which sends positive operators on $\mathcal{A}$ into positive operators on $\mathcal{B}$. A positive map ψ is decomposable. Otherwise, ψ is indecomposable. Decomposable maps can not detect Positive Partial Transpose entangled density operators. Indecomposable maps should detect at least one Positive Partial Transpose entangled density operator. The quantum channel(completely positive map) is positive partial transpose provided it's Choi matrix is positive partial transpose.

The study has a significant addition of mathematical knowledge and applications relevant to problems encountered in mathematical science and their related subjects, more specific in quantum information theories. The linear maps $\psi_{(\mu,c_1,\ldots,c_{n-1})}$ are completely positive maps. We believe the mathematical structure of these positive maps $\psi_{(\mu,c_1,\ldots,c_{n-1})}$ are useful in showing entanglement breaking using suitable indecomposable maps. The Choi matrices generated by the linear map $\Psi_{(\mu,c_1,c_2)}$ with block-matrix element transposition as unique addition will elicit new concepts in the study of completely positive matrix operators.

Chapter 2

LITERATURE REVIEW

2.1 Introduction

In this chapter, we highlighted related literature which are important in our study. A great number of studies have been carried on Positive maps on Hilbert spaces in general and on decomposition of positive maps in specific.

2.2 Positive and Completely positive maps

Positivity of matrices is a very useful and interesting property of positive maps. However, their characterization have been elusive and for various reasons especially positivity of the special class of completely positive linear maps. The concepts of positive linear map on C^*-algebras can be traced to middle 1950s with generalization of Kadison′s [45], [44] Schwartz inequality $\psi(A)^2 \leq \psi(A^2)$ for a unital positive map ψ, where A is a Hermitian matrix with characterizations of isometries of C^*- algebras. Choi [17] investigated 2-positive linear maps with special attention to completely positive linear maps an stated that, if ψ is a 2-positive map, then $\psi(A^*A) \geq \psi(A^*)\psi(A)$ for all $A \in \mathcal{A}$ (Corollary 2.8) implying every unital 2-positive map is a Schwartz map, however the converse is false . Choi [17] concluded that every 2-positive linear map is locally completely positive. That is, if ψ is 2-positive from $\mathcal{A}$ to $\mathcal{B}$, then for any $x \in \mathcal{B}$ the underlying space of $\mathcal{B}$, there is a completely positive linear map $\psi : \mathcal{A} \longrightarrow \mathcal{B}$ with $\|\psi_x\| = 1$, such that $\psi(A)x = \psi_x(A)x$ for $A \in \mathcal{A}$.

Woronowicz [105] defined a strong Kadison inequality with the assertion that $\psi(c) \geq \psi(b)^*\psi(b)$, whenever $b, c \in \mathcal{A}$ then b and b^* commute in the inequality $c \geq bb^*$ for every normalized positive map. Kirchberg [48] later showed that this was false. Tang gave a counterexample (in [97] Example 4) using a unital positive linear map showed that Woronowicz conjecture was not valid for a map from $\mathcal{M}_4$ to $\mathcal{M}_2$.

Russo [81] noted that for a unital C^*-algebra $\mathcal{A}$ is mapped by ψ a unital self-adjoint linear map into B, $\psi(A)$ is invertible for every invertible $\mathcal{A}$. Positive maps are self-adjoint maps with nonnegative spectrum because ψ is positive and consequently ψ is a Jordan homomorphism [80] where A a von Neumann algebra. However, Choi et al [11] gave a counterexample involving Toeplitz operators and showed that if the unital positive maps are invertibility preserving they are *-homomorphisms. The map is a Jordan homomorphism on condition that the range of ψ is a C^*-algebra.

Stinespring [84] introduced the concept of completely positive maps with the representation theorem from which many theories of completely positive maps has been advanced in last sixty years but still remains an open area for mathematical physicists and operator algebraists due to its application in quantum information.

Theorem 2.1. *([84], Stinespring Theorem.)*
Let $\mathcal{A}$ be a C^*-algebra with a unit, let $\mathcal{H}$ be a Hilbert space, and let ψ be a linear function from $\mathcal{A}$ to operators on $\mathcal{H}$. Then a necessary and sufficient condition that ψ have the form

$$\psi(a) = V^*\pi(a)V$$

for all $a \in \mathcal{A}$, where V is a bounded linear transformation from $\mathcal{H}$ to a Hilbert space $\mathcal{K}$ and ψ is a *-representation of $\mathcal{A}$ into operators on $\mathcal{K}$, is that ψ be completely positive.

On the other hand Arveson [2] stated the theorem that follows;

Theorem 2.2. *([2], Averson Theorem.)*
Let $\psi(\mathcal{A}, \mathcal{B})$ be a completely positive map and ϕ in $[0, \phi]$ a completely positive map such that $\psi \geq \phi$. The map $\psi \longrightarrow \psi_\phi$ is an affine order isomorphism of the partially ordered convex set of operators $\{\psi \in \pi(a) : 0 \leq \psi \leq \mathcal{I}_\mathcal{B}\}$ onto $[0, \phi]$ where $\psi(a) = V^*\pi(a)V$ and $\psi_\phi(a) = V^*\pi(a)\phi V$ for all $a \in \mathcal{A}$.

Stinespring ([84] characterized completely positive maps from $\mathcal{A}$ as a $*$-homomorphisms into $\mathcal{B}$. That is A linear map ψ from $\mathcal{A}$ to $\mathcal{B}$ is k-positive for every positive map $\psi \otimes \mathcal{I}_k$ from $\mathcal{A} \otimes \mathcal{M}_k$ to $\mathcal{B} \otimes \mathcal{M}_k$ is positive. ψ is completely if and only if it is k-positive for all $k = 1, 2, 3, \ldots$. The Stinespring Theorem commonly known as the Stinespring Dilation Theorem is generally stated as

Theorem 2.3. *([106], Stinespring's dilation theorem. Theorem 1.1.2)*
Let $\mathcal{A}$ be a unital C^*-algebra and $\psi : \mathcal{A} \longrightarrow \mathbf{B}(\mathcal{H})$.

(i). ψ is completely positive if and only if there exist a Hilbert space $\mathcal{K}$, a bounded linear operator $V : \mathcal{H} \longrightarrow \mathcal{K}$ and a $*-$homomorphism $\psi : \mathcal{A} \longrightarrow \mathbf{B}(\mathcal{K})$ such that $\psi(a) = V^*\pi(a)V$ for all $a \in \mathcal{A}$. Furthermore, $||V||^2 = ||\psi(1)||$.

(ii). ψ is completely copositive if and only if there exist a Hilbert space $\mathcal{K}$, a bounded linear operator $V : \mathcal{H} \to \mathcal{K}$ and an $*-$anti-homomorphism $\psi : \mathcal{A} \longrightarrow \mathbf{B}(\mathcal{K})$ such that $\psi(a) = V^*\pi(a)V$ for all $a \in \mathcal{A}$.

In addition, Averson [2] showed that for a $\mathcal{S}$ a norm-closed self-adjoint subspace of $\mathcal{A}$ containing the identity $\mathcal{I}_\mathcal{A}$ in $\mathcal{A}$, where $\mathcal{A}$ is a unital C*-algebra, each completely positive map from subspace $\mathcal{S}$ to a C^*-algebra $\mathcal{B}$ can be extended to a completely positive map from $\mathcal{A}$ to $\mathcal{B}$. The concept of complete positivity and positivity of linear maps correspond when the C^*-algebra is commutative. For every commutative C^*-algebra $\mathcal{A}$ and ψ is a positive operator-valued linear function on $\mathcal{A}$,ψ is completely positive map [84]. Stømer [91] also deduced that if either $\mathcal{A}$ or $\mathcal{B}$ is Abelian, then every positive map ψ from $\mathcal{A}$ to $\mathcal{B}$ is completely positive (or completely copositive).

Choi [15] observed that for a linear map ψ to attain complete positivity it has to be n-positive. It was enough to prove complete positivity by showing positivity of $\mathcal{I}_n \otimes \psi$ on a single element.

Theorem 2.4. *([42], Choi Theorem)*
Let $\psi : \mathcal{M}_n(\mathbb{C}) \longrightarrow \mathcal{M}_m(\mathbb{C})$ be a linear map. Then following are equivalent;

(i). ψ is $n-$positive.

(ii). the Choi matrix C_ψ is positive.

(iii). ψ is completely positive.

Choi [15] realized that a map ψ is completely positive provided it is k-positive where $k = \min\{n, m\}$. From this Choi characterized that;

$$\mathcal{P}_1(\mathcal{M}_n, \mathcal{M}_m) \supset \mathcal{P}_2(\mathcal{M}_n, \mathcal{M}_m) \supset \ldots \supset \mathcal{P}_k(\mathcal{M}_n, \mathcal{M}_m) = \mathcal{CP}(\mathcal{M}_n, \mathcal{M}_m)$$

where $\mathcal{P}_k$ and $\mathcal{CP}$ denote a k-positive map and completely positive map respectively. Choi [15] used this idea to describe the difference between k-positivity and $(k+1)$-positivity using the map $\psi_a A = aITr(A) - A$ which is k-positive but was not $(k+1)$-positive. Choi went further and noted that for a unital completely positive maps originating from a convex sets containing extreme points, where the extreme points are those ψ for the linearly independent set $\{V_i V_i^* : 1 \leq i \leq nk\}$. The map ψ is congruence on condition that ψ is given by $\psi(A) = V^* A V$ for all $A \in \mathcal{M}_n$, with V being an $n \times k$ matrix. Though it was conjectured that extreme rays of positive maps from $\mathcal{M}_n$ to $\mathcal{M}_m$ are all congruence maps, choi [15] established by a counterexample in biquadratic forms to disapprove this conjecture.

The Choi-Kraus theorem [42], [16] commonly known as the Choi's Theorem has been a pilar in the classification of completely positive linear maps. Choi [18] described a clear difference between positive and completely positive linear maps on C^*-algebras operators. Using an example Choi [18] constructed $(n-1)$-positive maps. The linear positive $(n-1)\mathcal{I}_{n^2} - (E_{ij})_{i\leq i,j\leq n}$ was shown to be $(n-1)$-positive but failed to be n-positive. Using The Stinespring's Dilation Theorem [106] , Kraus [53] showed that, for every completely positive that is trace preserving bounded linear map ψ from $\mathcal{M}_n$ to $\mathcal{M}_n$, there is a unital normal and completely positive bounded linear map $\bar{\psi}$ from $\mathcal{M}_n$ to $\mathcal{M}_n$ satisfying the relation $Tr(\psi(AB)) = Tr(A\bar{\psi}(B))$ for all $A, B \in \mathcal{M}_n$ where $\mathcal{M}_n$ are trace preserving maps with $\bar{\psi}$ as the dual map of ψ. Choi conquered with Kraus and stated for completely positive linear maps by the theorem below.

Theorem 2.5. *([6], Choi-Kraus Theorem) Let $\psi : \mathcal{M}_n \longrightarrow \mathcal{M}_m$ be a completely positive linear map. Then there exist $V_j \in \mathbb{C}^{n\times m}, 1 \leq j \leq nm$, such that*

$$\psi(A) = \sum_{j=1}^{nm} V_j^* A V_j.$$

The Kraus representation $\psi(A) = \sum_{j=1}^{l} V_j^* A V_j$ of ψ is not unique since the expression $[\psi(E_{jk})] = \sum V_i^* E_{jk} V_i$ is not unique, so $\{V_i\}$ is not uniquely determined. This additional condition on $\{V_i\}_{i=1}^{l}$ ensures that $\psi(A) = \sum_{j=1}^{l} V_j^* A V_j$ is a canonical expression for ψ. Hoyer [42], classified density matrices as trace preserving positive maps and realized that; For all $V_j \in \mathbb{C}^{n\times m}$, the map $\mathcal{M}_n$ to $\mathcal{M}_m$ define by $\psi(A) = \sum_{j=1}^{nm} V_j^* A V_j$ is trace-preserving if $\sum_{j=1}^{n} V_j^* V_j = I_n$, where I_n is the identity matrix on $\mathcal{M}_n$ and ψ is trace preserving.

Skowronek and Størmer [82] examined the norms of positive maps between two bounded Hilbert spaces $\mathcal{K}$ and $\mathcal{H}$ by constructing a linear map $TrA - \lambda\psi$ on $\mathbf{B}(\mathcal{K})$ for a completely positive map ψ of $\mathcal{K}$ into $\mathcal{H}$ and $\lambda > 0$. Størmer in [87] showed that each of these positive map is a positive scalar multiple of the map when $\lambda = 1$ for all positive maps. Further, Størmer used these maps was to show that some maps are k-positive but are not $(k+1)$-positive. On the other hand, Tomiyama [100] gave the an answer to question raised by Choi in [15] by describing positive maps for which n-positivity correspond with $(k+1)$-positivity. Tomiyama [100] showed that, for each n-positive map form $\mathcal{A}$ to $\mathcal{B}$ is $k+1$-positive provided either $\mathcal{A}$ or $\mathcal{B}$ has irreducible representations whose dimensions less than n and every n-positive map is automatically completely positive. Takasaki and Tomiyama [94] investigated geometric relations by examining three distinct positive maps in the M_n matrix algebra with a look at their the transpose map and the completely positive map and concluded that;

Theorem 2.6. *([94]) For a nonnegative number α with $0 < \alpha <_1$, the map $\alpha\psi + (1-\alpha)\tau(n)$ is $(n-1)$-positive but not n-positive when $\frac{1}{2} \leq \alpha \leq 1$, completely positive when $\frac{1}{n} \leq \alpha \leq \frac{1}{2}$ and it is positive when $0 \leq \alpha \leq \frac{1}{n}$.*

For all rank one orthogonal projections $W \in \mathcal{M}_k(\mathcal{M}_n)$, where $(\mathcal{I}_k \otimes \psi)(W) \geq 0$. Equivalently, for all orthonormal vetors $\vec{x} = (x_i, ..., x_k) \in \mathbb{C}^n$, the operator matrix $[\psi(x_i x_j)]_{1 \leq i,j \leq k} \geq 0$ imply that ψ is k-positive [41]. That is,

Theorem 2.7. *([41]) Let suppose $\psi : \mathcal{M}_n(\mathbb{C}) \longrightarrow \mathcal{M}_m(\mathbb{C})$ is a linear map continuous under strong operator topology. The following are equivalent.*

(i). ψ is k-positive, i.e., $I_k \otimes \psi$ is positive.

(ii). $(\mathcal{I}_k \otimes \psi)(W) \geq 0$ for all rank one othogonal projections $W \in \mathcal{M}_k(\mathcal{M}_n)$.

(iii). For all orthonormal subset $x = (x_1, \ldots, x_k) \in \mathbb{C}^n$, the operator matrix defined by $\psi(X) = [\psi(x_i x_j)]_{1 \leq i,j \leq k}$ is positive semi-definite.

The structure of completely positive maps can be understood through the Kraus representation. Moreover Choi's theorem provides a technique to clearly visualize their structures well but as one gets deeper into the study of positive maps, the situation becomes much less clear due to the lack of a complete structural representation of these maps.

2.3 Decomposable and Indecomposable Positive maps

Choi [12] introduced the concept of decomposable positive maps with the quest to answer the question by Hilbert, 1888 that asked '*Which positive semidefinite polynomial can be written as a sum of two squares?*' That is, if all positive semidefinite homogeneous polynomials can be stated as a sum of squares of homogeneous polynomials. Choi [16] realized that there existed a positive semidefinite biquadratic form which could not be decomposed as the sum of squares of bilinear forms (i.e completely positive and completely copositive maps) by giving concrete counterexample for the case of map from $\mathcal{M}_3$ to $\mathcal{M}_3$. In [12] and [15] Choi defined structures of completely positive linear maps between complex matrix algebras for decomposition of positive maps. One of the basic problems about the structure of the set positive maps is whether they could be decomposed as an algebraic sum of some simpler classes of positive maps. Two sets of positive maps were then considered; the class of completely positive maps and the class of completely copositive maps. That is, a positive linear map ψ from $\mathcal{M}_n(\mathbb{C}$ to $\mathcal{M}_n(\mathbb{C})$ is decomposable if expressed as the sum of completely positive map ψ_1 and completely copositive map ψ_2, where ψ_1 is $k-$positive and ψ_2 $k-$copositive for all $k \in \mathbb{N}$, respectively [50].

The first example of an indecomposable positive linear map was constructed by Choi [16]. In particular, the map ψ from $\mathcal{M}_3$ to$\mathcal{M}_3$ defined by

$$x_{ij} \longrightarrow \begin{pmatrix} x_{11}+x_{22} & -x_{12} & -x_{33} \\ -x_{21} & x_{22}+x_{33} & -x_{23} \\ -x_{31} & -x_{32} & x_{33}+x_{11} \end{pmatrix} \tag{2.3.1}$$

as a positive map that does not admit an expression $\psi(x) = \sum V_i^* X V_i$ where V_i is a 3×3 matrix.

Results by Størmer [86] and Woronowicz [105] found that, if $nm \leq 6$, all positive maps ψ from $\mathcal{M}_n$ to $\mathcal{M}_m$ are decomposable but this is not true in higher dimensions. The decomposition of every linear positive map is not possible [16]. In [13], a counter example is given to justify this theorem. Woronowicz [105] also gives counter example through computations to show that some positive maps are indecomposable. However, they concluded that the decomposition of positive maps from $\mathcal{M}_n(\mathbb{C})$ to $\mathcal{M}_m(\mathbb{C})$ is possible when $mn \leq 6$. For ψ a faithful 2-positive unital projection on a C*-algebra $\mathcal{A}$, such that the self-adjoint part of the range of ψ is a Jordan C^*-algebra of $\mathcal{A}$. Robertson [79] constructed the first explicit example of an indecomposable positive linear map from $\mathcal{M}_2$ into $\mathcal{M}_n$ and (in Theorem 2.4,) showed that if ψ is the sum of 2-positive and 2-copositive maps then ψ is decomposable.

Størmer [92], [88] showed that a map ψ from $\mathcal{A}$ to $\mathbf{B}(\mathcal{H})$ is decomposable if there are a Hilbert space $\mathcal{K}$, a Jordan morphism π from $\mathcal{A}$ to $\mathbf{B}(\mathcal{K})$, and a bounded linear operator W from $\mathcal{H}$ to $\mathcal{K}$, such that $\psi(A) = W^*\pi(A)W$ for every $A \in \mathcal{A}$. Størmer characterizes decomposable maps in the spirit of Stinespring dilation theorem by,

Theorem 2.8. *([92], Theorem 1). Let $\psi : \mathcal{A} \longrightarrow \mathbf{B}(\mathcal{H})$ be a linear map. Then ψ is decomposable if and only if for all $k \in \mathbb{N}$ whenever (a_{ij}) and (a_{ji}) belong to $\mathcal{M}_k(\mathcal{A})^+$ then $[\psi(a_{ij})] \in \mathcal{M}_k(\mathbf{B}(\mathcal{H}))^+$.*

For a linear map $\Psi_{A,B}$ where $A, B \in \mathcal{M}_n$, on $\mathcal{M}_n$, Li and Woerdeman [59] defined a decomposable positive map $\Psi_{A,B} = I \circ X + A \circ X + B \circ X^T$ where X satisfy the condition $\Psi(X_{ii}) = X_{ii}$ and showed that every positive map of the form the map $\Psi_{A,B}$ are hermitian matrices with zero diagonals on $\mathcal{M}_n$. It was concluded that these maps are decomposable if and only if $n \leq 3$ and used an example to show that for the case of $n \geq 4$ that map $\Psi_{A,B}$ is indecomposable.

Størmer [88] showed that a positive unital idempotent map, of a finite dimensional C^*-algebra into itself is indecomposable if and only if it is atomic. That is, it is not the sum of a 2-positive and a 2-copositive map. Following the idea of Choi et al [97] obtain a decomposition theorem for k-positive linear maps from $\mathcal{M}_n(\mathbb{C})$ to $\mathcal{M}_m(\mathbb{C})$, where $2 \leq k < \min\{n, m\}$ and gave an affirmative answer to Kye conjecture [54] that every 2-positive linear map from $\mathcal{M}_3(\mathbb{C})$ to $\mathcal{M}_3(\mathbb{C})$ is decomposable.

Theorem 2.9. *([106], Theorem 1.1) Every 2-positive (respectively 2-copositive) linear map in $\mathbf{B}(\mathcal{M}_3(\mathbb{C}), \mathcal{M}_3(\mathbb{C}))$ is decomposable.*

Tanahashi and Tomiyama [96] constructed a series of linear maps as the extension of Choi's map and introduced the concept of atomic map with a stronger indecomposability and showed that Choi's map is atomic. Ha [27] obtain more examples of atomic maps by representation for positive projections onto spin factors. The projections are uniquely determined by the dimension of the spin factors. Osaka [71] and [73] gives an example of atomic map in $\mathcal{M}_n$ where $n \geq 4$. All generalized indecomposable Choi maps [30], [96] are known to be atomic.

Indecomposable maps have been considered as a huge obstacle in getting a canonical form for a positive map. Consider a map defined on $\mathcal{M}_3$, depending on three non-negative parameters a, b, c,

$$\psi_{(a,b,c)}(X) = \begin{pmatrix} ax_{11} + bx_{33} + cx_{22} & -x_{12} & -x_{13} \\ -x_{21} & cx_{11} + ax_{33} + bx_{22} & -x_{23} \\ -x_{31} & -x_{32} & bx_{11} + cx_{33} + ax_{22} \end{pmatrix}$$

where $X \in \mathcal{M}_3$. The map $\psi_{(2,0,2)}(X)$ was the first example of a indecomposable map [16]. Choilam [14] noted that the map was indecomposable and extremal in the cone of positive maps by an argument involving the associated biquadratic form, $F(x, y) = \psi\langle (x^*x)z, z\rangle$ for all $x, z \in \mathbb{C}^n$.

Chruściński [19] provided a characterization of important classes of positive maps in finite dimensional matrix algebras due to the Choi-Jamiolkowski [87] isomorphism and their corresponding classes of indecomposable maps and showed that a positive partial transpose map is entangled if and only if there exists an indecomposable map. Using the Choi-like maps in $\mathcal{M}_3(\mathbb{C})$, Chruściński [21] gave a generalizations in $\mathcal{M}_n(\mathbb{C})$ and the Robertson map [79] in $\mathcal{M}_4(\mathbb{C})$ together with its generalizations in $\mathcal{M}_{2k}(\mathbb{C})$) and proceeded to discuss several properties entanglement theory related to these maps. Chruściński1 and Sarbicki [21] analyzed linear positive maps from $\mathbf{B}(\mathcal{K})$ to $\mathbf{B}(\mathcal{H})$ then provided a sufficient condition where this map is exposed by the strong spanning property that makes it sufficient for them to be optimal. In addition they showed that this condition was necessary if the linear maps is decomposable when their dimension is 2. The study was extended in [20] where a class of positive linear maps from $\mathbf{B}(\mathbb{C}^{2n})$ to $\mathbf{B}(\mathbb{C}^{2n})$ was constructed and shown that are exposed and that the maps reproduce the well known Robertson maps which are extremal and are also exposed giving a class of exposed indecomposable positive maps in the algebra of $2n \times 2n$ complex matrices with $n \geq 2$.

Robertson [79] and Stømer [92] constructed extreme maps for $n = 2, 3, 4$ with adjustment on diagonal element when $n = 3$ and negating the off diagonal elements. The structures of positive cones of these maps were noted to be complicated for positive linear maps in the complex field. Kye [56] studied these extreme maps with the context of Hadamard products in the three-dimensional case and found that every positive linear map of this type is decomposable. Further Kye gave a characterization for the positivity of these maps when real coefficients for positive linear maps between matrix algebras with fix diagonal elements. Kim and Kye [47] showed that a positive linear map on $\mathcal{M}_n$ that leaves invariant the diagonal entries is decomposable if $n = 3$, however this fails in when $n = 4$. Li Chi-Kwong and Woerdeman [59] showed that every completely positive map that leaves the diagonal entries invariant of all diagonal entries equal to the map ψ_A from $\mathcal{M}_n$ to $\mathcal{M}_n$ of the form from X to $A \circ X$ is completely positive. Osaka [72] gave a large class of extremal positive maps in $\mathcal{M}_3(\mathbb{C})$ that are neither 2-positive nor 2-copositive and further described the algebraic structure of the set of all positive linear maps in $\mathcal{M}_3(\mathbb{C})$ where it is shown that the maps constructed in [56] are decomposable using atomic concept.

Breuer [9] and Hall [33] independently generalized the Breuer-Hall maps [9]

$$\psi_n^U = \frac{1}{2n-2}((TrX)I - X - UX^TU^*)$$

on $\mathcal{M}_{2n}$ as reduction map. Letting U to be an antisymmetric unitary on $\mathbb{C}^{2n}$. Breuer and Hall showed that this map ψ_n^U is positive and indecomposable.

Robertson [79] used this concept to create an example of an indecomposable positive map on $\mathcal{M}_4$. The Robertson, map ψ from $\mathcal{M}_4$ to $\mathcal{M}_4$ is given by

$$\psi(x_{ij}) = \begin{pmatrix} x_{33}+x_{44} & 0 & x_{13}+x_{42} & x_{14}-x_{32} \\ 0 & x_{33}+x_{44} & x_{23}-x_{41} & x_{24}+x_{31} \\ x_{31}+x_{24} & x_{32}-x_{14} & x_{11}+x_{22} & 0 \\ x_{41}-x_{23} & x_{42}+x_{13} & 0 & x_{11}+x_{22} \end{pmatrix}$$

Cho, Kye, and Lee [10] generalized the idea of Choi maps by constructing a class of parametric linear maps on $\mathcal{M}_3(\mathbb{C})$ and looked at positivity, completely positivity and decomposability in relation with positive semidefinite biquadratic form where they defined the map $\psi_{(a,b,c)}$ from $\mathcal{M}_3(\mathbb{C})$ to $\mathcal{M}_3(\mathbb{C})$ given by $\psi_{(a,b,c)}(X)$ is given by the matrix:

$$\begin{pmatrix} \delta x_{11}+bx_{22}+cx_{33} & -x_{12} & -x_{13} \\ -x_{21} & \delta x_{22}+bx_{33}+cx_{11} & -x_{23} \\ -x_{31} & -x_{32} & \delta x_{33}+bx_{11}+cx_{22} \end{pmatrix}$$

where $X \in \mathcal{M}_3$, $\delta = (a-1)$ and a, b, c are nonnegative real numbers and concluded in [10] that the linear map $\psi_{(a,b,c)}$ is positive if and only if; $a \geq 1$, $a+b+c=1$, $bc \geq (2-a)^2, 1 \leq a \leq 2$, completely positive if and only if $a \geq 3$, decomposable if and only if $a \geq 1$, $bc \geq \left(\frac{3-a}{2}\right)^2, a \in [1,3]$ and 2-positive if and only if $a \geq 3$, or $2 \leq a < 3$ and $bc = (3-a)(b+c) > 0$ and completely copositive if and only if it is 2-copositive if and only if $a \geq 1$, $bc \geq 1$.

The map of the form $\psi_{(a,b,c;\theta)}$ and its variants was investigated by Ha [31] and Osaka [72] in various contexts was fond to be separable if and only if it is partially positive transpose, while [92] considered $\psi_{(1,b,\frac{1}{b},\pi)}$ which turned out to be indecomposable. For nonnegative real numbers a, b, c and $-\pi \leq \theta \leq \pi$, they consider the map $\psi_{(a,b,c;\theta)}$ between $\mathcal{M}_3$ defined by

$$\begin{pmatrix} ax_{11}+bx_{22}+cx_{33} & -e^{i\theta}x_{12} & -e^{i\theta}x_{13} \\ -e^{i\theta}x_{21} & cx_{11}+ax_{22}+bx_{33} & -e^{i\theta}x_{23} \\ -e^{i\theta}x_{31} & -e^{i\theta}x_{32} & bx_{11}+cx_{22}+ax_{33} \end{pmatrix}$$

A family of indecomposable maps for an arbitrary finite dimension $n = 3$ was constructed by Kossakowski [50]. Several methods of construction of indecomposable maps have been proposed by Kim and Kye [47] , Osaka [73], [72], [71] and Tang [97] most of which are in the context of quantum entanglement.

Majewski and Marciniak [67] used the extremal unital positive map from $\mathcal{M}_2$ to $\mathcal{M}_2$;

$$\begin{pmatrix} x_{11} & \alpha x_{12}+\beta x_{21} \\ \bar{\alpha}x_{21}+\bar{\beta}x_{12} & \gamma x_{11}+\varepsilon x_{12}+\bar{\varepsilon}x_{21}+\delta x_{22} \end{pmatrix}$$

as defined by Størmer [92] from $\mathcal{M}_2$ to $\mathcal{M}_2$ to construct concrete decomposable maps and showed that in most cases the decomposition is unique. Majewski and Marciniak [68] considered analysis of the maps from $\mathcal{M}_2$ to $\mathcal{M}_3$ based on Choi matrix method in which they gave a generalized Choi matrix for the positive maps from $\mathcal{M}_2$ to $\mathcal{M}_{n+1}$ where $n \geq 1$ giving conditions under which they are decomposable. The general problem of describing all positive maps and their decomposition remains open.

Augusiak and Stasi´nska [3] discuss some general connections between the notions of positive map, weak majorization and entropic inequalities in the context of detection of entanglement among bipartite quantum systems based on the fact that any positive map ψ from $\mathcal{M}_n(\mathbb{C})$ to $\mathcal{M}_n(\mathbb{C})$ can be written as the difference between two completely positive maps.

The literature here in gave valuable insight that guided this study. The fundamental results obtained emanate from the question asked by Yang et al [106] in their paper titled "*All 2-positive linear maps from $\mathcal{M}_3(\mathbb{C})$ to $\mathcal{M}_3(\mathbb{C})$ are decomposable*".

Chapter 3

RESEARCH METHODOLOGY

3.1 Introduction

In this chapter, some concepts in matrix algebra have been presented to make this study self-sufficient. More specifically, important concepts to explain the connections linking tensor products and linear positive maps.

3.2 Positive semidefinite matrices

A matrix A is said to be orthogonal if $AA^T = I$ or, $A^T A = I$ with the rows (or the columns) of A forming orthonormal basis of $\mathbb{R}^n$. A matrix A is said to be symmetric if $A = A^T$. The spectral decomposition theorem being an important theorem about real symmetric matrices it decomposes a square positive matrix to a diagonal matrix with all non-negative diagonal entries(eigenvalues).

Theorem 3.1. *([57], Theorem 1.7.1)* ***(Spectral decomposition theorem)*** *Any real symmetric matrix $A \in M_n$ can be decomposed as $A = \sum_{i=1}^{n} \lambda_i v_i v_i^*$, where λ_i's are the eigenvalues of A and the vectors $v_i \in \mathbb{R}^n$ are the corresponding eigenvectors which form an orthonormal basis of $\mathbb{R}^n$.*

The matrix $A = VDV^T$, where λ_i's are the diagonal entries on the diagonal matrix D and the matrix V is orthogonal. Similarly, by the singular value decomposition of A is decomposed to $A = UDU^*$, where U is a unitary and diagonal matrix D whose diagonal entries λ_i's are the nonnegative eigenvalues of A.

Theorem 3.2. *([35], Theorem 1.36) Let $A \in \mathbf{B}(\mathcal{H})$ be an operator. The following conditions are equivalent.*

(i). A is positive semidefinite, which is defined by the property: $z^*Az \geq 0$ for all $z \in \mathbb{R}^n$.

(ii). $A = A^*$ and $\sigma(A) = [0, \infty)$.

(iii) $A = B^*B$ for some operator $B \in \mathbf{B}(\mathcal{H})$.

A symmetric matrix A is also positive semidefinite if the real number z^*Az is non-negative for all $z \in \mathbb{R}^n$. If, furthermore, $z^*Az > 0$ then A is a positive definite matrix. In the case the symmetric matrix A is a block matrix, we have;

Theorem 3.3. *([46], Theorem 3.8) Given a symmetric matrix*

$$M = \begin{pmatrix} A & C^T \\ C & B \end{pmatrix},$$

with B not necessarily invertible, the matrix A is positive semidefinite on conditions that following hold:

(i). B is positive semidefinite,

(ii). the Schur complement $A/B = A - CB^*C \geq 0$, and

(ii). $(I - BB^*)C^* = 0$.

The spectrum of A ($\sigma(A) \in \mathbb{C}$), is the multiset of all eigenvalues of A counted with multiplicities. The positivity of A implies it is Hermitian.

Theorem 3.4. *Let A be a $n \times n$ hermitian matrix. Then, the following statements are equivalent:*

(i). A is positive semidefinite.

(ii). Every principal submatrix of A is positive semidefinite.

(iii). Every principal subdeterminant of A is nonnegative.

Let $A \in \mathbb{C}^{n\times n}$ be a Hermitian positive definite matrix. Then there exists a unique upper triangular $L \in \mathbb{C}^{n\times n}$ where the diagonal elements of L are real and positive such that $A = L^*L$. That is, A has an LDL^T-factorization if and only if $A = A^T$ and principal submatrices A_k of A are all nonsingular for $k = 1, ..., n-1$. When A is positive definite and L has all diagonal entries positive, the Cholesky decomposition $A = LL^T$ is unique.

Theorem 3.5. *([36], Theorem 10.9) Let $A \in \mathbb{C}^{n\times n}$ be positive semidefinite of rank k.*

(i). There exists an upper triangular L with nonnegative diagonal entries such that $A = L^*L$.

(ii). There is a permutation matrix P such that P^TAP has a unique Cholesky factorization of the form

$$P^TAP = L^*L, \text{ where } L = \begin{pmatrix} L_{11} & L_{12} \\ 0 & 0 \end{pmatrix}$$

Theorem 3.6. *([35], Theorem 2.3) Let S be an invertible matrix. The self-adjoint block matrix $M = \begin{pmatrix} S & P \\ P^* & Q \end{pmatrix}$*

(i). is positive if and only if S is positive and $P^*S^{-1}P \leq Q$.

(ii). $\det M = (\det S)\det(Q - P^*S^{-1}P)$.

Remark 3.7. For $M = \begin{pmatrix} s & \vec{p} \\ \vec{p}^* & Q \end{pmatrix}$. In case $s \in \mathbb{R}^+$ and not matrix. Then from Theorem 3.6,

$$\det M = (\det S)\det(Q - P^*S^{-1}P) = s\det(Q - \vec{p}^*s^{-1}\vec{p}) \geq 0$$

if and only if $sQ - \vec{\vec{p}}^*\vec{p} \geq 0$.

Altering the arrangement of the elements of the matrix in Theorem. 3.6 we have the following,

Theorem 3.8. *([5], Lemma 8.2.6). Let $A \in \mathbb{F}^{n\times n}, c \in \mathbb{F}^n$, and $b \in \mathbb{R}$, and define*

$$\mathcal{A} = \begin{pmatrix} A & c \\ c^* & b \end{pmatrix}$$

Then, the following statements are equivalent:

(i). If A is positive semidefinite. Then either $b = 0$ and $b = 0$, or $b > 0$ and $cc^* \leq bA$. Furthermore, the following statements are equivalent:

(i). A is positive definite, and $c^*A^{-1}c \leq a$.

(iii) $b > 0$ and $xx^* \leq bA$. In this case, $\det \mathcal{A} = \det A. \det(b - c^*A^{-1}c)$.

Theorem 3.9. *([46], The Cauchy-Binet formula. Theorem 4.5) Let A and B be matrices of size $n \times m$ and $m \times n$, respectively, and $n \leq m$. Then*

$$\det AB = \sum_{1\leq k_1<k_2<\ldots<k_n\leq m} A_{k_1\ldots k_n}B^{k_1\ldots k_n},$$

where $A_{k_1 \dots k_n}$ is the minor obtained from the columns of A whose numbers are $k_1, \dots, k_n$ and $B^{k_1, \dots, k_n}$ is the minor obtained from the rows of B whose numbers are $k_1, \dots, k_n$.

Remark 3.10. If $A, B \in M_{n \times n}$, then it is clear that $\det(AB) = \det(A)\det(B)$. For a case where A is $n \times k$ matrix and B a $k \times m$ matrix, then the resulting matrix product AB is a square matrix of dimension km. It is noted that the Cauchy-Binet formula applies in the generalized Cauchy-Binet case in [38] Theorem 6 .

3.3 Tensor products

3.3.1 Eigenvalues of tensor product

The tensor product $A \otimes B$ is positive semidefinite if and only if A and B are both positive semidefinite or both are negative semidefinite. This follows from the fact that given the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_i$ for A and $\mu_1, \mu_2, ..., \mu_j$ for B; the eigenvalues of $A \otimes B$ are $\lambda_i \mu_j$ for all i, j. By the spectral theorem, if x_i and y_i are the orthonormal sets of the eigenvectors for A and B respectively with their corresponding eigenvalues λ_i and μ_j, for $1 \le i \le n$ and $1 \le j \le m$, then

$$(A \otimes B)(x_i \otimes y_j) = \lambda_i \mu_j (x_i \otimes y_j),$$

where $x_i \otimes y_j$ are the eigenvectors and $\lambda_i \mu_j$ are the eigenvalues of $A \otimes B$. Similarly, if $A \in \mathcal{M}_n$ and $B \in \mathcal{M}_m$ are Hermitian (positive), then $A \otimes B$ is also Hermitian.

On the tensor product space, the matrix acts on the vectors, so that $v \mapsto Av$, but $w \mapsto w$. This matrix is written as $A \otimes I$, where I is the identity matrix.

Let λ_i's and μ_j's be the eigenvalues of A and B respectively. By the spectral theorem, if x_i and y_i are the orthonormal sets of the eigenvectors for A and B respectively with their corresponding eigenvalues λ_i and μ_j, for $1 \leq i \leq n$ and $1 \leq j \leq m$, then

$$\begin{aligned}(A \oplus B)(x_i \otimes y_j) &= (A \otimes I_m)(x \otimes y) + (I_n \otimes B)(x \otimes y) \\ &= (Ax \otimes y) + (x \otimes By) = (\lambda x \otimes y) + (x \otimes \mu y) \\ &= \lambda(x \otimes y) + \mu(x \otimes y) = (\lambda + \mu)(x \otimes y)\end{aligned}$$

where $x_i \otimes y_j$ are the eigenvectors and $\lambda_i + \mu_j$ are the eigenvalues of $A \oplus B$.

Like the tensor product, the direct sum $A \oplus B$ is positive semidefinite if and only if A and B are both positive semidefinite or both are negative semidefinite. Similarly, if $A \in \mathcal{M}_n$ and $B \in \mathcal{M}_m$ are Hermitian, then $A \oplus B$ is also Hermitian.

3.3.2 Partial transposition of tensor product

The partial transpose map apples the usual matrix transpose to one half of the space $\mathcal{M}_n \otimes \mathcal{M}_m$. The partial transpose is the linear maps $\mathcal{I}_n \otimes \tau$, $\tau \otimes \mathcal{I}_n$ and $\tau_n \otimes \psi$ acting on $\mathcal{M}_n \otimes \mathcal{M}_m$ such that,

$$\begin{aligned}(\mathcal{I}_n \otimes \tau)(A \otimes B) &= A \otimes B^T, \\ (\tau \otimes \mathcal{I}_n)(A \otimes B) &= A^T \otimes B, \\ (\tau_n \otimes \psi)(A \otimes B) &= A^T \otimes \psi(B).\end{aligned}$$

If A and B are positive semidefinite matrices in $\mathcal{M}_n$ and $\mathcal{M}_m$ respectively. Then by tensor product properties $A \otimes B^T$, $A^T \otimes B$ and $A^T \otimes \psi(B)$ are also positive semidefinite. We have that $(A \otimes B)^\Gamma$ denotes the partial transpose of $(A \otimes B)$ with respect to the first component A.

3.3.3 Block matrix canonical shuffling

Let us consider $\mathcal{M}_m(\mathcal{M}_n)$ for a C^*-algebra $A = (A_{ij})_{i,j=1}^m$ where A_{ij} is in $\mathcal{M}_n$. Thus $A_{ij} = (a_{i,j,k,l})_{k,l=1}^n$ with $a_{i,j,k,l} \in \mathcal{M}_n$. Setting $B_{kl} = (a_{i,j,k,l})_{i,j=1}^m$ as an element of $\mathcal{M}_m(\mathcal{M}_n)$ and thus $B = (B_{kl})_{k,l=1}^n$ in $\mathcal{M}_n(\mathcal{M}_m)$. Now $\mathcal{M}_m(\mathcal{M}_n)$ and $\mathcal{M}_n(\mathcal{M}_m)$ are both isomorphic to $\mathcal{M}_{mn}$ by deleting the extra parentheses. With this identifiers $\mathcal{M}_m$ and $\mathcal{M}_n$ are unitarily equivalent elements of $\mathcal{M}_{mn}$. This is observed if we regard A as an element of $\mathcal{M}_{mn}$, say $A = (C_{st})_{s,t=1}^{mn}$, then $C_{st} = a_{i,j,kl}$ where $s = n(i-1)+k$ and $t = n(j-1)+l$ while if we regard B as an element of $\mathcal{M}_{mn}$, say $B = (d_{st})_{s,t=1}^{mn}$, then $d_{st} = a_{i,j,kl}$ where $s = m(k-1)+i$ and $t = m(j-1)+j$. Now, let $\{E_{ij}\}_{i,j=1}^m$ and $\{F_{kl}\}_{k,l=1}^n$ denote the standard matrix units for $\mathcal{M}_m$ and $\mathcal{M}_n$ respectively. The element A of $\mathcal{M}_m(\mathcal{M}_n) \simeq \mathcal{M}_n \otimes \mathcal{M}_m$ is just

$$A = (a_{i,j,k,l})_{i,j=1}^n = (A_{ij})_{k,l=1}^m, \tag{3.3.1}$$

where $A_{ij} \in \mathcal{M}_n$ and $(A_{ij}) \in \mathcal{M}_m$. On the other hand ,

$$B = (a_{i,j,k,l})_{i,j=1}^m = (B_{kl})_{k,l=1}^n, \tag{3.3.2}$$

where $B_{kl} \in \mathcal{M}_m$ and $(B_{ij}) \in \mathcal{M}_n(\mathcal{M}_m)$. Since the above operation is for passing from $\mathcal{M}_m(\mathcal{M}_n)$ to $\mathcal{M}_m \otimes \mathcal{M}_n$ is just a permutation, it is a *-isomorphism. This operation passing from $\mathcal{M}_m(\mathcal{M}_n)$ through to $\mathcal{M}_m \otimes \mathcal{M}_n$ is just a permutation, it is a $*$-isomorphism. This $*$-isomorphism is simply canonical shuffle. It is important to note that since the canonical shuffle is a $*$-isomorphism, it preserves norm and positivity.

Naturally a block matrix $\mathcal{M}_n(\mathcal{M}_{n+1})$ identify with matrix $\mathcal{M}_{n(n+1)}$ if the corresponding entries are the same. Similarly the multiplication and *-operation on $\mathcal{M}_n(\mathcal{M}_{n+1})$ become the usual multiplication and $*$-operation on $\mathcal{M}_{n(n+1)}$. The identification defines a $*$-isomorphism. Hence, the unique norm on $\mathcal{M}_n \otimes \mathcal{M}_{n+1}$ is the norm obtained by this identification with $\mathcal{M}_{n(n+1)}$. An element of $\mathcal{M}_n(\mathcal{M}_{n+1})$ will be positive if and only if the corresponding matrix in $\mathcal{M}_{n(n+1)}$ is positive.

3.3.4 Tensor product maps

The tensor product space $\mathbb{C}^n \otimes \mathbb{C}^k$ is identical with the space of block matrices $\{E_{ij}\}^{n,k} \in M_{n,k}$ making a basis of $\mathcal{M}_{n,k}$ with a tensor representation of the canonical bases $\{e_i\}^n$, $\{e_j\}^k$ of $\mathbb{C}^n$ and $\mathbb{C}^k$ respectively such that $E_{ij}^{n,k} = e_i^n \otimes e_j^k, i = 1, \dots, n, j = 1, \dots, k$ The matrix $[A_{ij}]^{n,m}$ identifies with the elements of a block matrix $\mathcal{M}_n \otimes \mathcal{M}_k$. Let $\psi : \mathcal{M}_m \longrightarrow \mathcal{M}_p$ and $\psi : \mathcal{M}_n \longrightarrow \mathcal{M}_k$. The tensor product map $\psi \otimes \psi : \mathcal{M}_m \otimes \mathcal{M}_n \longrightarrow \mathcal{M}_p \otimes \mathcal{M}_k$ is given by $(\psi \otimes \psi)(A \otimes X) = \psi(A) \otimes \psi(X)$.

Let $\mathcal{I}_n : M_n \longrightarrow M_n$ be an identity map $\mathcal{I}_n(A) = A$ for any $A \in M_n$. The map $\mathcal{I}_n \otimes \psi : M_n \otimes M_m \longrightarrow M_p \otimes M_k$ is given by $(\mathcal{I}_n \otimes \psi)(A \otimes X) = A \otimes \psi(X)$. The set of matrices $A \otimes X \in M_m(M_n)$ by extension [42] is uniquely defined by linearity with the standard bases e_i and e_j for $\mathbb{C}^n$, the set $e_i e_j^* = E_{ij}$ representing a basis for M_n with $(\mathcal{I}_m \otimes \psi)(E_{ij} \otimes X_{ij}) = E_{ij} \otimes \psi(X_{ij})$. Every matrix $[A_{ij}] \in \mathcal{M}_k(\mathbb{C}) \otimes \mathcal{M}_n(\mathbb{C})$ is written as $[A_{ij}] = \sum_i A \otimes X$, where $A \in \mathcal{M}_k(\mathbb{C}), x \in \mathcal{M}_n(\mathbb{C})$. The map $\mathcal{I}_k \otimes \psi$ is defined linearly through $(\mathcal{I}_k \otimes \psi)(A \otimes X) = A \otimes \psi(X)$.

3.4 Mathematica

The use of *Mathematica software* has been employed in the analysis of matrix determinant and eigenvalues in the study. Matrix computations are an essential part of linear algebra. Mathematica provides a wide range of functions for carrying out matrix computations. these include; eigensystem, solving linear systems, matrix decompositions, determinants of matrices among others. Mathematica does not distinguish between row and column vectors. It is frequently useful to refer to the components of a vector, the entries of a matrix, or the rows of a matrix. Mathematica has an indexing operator.

Parentheses are reserved for algebraic grouping, square brackets for function evaluation, and curly brackets for lists and double square brackets are used for the indexing operator. Problem of computing the eigenvalues of a square matrix is equivalent to the problem of finding the roots of an $n - th$ degree polynomial. Mathematica is not used only to find the roots of the characteristic polynomial; the output is simply the generated desired eigenvalues are the roots of the n-degree polynomial. Mathematica was used to estimate the numerical roots and performs a numerical computation instead of a symbolic computation whenever the input matrix has floating point entries instead of symbolic entries.

Chapter 4

RESULTS AND DISCUSSION

4.1 Introduction

In this chapter, we constructed a linear positive map $\psi_{(\mu,c_1,\ldots,c_{n-1})}$ and gave the values for the parameters $c_1,\ldots,c_{n-1}$ and μ for $n = 2,3,4$. for which the maps are positive, 2-positive and completely positive and decomposable.

4.2 Construction of the positive map $\psi_{(\mu,c_1,\ldots,c_{n-1})}$

Let $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ be a column vector in $\mathbb{C}^n$ and $\vec{x}^* = \begin{pmatrix} \bar{x}_1 & \ldots & \bar{x}_n \end{pmatrix}$ denotes the conjugate transpose of the vector $\vec{x}$. We define the norm of $\vec{x}$ by $||x|| = (x_1^2 + \ldots + x_n^2)^{\frac{1}{2}}$ It is clear that;

$$\vec{x}^*\vec{x} = \begin{pmatrix} \bar{x}_1 & \ldots & \bar{x}_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1^2 + \ldots + x_n^2 = ||x||^2$$

while,

$$\vec{x}\vec{x}^* = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \begin{pmatrix} \bar{x}_1 & \dots & \bar{x}_n \end{pmatrix} = \begin{pmatrix} x_1\bar{x}_1 & \dots & x_1\bar{x}_n \\ \vdots & \ddots & \vdots \\ x_n\bar{x}_1 & \dots & x_n\bar{x}_n \end{pmatrix} \in \mathbb{C}^{n\times n}. \tag{4.2.1}$$

By the definition of positive semidefiniteness the matrix, $x_i\bar{x}_j$ is positive. We denote this matrix by X. Since X is positive semidefinite, then all it's principal minors(eigenvalues) are nonnegative. The diagonal elements of the matrix X are such that $x_i\bar{x}_i = |x_i|$ are positive real numbers. In this study we will denote the diagonal entries $x_i\bar{x}_i \in \mathbb{R}$ by α_n.

Let $\mu, c_1, \dots, c_{n-1} \in \mathbb{R}$ such that $c_1, \dots, c_{n-1} \geq 0$, $0 < \mu < 1$ and $r \in \mathbb{N}$. We define the family of positive maps $\psi_{(\mu,c_1,\dots,c_{n-1})}(X)$ where $X \in \mathcal{M}_n(\mathbb{C})$ as follows:

$$\psi_{(\mu,c_1,\dots,c_{n-1})} : \mathcal{M}_n(\mathbb{C}) \longrightarrow \mathcal{M}_{n+1}(\mathbb{C}).$$

$$X \mapsto \begin{pmatrix} P_1 & -c_1x_1\bar{x}_2 & -c_2x_1\bar{x}_3 & \cdots & -c_{n-2}x_1\bar{x}_{n-1} & 0 & -\mu x_1\bar{x}_n \\ -c_1x_2\bar{x}_1 & P_2 & -c_2x_2\bar{x}_3 & \cdots & -c_{n-2}x_2\bar{x}_{n-1} & -c_{n-1}x_2\bar{x}_n & 0 \\ -c_2x_3\bar{x}_1 & -c_2x_3\bar{x}_2 & P_3 & \cdots & -c_{n-2}x_3\bar{x}_{n-1} & -c_{n-1}x_3\bar{x}_n & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \\ -c_{n-2}x_{n-1}\bar{x}_1 & -c_{n-2}x_{n-1}\bar{x}_2 & -c_{n-2}x_{n-1}\bar{x}_3 & \cdots & P_{n-1} & -c_{n-1}x_{n-1}\bar{x}_n & 0 \\ 0 & -c_{n-1}x_n\bar{x}_2 & -c_{n-1}x_n\bar{x}_3 & \cdots & -c_{n-1}x_n\bar{x}_{n-1} & P_n & 0 \\ -\mu x_n\bar{x}_1 & 0 & 0 & \cdots & 0 & 0 & P_{n+1} \end{pmatrix},$$

where

$$
\begin{aligned}
P_1 &= \mu^{-r}(\alpha_1 + \alpha_2 c_1 \mu^r + \ldots + \alpha_{n-1} c_{n-2} \mu^r + \alpha_n c_{n-1} \mu^r) \\
P_2 &= \mu^{-r}(\alpha_2 + \alpha_3 c_1 \mu^r + \ldots + \alpha_n c_{n-2} \mu^r + \alpha_1 c_{n-1} \mu^r) \\
P_3 &= \mu^{-r}(\alpha_3 + \alpha_4 c_1 \mu^r + \ldots + \alpha_1 c_{n-2} \mu^r + \alpha_2 c_{n-1} \mu^r) \\
\vdots &= \vdots \\
P_n &= \mu^{-r}(\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_n) \\
P_{n+1} &= \mu^{-r}(\alpha_n + \alpha_1 c_1 \mu^r + c_2 \alpha_2 \mu^r + \ldots + \alpha_{n-1} c_{n-1} \mu^r)
\end{aligned}
$$

4.3 Positivity

A crucial problem in applications of positive maps is checking whether or not they are positive. Determining that a linear map is positive is equivalent to detecting nonnegativity of biquadratic forms. It is known that there exist a positive semidefinite biquadratic form which is not a sum of squares of bilinear forms ([16], Theorem 1). We show the positivity of the map ψ by expressing the positive semidefinite matrix $\psi(X)$ as a sum of squares of bilinear forms.
The linear map $\psi_{(\mu, c_1, \ldots, c_{n-1})}$ is uniquely determined by the polynomial function;

$$F(v_1, \ldots, v_{n+1}, t) \ := \ \vec{v}[\psi_{(c_1, \ldots, c_{n-1})}(X)]\vec{v}^T \tag{4.3.1}$$

as a biquadratic function of vector $v \in \mathbb{R}^{n+1}$ and $t \in \mathbb{C}$. The linear map $\psi_{(\mu, c_1, \ldots, c_{n-1})}$ is positive if and only if $F(v_1, \ldots, v_{n+1}, t)$ is positive semidefinite (a sum of squares).

4.3.1 Positivity of the linear map $\psi_{(\mu,c_1)}$

Lemma 4.1. *Let $0 < \mu \leq 1$ and $c_1 \geq 0$. Then the function*

$$F(v_1, v_2, v_3, t) = \mu^{-r}(1 + c_1|t|\mu^r)v_1^2 + \mu^{-r}(1 + |t|)v_2^2 + \mu^{-r}(c_1\mu^r + |t|)v_3^2 - 2v_1v_3\mu\text{Re}(t) \qquad (4.3.2)$$

is positive semidefinite for all $v_1, v_2, v_3 \in \mathbb{R}$ and $t \in \mathbb{C}$.

Proof. If $v_1 = 0$, we have that;

$$F(0, v_2, v_3, t) = \mu^{-r}(1 + |t|)v_2^2 + \mu^{-r}(c_1\mu^r + |t|)v_3^2 \geq 0.$$

Assume $F(v_1, v_2, v_3, t) < 0$ and $v_1 \neq 0$ and. Since $0 < \mu \leq 1$. By completing squares; $F(v_1, v_2, v_3, t)$

$$\begin{aligned}
&= \mu^{-r}(1 + c_1|t|\mu^r)v_1^2 + \mu^{-r}(1 + |t|)v_2^2 + \mu^{-r}(c_1\mu^r + |t|)v_3^2 - 2v_1v_3\mu\text{Re}(t) \\
&= c_1|t|v_1^2 + \mu^{-r}(1 + |t|)v_2^2 + \mu^{-r}c_1v_3^2 + (\mu^{-r}v_1^2 - 2v_1v_3\mu\text{Re}(t) + \mu^{-r}|t|v_3^2) \\
&= c_1|t|v_1^2 + \mu^{-r}(1 + |t|)v_2^2 + \mu^{-r}c_1v_3^2 \\
&\quad +\mu^{-r}[(v_1 - \mu^{1+r}\text{Re}(t)v_3)^2 + (|t| - \mu^{2+2r}\text{Re}(t)^2)v_3^2] \\
&< 0.
\end{aligned}$$

Letting $t = a + ib \in \mathbb{C}$. Since $c_1 \geq 0$ and $0 < \mu \leq 1$. From the coefficient of v_3^2,

$$\begin{aligned}
|t| - \mu^{2+2r}\text{Re}(t)^2 &= |a|^2 + |b|^2 - \mu^{2+2r}|a|^2 \\
&= |a|^2(1 - \mu^{2+2r}) + |b|^2 \\
&\geq 0.
\end{aligned}$$

Thus $F(v_1, v_2, v_3, t) < 0$ is a contradiction. Hence, $F(v_1, v_2, v_3, t) \geq 0$ for every $v_1, v_2, v_3 \in \mathbb{R}$ and $t \in \mathbb{C}$. □

Proposition 4.2. *The linear map $\psi_{(\mu,c_1)}$ is positive.*

Proof. We need to show that,

$$\psi\left(\begin{pmatrix} s \\ t \end{pmatrix}\begin{pmatrix} \bar{s} & \bar{t} \end{pmatrix}\right) \in \mathcal{M}_3^+$$

for every $s, t \in \mathbb{C}$ is,

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}^T \begin{pmatrix} \mu^{-r}|s| + c_1|t| & 0 & -\mu s\bar{t} \\ 0 & \mu^{-r}(|s|+|t|) & 0 \\ -\mu t\bar{s} & 0 & c_1|s| + \mu^{-r}|t| \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \geq 0 \tag{4.3.3}$$

for every $v_1, v_2, v_3 \in \mathbb{R}$ and $s, t \in \mathbb{C}$.

Let $s = 0$. Then,

$$c_1|t|v_1^2 + \mu^{-r}|t|v_2^2 + \mu^{-r}|t|v_3^2 \geq 0.$$

If $s \neq 0$. Assume that $s = 1$. Then,

$$z^T\psi\left(\begin{pmatrix} 1 \\ t \end{pmatrix}\begin{pmatrix} 1 & \bar{t} \end{pmatrix}\right)z$$

yields the inequality.

$$(\mu^{-r} + c_1|t|)v_1^2 + \mu^{-r}(1+|t|)v_2^2 + (c_1 + \mu^{-r}|t|)v_3^2 - 2v_1v_3\mathrm{Re}(t) \geq 0.$$

By Lemma 4.1, $\psi_{(\mu,c_1)}$ is positive. □

4.3.2 Positivity of the linear maps $\psi_{(\mu,c_1,c_2)}$

We characterize the positivity of the map $\psi_{(\mu,c_1,c_2)}$ for $v = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$ and $t \in \mathbb{C}$.

Lemma 4.3. *Let $0 < \mu \leq 1$ and $c_1, c_2 \geq 0$. Then the function*

$$F(v_1, v_2, v_3, v_4, t) = \mu^{-r}(1 + c_1\mu^r + c_2|t|\mu^r)v_1^2 + \mu^{-r}(1 + c_1|t|\mu^r + c_2\mu^r)v_2^2$$
$$+\mu^{-r}(2 + |t|)v_3^2 + \mu^{-r}(|t| + c_1\mu^r + c_2\mu^r)v_4^2 - 2c_1v_1v_2 - 2c_2\text{Re}(t)v_2v_3 - 2\mu\text{Re}(t)v_1v_4$$

is positive semidefinite for every $v_1, v_2, v_3, v_4 \in \mathbb{R}$ and $t \in \mathbb{C}$ provided it satisfies the inequalities:

$$\mu^{-r} > c_1. \tag{4.3.4}$$

$$\mu^{-r} > c_2. \tag{4.3.5}$$

$$c_2 \geq c_1. \tag{4.3.6}$$

Proof. If $v_1 = 0$. Then,
$F(0, v_2, v_3, v_4, t)$

$$\begin{aligned}
&= \mu^{-r}(1 + c_1|t|\mu^r + c_2\mu^r)v_2^2 + \mu^{-r}(2 + |t|)v_3^2 + \mu^{-r}(|t| + c_1\mu^r + c_2\mu^r)v_4^2 - 2c_2\text{Re}(t)v_2v_3 \\
&= (c_1|t| + c_2)v_2^2 + 2\mu^{-r}v_3^2 + \mu^{-r}(|t| + c_1\mu^r + c_2\mu^r)v_4^2 + (\mu^{-r}v_2^2 - 2v_2v_3c_2\text{Re}(t) + \mu^{-r}|t|v_3^2) \\
&= (c_1|t| + c_2)v_2^2 + \mu^{-r}(|t| + c_1\mu^r + c_2\mu^r)v_4^2 + \mu^{-r}(v_2 - \mu^r c_2\text{Re}(t)v_3)^2 \\
&\quad +(2\mu^{-r} + \mu^{-r}|t| - \mu^r c_2^2\text{Re}(t)^2)v_3^2.
\end{aligned}$$

$F(0, v_2, v_3, v_4, t)$ is positive when the coefficient of v_3^2 satisfies the inequality,

$$\mu^{-2r}(2 + |t|) - c_2^2\Re^2(t)^2 \geq 0. \tag{4.3.7}$$

Letting $t = a + ib$. We have that,

$$\begin{aligned}
\mu^{-r}(2 + |t|) - \mu^r c_2^2\Re^2(t)^2 &= 2\mu^{-2r} + (\mu^{-2r}(|a|^2 + |b|^2) - x^2c_2^2) \\
&= 2\mu^{-2r} + \mu^{-2r}|b|^2 + |a|^2(\mu^{-2r} - c_2^2)
\end{aligned}$$

is positive whenever $\mu^{-r} \geq c_2$ holds.

If $v_2 = 0$. Then ,

$F(v_1, 0, v_3, v_4, t)$

$$\begin{aligned}
&= \mu^{-r}(1 + c_1\mu^r + c_2|t|\mu^r)v_1^2 + \mu^{-r}(2 + |t|)v_3^2 + \mu^{-r}(|t| + c_1\mu^r + c_2\mu^r)v_4^2 - 2\mu\text{Re}(t)v_1v_4 \\
&= \mu^{-r}(c_1\mu^r + c_2|t|\mu^r)v_1^2 + \mu^{-r}(2 + |t|)v_3^2 + \mu^{-r}(c_1\mu^r + c_2\mu^r)v_4^2 \\
&\quad +(\mu^{-r}v_1^2 - 2v_1v_4\mu\text{Re}(t) + \mu^{-r}|t|v_4^2) \\
&= \mu^{-r}(c_1\mu^r + c_2|t|\mu^r)v_1^2\mu^{-r}(2 + |t|)v_3^2 + \mu^{-r}(c_1\mu^r + c_2\mu^r)v_4^2 \\
&\quad +\mu^{-r}(v_1 - \mu^{1+r}\text{Re}(t)v_4)^2 + \mu^{-r}(|t| - \mu^{2+2r}\text{Re}(t)^2)v_4^2 \\
&\geq 0.
\end{aligned}$$

If $v_3 = 0$. Then ,

$F(v_1, v_2, 0, v_4, t)$

$$\begin{aligned}
&= \mu^{-r}(1 + c_1\mu^r + c_2|t|\mu^r)v_1^2 + \mu^{-r}(1 + c_1|t|\mu^r + c_2\mu^r)v_2^2 \\
&\quad +\mu^{-r}(|t| + c_1\mu^r + c_2\mu^r)v_4^2 - 2c_1v_1v_2 - 2\mu\text{Re}(t)v_1v_4 \\
&= c_2|t|v_1^2 + \mu^{-r}(1 + c_2\mu^r)v_2^2 + (c_1 + c_2)v_4^2 \\
&\quad +(\mu^{-r}v_1^2 - 2v_1v_4\mu\text{Re}(t) + \mu^{-r}|t|v_4^2) + c_1(v_1^2 - 2v_1v_2 + |t|v_2^2) \\
&= c_2|t|v_1^2 + \mu^{-r}(1 + c_2\mu^r)v_2^2 + (c_1 + c_2)v_4^2 + \mu^{-r}(v_1 - \mu^{1+r}\text{Re}(t)v_4)^2 \\
&\quad +\mu^{-r}(|t| - \mu^{2+2r}\text{Re}(t)^2)v_4^2 + c_1(v_1 - v_2)^2 + c_1(|t| - 1)v_2^2.
\end{aligned}$$

From the coefficients of v_2^2,

$$\mu^{-r} + c_2 + c_1(|t| - 1) \;=\; (\mu^{-r} - c_1) + c_2 + c_1|t| \geq 0. \tag{4.3.8}$$

Therefore, $F(v_1, v_2, 0, v_4, t)$ is positive whenever $\mu^{-r} - c_1 \geq 0$ holds. If $v_4 = 0$. Then,

$F(v_1, v_2, v_3, 0, t)$

$$\begin{aligned}
&= \mu^{-r}(1 + c_1\mu^r + c_2|t|\mu^r)v_1^2 + \mu^{-r}(1 + c_1|t|\mu^r + c_2\mu^r)v_2^2 \\
&\quad +\mu^{-r}(2 + |t|)v_3^2 - 2c_1v_1v_2 - 2c_2\text{Re}(t)v_2v_3 \\
&= \mu^{-r}(1 + c_2|t|\mu^r)v_1^2 + c_2v_2^2 + 2\mu^{-r}v_3^2 + c_1(v_1^2 - 2v_1v_2 + |t|v_2^2) \\
&\quad +(\mu^{-r}v_2^2 - 2c_2\text{Re}(t)v_2v_3 + \mu^{-r}|t|v_3^2) \\
&= \mu^{-r}(1 + c_2|t|\mu^r)v_1^2 + 2\mu^{-r}v_3^2 + c_2v_2^2 + c_1(v_1 - v_2)^2 + c_1(|t| - 1)v_2^2 \\
&\quad +\mu^{-r}(v_2 - \mu^r c_2\text{Re}(t)v_3)^2 + (\mu^{-r}|t| - \mu^r c_2^2\text{Re}(t)^2)v_3^2.
\end{aligned}$$

From coefficients of v_2^2 we have that,

$$c_2 + c_1|t| - c_1 \geq 0$$

Thus, $F(v_1, v_2, v_3, 0, t)$ is positive provided $c_2 - c_1 \geq 0$.
Let $v_i \neq 0, i = 1, 2, 3, 4$ and assume that there exist $v_1, v_2, v_3, v_4 \in \mathbb{R}$ and $t \in \mathbb{C}$ such that $v_1 \neq 0$ and $F(v_1, v_2, v_3, v_4, t) < 0$. Since $0 < \mu \leq 1$ and $c_1, c_2 \geq 0$. Then,
$F(v_1, v_2, v_3, v_4, t)$

$$\begin{aligned}
&= \mu^{-r}(1 + c_1\mu^r + c_2|t|\mu^r)v_1^2 + \mu^{-r}(1 + c_1|t|\mu^r + c_2\mu^r)v_2^2 + \mu^{-r}(2 + |t|)v_3^2 \\
&\quad +\mu^{-r}(|t| + c_1\mu^r + c_2\mu^r)v_4^2 - 2c_1v_1v_2 - 2c_2\text{Re}(t)v_2v_3 - 2\mu\text{Re}(t)v_1v_4 \\
&= \mu^{-r}v_1^2 + \mu^{-r}v_2^2 + 2\mu^{-r}v_3^2 + (c_1 + c_2)\mu^{-r}v_4^2 + c_1(v_1 - v_2)^2 + c_1(|t| - 1)v_2^2 \\
&\quad +\mu^{-r}(v_2 - \mu^r c_2\text{Re}(t)v_3)^2 + (\mu^{-r}|t| - \mu^r c_2^2\text{Re}(t)^2)v_3^2 \\
&\quad +\mu^{-r}(v_1 - \mu^{1+r}\text{Re}(t)v_4)^2 + (\mu^{-r}|t| - \mu^{2+2r}\text{Re}(t)^2)v_4^2 \\
&< 0.
\end{aligned}$$

This is a contradiction when the inequalities (4.3.7) and (4.3.8) holds . Thus $F(v_1, v_2, v_3, v_4, t) \geq 0$ for every $v_1, v_2, v_3, v_4 \in \mathbb{R}$ and $t \in \mathbb{C}$. □

Proposition 4.4. *The linear map $\psi_{(\mu,c_1,c_2)}$ is positive if the conditions in Lemma 4.3* *are satisfied.*

Proof. We show that,

$$\psi\left(\begin{pmatrix} q \\ s \\ t \end{pmatrix}\begin{pmatrix} \bar{q} & \bar{s} & \bar{t} \end{pmatrix}\right) \in \mathcal{M}_4^+$$

for every $q, s, t \in \mathbb{C}$ is;

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}^T \begin{pmatrix} P_1 & -c_1 q\bar{s} & 0 & -\mu q\bar{t} \\ -c_1 s\bar{q} & P_2 & -c_2 s\bar{t} & 0 \\ 0 & -c_2 t\bar{s} & P_3 & 0 \\ -\mu t\bar{q} & 0 & 0 & P_4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \geq 0 \qquad (4.3.9)$$

where,

$$\begin{aligned} P_1 &= \mu^{-r}(|q| + c_1|s|\mu^r + c_2|t|\mu^r) \\ P_2 &= \mu^{-r}(|s| + c_1|t|\mu^r + c_2|q|\mu^r) \\ P_3 &= \mu^{-r}(|q| + |s| + |t|) \\ P_4 &= \mu^{-r}(|t| + c_1|q|\mu^r + c_2|s|\mu^r) \end{aligned}$$

for every $v_1, v_2, v_3, v_4 \in \mathbb{R}$ and $q, s, t \in \mathbb{C}$.

Taking $q = s = 0$.

$$F(v_1, v_2, v_3, v_4, t) = c_2\mu^{-r}|t|v_1^2 + c_1|t|v_2^2 + \mu^{-r}|t|v_3^2 + \mu^{-r}|t|v_4^2 \geq 0.$$

If $q = 0$, since $0 < \mu \leq 1$, by inequality (4.3.7),

$F(v_1, v_2, v_3, v_4, t)$

$$\begin{aligned} &= (c_1 + c_2|t|)v_1^2 + \mu^{-r}(1 + c_1|t|)v_2^2 + \mu^{-r}(1 + |t|)v_3^2 + (\mu^{-r}|t| + c_2)v_4^2 - 2c_2\mathrm{Re}(t)v_2v_3 \\ &\geq 0. \end{aligned}$$

If $s = 0$,

$F(v_1, v_2, v_3, v_4, t)$

$$\begin{aligned} &= \mu^{-r}(1 + c_2|t|)v_1^2 + (c_1|t| + c_2)v_2^2 + \mu^{-r}(1 + |t|)v_3^2 + \mu^{-r}(|t| + c_1\mu^r)v_4^2 - 2\mu\mathrm{Re}(t)v_1v_4 \\ &= c_2|t|v_1^2 + (c_1|t| + c_2)v_2^2 + \mu^{-r}(1 + |t|)v_3^2 + c_1v_4^2 + \mu^{-r}(v_1 - \mu^{1+r}\mathrm{Re}(t)v_4)^2 \\ &\quad +(\mu^{-r}|t| - \mu^{2+r}\mathrm{Re}(t)^2)v_4^2 \\ &\geq 0. \end{aligned}$$

If q and s are not equal to zero. Assume that $q = s = 1$. Then, by Lemma 4.3

$$z^T\psi\left(\begin{pmatrix} 1 \\ 1 \\ t \end{pmatrix} \begin{pmatrix} 1 & 1 & \bar{t} \end{pmatrix}\right)z$$

is positive for every $v = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$ and $t \in \mathbb{C}$ □

4.3.3 Positivity of the linear maps $\psi_{(\mu,c_1,c_2,c_3)}$

Lemma 4.5. *Let $0 < \mu \leq 1$ and $c_1, c_2, c_3 \geq 0$. Then the function*

$F(v_1, v_2, v_3, v_4, v_5, t) = \mu^{-r}(1+c_1\mu^r+c_2\mu^r+c_3|t|\mu^r)v_1^2+\mu^{-r}(1+c_1\mu^r+c_2|t|\mu^r+c_3\mu^r)v_2^2$
$+\mu^{-r}(1 + c_1|t|\mu^r + c_2\mu^r + c_3\mu^r)v_3^2 + \mu^{-r}(3 + |t|)v_4^2 + \mu^{-r}(|t| + c_1\mu^r + c_2\mu^r + c_3\mu^r)v_5^2$
$-2c_1v_1v_2 - 2c_2v_1v_3 - 2c_2v_2v_3 - 2c_3\mathrm{Re}(t)v_2v_4 - 2c_3Re(t)v_3v_4 - 2\mu\mathrm{Re}(t)v_1v_5$

is positive semidefinite for every v_1, v_2, v_3, v_4, v_5 and $t \in \mathbb{C}$ whenever it satisfies the *inequalities,*

$$\mu^{-r} \geq 2c_3, \tag{4.3.10}$$

$$\mu^{-r} \geq 2c_1, \tag{4.3.11}$$

$$\mu^{-r} \geq c_2, \tag{4.3.12}$$

$$c_1\mu^{-r} \geq c_2^2. \tag{4.3.13}$$

Proof. If $v_1 = 0$. Then,

$F(0, v_2, v_3, v_4, v_5, t)$

$= \mu^{-r}(1 + c_1\mu^r + c_2|t|\mu^r + c_3\mu^r)v_2^2 + \mu^{-r}(1 + c_1|t|\mu^r + c_2\mu^r + c_3\mu^r)v_3^2 + \mu^{-r}(3 + |t|)v_4^2$

$+\mu^{-r}(|t| + c_1\mu^r + c_2\mu^r + c_3\mu^r)v_5^2 - 2c_2v_2v_3 - 2c_3\mathrm{Re}(t)v_2v_4 - 2c_3\mathrm{Re}(t)v_3v_4$

$= \mu^{-r}(1 + c_1\mu^r)v_2^2 + \mu^{-r}(1 + c_1|t|\mu^r)v_3^2 + 3\mu^{-r}v_4^2 + \mu^{-r}(|t| + c_1\mu^r + c_2\mu^r + c_3\mu^r)v_5^2$

$+c_2(v_3 - c_2)^2 + c_2(|t| - 1)v_2^2 + c_3(v_2 - \mathrm{Re}(t)v_4)^2 + (\mu^{-r}\frac{|t|}{2} - c_3\mathrm{Re}(t)^2)v_4^2$

$+c_3(v_3 - \mathrm{Re}(t)v_4)^2 + (\mu^{-r}\frac{|t|}{2} - c_3\mathrm{Re}(t)^2)v_4^2.$

From the coefficients of v_2^2 and v_4^2 we have,

$$\mu^{-r} + c_1 + c_2|t| - c_2 \quad = \quad (\mu^{-r} - c_2) + c_1 + c_2|t|$$

and

$$3\mu^{-r} + \mu^{-r}|t| - 2c_3\mathrm{Re}(t)^2 \quad = \quad 3\mu^{-r} + \mu^{-r}(|a|^2 + |b|^2) - 2c_3|a|^2$$

respectively. The function $F(0, v_2, v_3, v_4, v_5, t) \geq 0$ whenever it satisfies the inequalities, $\mu^{-r} \geq c_2$ and $\mu^{-r} \geq 2c_3$.

If $v_2 = 0$. Then ,

$F(v_1, 0, v_3, v_4, v_5, t)$

$$\begin{aligned}
&= \mu^{-r}(1 + c_1\mu^r + c_2\mu^r + c_3|t|\mu^r)v_1^2 + \mu^{-r}(1 + c_1|t|\mu^r + c_2\mu^r + c_3\mu^r)v_3^2 \\
&\quad +\mu^{-r}(3 + |t|)v_4^2 + \mu^{-r}(|t| + c_1\mu^r + c_2\mu^r + c_3\mu^r)v_5^2 \\
&\quad -2c_2v_1v_3 - 2c_3\mathrm{Re}(t)v_3v_4 - 2\mu\mathrm{Re}(t)v_1v_5 \\
&= (c_1 + c_3|t|)v_1^2 + (c_1|t| + c_3)v_3^2 + 3\mu^{-r}v_4^2 + (c_1 + c_2 + c_3)v_5^2 + c_2(v_1 - v_3)^2 \\
&\quad +\mu^{-r}(v_3 - \mu^r c_3\mathrm{Re}(t)v_4)^2 + (\mu^{-r}|t| - \mu^r c_3^2\mathrm{Re}(t)^2)v_4^2 \\
&\quad +\mu^{-r}(v_1 - \mu^{1+r}\mathrm{Re}(t)v_5)^2 + (\mu^{-r}|t| - \mu^{2+r}\mathrm{Re}(t)^2)v_5^2 \\
&\geq 0
\end{aligned}$$

whenever the coefficients of v_4^2 satisfies the inequality

$$\begin{aligned}
\mu^{-2r}(3 + |t|) - c_3^2\mathrm{Re}(t)^2 &= 3\mu^{-2r} + \mu^{-2r}(|a|^2 + |b|^2) - c_3^2|a|^2 \\
&\geq 0
\end{aligned} \tag{4.3.14}$$

whenever (4.3.10) holds.

If $v_3 = 0$. Then ,

$F(v_1, v_2, 0, v_4, v_5, t)$

$$\begin{aligned}
&= \mu^{-r}(1 + c_1\mu^r + c_2\mu^r + c_3|t|\mu^r)v_1^2 + \mu^{-r}(1 + c_1\mu^r + c_2|t|\mu^r + c_3\mu^r)v_2^2 \\
&\quad +\mu^{-r}(3 + |t|)v_4^2 + \mu^{-r}(|t| + c_1\mu^r + c_2\mu^r + c_3\mu^r)v_5^2 \\
&\quad -2c_1v_1v_2 - 2c_3\mathrm{Re}(t)v_2v_4 - 2\mu\mathrm{Re}(t)v_1v_5 \\
&= (c_2 + c_3|t|)v_1^2 + (c_1 + c_2|t|)v_2^2 + 3\mu^{-r}v_4^2 + (c_1 + c_2 + c_3)v_5^2 + c_1(v_1 - v_2)^2 \\
&\quad +(c_3 - c_1)v_2^2 + \mu^{-r}(v_2 - \mu^r c_3\mathrm{Re}(t)v_4)^2 + (\mu^{-r}|t| - \mu^r c_3^2\mathrm{Re}(t)^2)v_4^2 \\
&\quad +\mu^{-r}(v_1 - \mu^{1+r}\mathrm{Re}(t)v_5)^2 + (\mu^{-r}|t| - \mu^{2+r}\mathrm{Re}(t)^2)v_5^2 \\
&\geq 0
\end{aligned}$$

with the coefficients of v_4^2 satisfying the inequality (4.3.10).

If $v_4 = 0$. Then,

$F(v_1, v_2, v_3, 0, v_5, t)$

$$\begin{aligned}
&= \mu^{-r}(1 + c_1\mu^r + c_2\mu^r + c_3|t|\mu^r)v_1^2 + \mu^{-r}(1 + c_1\mu^r + c_2|t|\mu^r + c_3\mu^r)v_2^2 \\
&\quad + \mu^{-r}(1 + c_1|t|\mu^r + c_2\mu^r + c_3\mu^r)v_3^2 + \mu^{-r}(|t| + c_1\mu^r + c_2\mu^r + c_3\mu^r)v_5^2 \\
&\quad - 2c_1v_1v_2 - 2c_2v_1v_3 - 2c_2v_2v_3 - 2\mu\mathrm{Re}(t)v_1v_5 \\
&= c_3|t|v_1^2 + c_1v_2^2 + \mu^{-r}(1 + c_1|t|\mu^r + c_3\mu^r)v_3^2 + \mu^{-r}(|t| + c_1\mu^r + c_2\mu^r + c_3\mu^r)v_5^2 \\
&\quad + c_1(v_1 - v_2)^2 + c_2(v_1 - v_3)^2 + \mu^{-r}(v_3 - \mu^r c_2 v_2)^2 + (c_2|t| - \mu^r c_2^2)v_2^2 + \\
&\quad \mu^{-r}(v_1 - \mu^{1+r}\mathrm{Re}(t)v_5)^2 + (\mu^{-r}|t| - \mu^{2+r}\mathrm{Re}(t)^2)v_5^2 \\
&\geq 0
\end{aligned}$$

The function $F(v_1, v_2, v_3, 0, v_5, t)$ is positive if the coefficients of v_2^2 satisfies the inequality (4.3.13).

If $v_5 = 0$. Then,

$F(v_1, v_2, v_3, v_4, 0, t)$

$$\begin{aligned}
&= \mu^{-r}(1 + c_1\mu^r + c_2\mu^r + c_3|t|\mu^r)v_1^2 + \mu^{-r}(1 + c_1\mu^r + c_2|t|\mu^r + c_3\mu^r)v_2^2 \\
&\quad + \mu^{-r}(1 + c_1|t|\mu^r + c_2\mu^r + c_3\mu^r)v_3^2 + \mu^{-r}(3 + |t|)v_4^2 \\
&\quad - 2c_1v_1v_2 - 2c_2v_1v_3 - 2c_2v_2v_3 - 2c_3\mathrm{Re}(t)v_2v_4 - 2c_3Re(t)v_3v_4 \\
&= \mu^{-r}(1 + c_3|t|\mu^r)v_1^2 + c_3v_2^2 + \mu^{-r}v_3^2 + 3\mu^{-r}v_4^2 + c_1(v_1 - v_2)^2 + c_2(v_1 - v_3)^2 \\
&\quad + c_2(|t|v_2 - v_3)^2 + (c_1|t| - c_2)v_3^2 + \mu^{-r}(v_2 - c_1\mu^r\mathrm{Re}(t)v_4)^2 \\
&\quad + (\mu^{-r}\frac{|t|}{2} - \mu^r c_1^2\mathrm{Re}(t)^2)v_4^2 + c_3(v_3 - \mathrm{Re}(t)v_4)^2 + (\mu^{-r}\frac{|t|}{2} - c_3\mathrm{Re}(t)^2)v_4^2
\end{aligned}$$

The function $F(v_1, v_2, v_3, v_4, 0, t)$ is positive whenever the coefficients of v_3^2 satisfies (4.3.12) while the coefficients v_4^2 satisfies the inequalities (4.3.10) and (4.3.11).

Let $v_i \neq 0, i = 1,2,3,4,5$ and assume that there exist $v_1, v_2, v_3, v_4, v_5 \in \mathrm{Re}al$ and $t \in \mathbb{C}$ such that $v_1 \neq 0$ and $F(v_1, v_2, v_3, v_4, v_5, t) < 0$. Since $0 < \mu \leq 1$ and $c_1, c_2 \geq 0$. Then, $F(v_1, v_2, v_3, v_4, v_5, t)$

$$\begin{aligned}
&= \mu^{-r}(1 + c_1\mu^r + c_2\mu^r + c_3|t|\mu^r)v_1^2 + \mu^{-r}(1 + c_1\mu^r + c_2|t|\mu^r + c_3\mu^r)v_2^2 \\
&\quad +\mu^{-r}(1 + c_1|t|\mu^r + c_2\mu^r + c_3\mu^r)v_3^2 + \mu^{-r}(3 + |t|)v_4^2 \\
&\quad +\mu^{-r}(|t| + c_1\mu^r + c_2\mu^r + c_3\mu^r)v_5^2 \\
&\quad -2c_1v_1v_2 - 2c_2v_1v_3 - 2c_2v_2v_3 - 2c_3\mathrm{Re}(t)v_2v_4 - 2c_3Re(t)v_3v_4 - 2\mu\mathrm{Re}(t)v_1v_5 \\
&= c_3|t|v_1^2 + \mu^{-r}v_2^2 + \mu^{-r}v_3^2 + 3\mu^{-r}v_4^2 + (c_1 + c_2 + c_3)v_5^2 + c_1(v_1 - v_2)^2 \\
&\quad +c_2(c_1 - c_3)^2 + c_2(|t|v_2 - v_3)^2 + (c_1|t| - c_2)v_3^2 + c_3(v_2 - \mathrm{Re}(t)v_4)^2 \\
&\quad +(\mu^{-r}\frac{|t|}{2} - c_3\mathrm{Re}(t)^2)v_4^2 + c_3(v_3 - \mathrm{Re}(t)v_4)^2 + (\mu^{-r}\frac{|t|}{2} - c_3\mathrm{Re}(t)^2)v_4^2 \\
&\quad +\mu^{-r}(v_1 - \mu^{1+r}\mathrm{Re}(t)v_5)^2 + (|t|\mu^{-r} - \mu^{2+r}\mathrm{Re}(t)^2)v_5^2 \\
&< 0.
\end{aligned}$$

This is a contradiction when the inequalities (4.3.10) and (4.3.12) holds .
Thus, $F(v_1, v_2, v_3, v_4, v_5, t) \geq 0$ for every $v_1, v_2, v_3, v_4, v_5 \in \mathrm{Re}al$ and $t \in \mathbb{C}$ □

Proposition 4.6. *The linear map $\psi_{(\mu,c_1,c_2,c_3)} : \mathcal{M}_4 \longrightarrow \mathcal{M}_5$ is positive provided Lemma 4.5 is satisfied.*

Proof. We show that,

$$\psi_{(\mu,c_1,c_2,c_3)}\left(\begin{pmatrix} q \\ s \\ u \\ t \end{pmatrix}\begin{pmatrix} \bar{q} & \bar{s} & \bar{u} & \bar{t} \end{pmatrix}\right) \in \mathcal{M}_5^+$$

for every $q, s, u, t \in \mathbb{C}$.

That is,

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix}^T \begin{pmatrix} p_1 & -c_1 q\bar{s} & -c_2 q\bar{u} & 0 & -\mu q\bar{t} \\ -c_1 s\bar{q} & p_2 & -c_2 s\bar{u} & -c_3 s\bar{t} & 0 \\ -c_2 u\bar{q} & -c_2 u\bar{s} & p_3 & -c_3 u\bar{t} & 0 \\ 0 & -c_3 t\bar{s} & -c_3 t\bar{u} & p_4 & 0 \\ -\mu t\bar{q} & 0 & 0 & 0 & p_5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix} \geq 0 \qquad (4.3.15)$$

where,

$$\begin{aligned} p_1 &= \mu^{-r}(|q| + |s|c_1\mu^r + |u|c_2\mu^r + c_3|t|\mu^r) \\ p_2 &= \mu^{-r}(|s| + |u|c_1\mu^r + c_2|t|\mu^r + |q|c_3\mu^r) \\ p_3 &= \mu^{-r}(|u| + c_1|t|\mu^r + |q|c_2\mu^r + |s|c_3\mu^r) \\ p_4 &= \mu^{-r}(|q| + |s| + |u| + |t|) \\ p_5 &= \mu^{-r}(|t| + |q|c_1\mu^r + |s|c_2\mu^r + |u|c_3\mu^r) \end{aligned}$$

for every $v_1, v_2, v_3, v_4, v_5 \in \mathbb{R}$ and $q, s, u, t \in \mathbb{C}$.

Taking $q = s = u = 0$,

$$c_3|t|v_1^2 + c_2|t|v_2^2 + c_1|t|v_3^2 + \mu^{-r}|t|v_4^2 + \mu^{-r}|t|v_5^2 \geq 0.$$

If $q = 0$, given that $0 < \mu \leq 1$. Then,

$$
\begin{aligned}
& (c_1 + c_2 + c_3|t|)v_1^2 + \mu^{-r}(1 + c_1\mu^r + c_2|t|\mu^r)v_2^2 + \mu^{-r}(1 + c_1\mu^r + c_3\mu^r)v_3^2 \\
& +\mu^{-r}(2 + |t|)v_4^2 + \mu^{-r}(|t| + c_2\mu^r + c_3\mu^r)v_5^2 \\
& -2c_2v_2v_3 - 2c_3\mathrm{Re}(t)v_2v_4 - 2c_3\mathrm{Re}(t)v_3v_4 \\
= \; & (c_1 + c_2 + c_3|t|)v_1^2 + c_1v_2^2 + \mu^{-r}v_3^2 + 2\mu^{-r}v_4^2 + \mu^{-r}(|t| + c_2\mu^r \\
& +c_3\mu^r)v_5^2 + c_2(|t|v_2 - v_3)^2 + (\frac{c_1}{c_2} - 1)v_2^2 + \mu^{-r}(v_2 - \mu^r c_3\mathrm{Re}(t)v_4)^2 \\
& +(\mu^{-r}\frac{|t|}{2} - \mu^r c_3^2\mathrm{Re}(t)^2)v_4^2 + c_3(v_3 - \mathrm{Re}(t)v_4)^2 + (\mu^{-r}\frac{|t|}{2} - c_3\mathrm{Re}(t)^2)v_4^2
\end{aligned}
$$

is positive by inequality (4.3.10) and (4.3.12).

If $s = 0$. Since $0 < \mu \leq 1$. Then,

$$
\begin{aligned}
& \mu^{-r}(1 + c_2\mu^r + c_3|t|\mu^r)v_1^2 + (c_1 + c_2|t| + c_3)v_2^2 + \mu^{-r}(1 + c_1|t|\mu^r + c_2\mu^r)v_3^2 \\
& +\mu^{-r}(2 + |t|)v_4^2 + \mu^{-r}(|t| + c_1\mu^r + c_3\mu^r)v_5^2 \\
& -2v_2v_3c_2 - 2v_3v_4c_3\mathrm{Re}(t) - 2v_1v_5\mu\mathrm{Re}(t) \\
= \; & c_3|t|v_1^2 + (c_1 + c_2|t| + c_3)v_2^2 + c_1|t|v_3^2 + 2\mu^{-r}v_4^2 + (c_1 + c_3)v_5^2 + c_2(v_1 - v_3)^2 \\
& +\mu^{-r}(v_3 - \mu^r c_3\mathrm{Re}(t)v_4)^2 + (\mu^{-r}|t| - \mu^r c_3^2\mathrm{Re}(t)^2)v_4^2 \\
& +\mu^{-r}(v_1 - \mu^{1+r}\mathrm{Re}(t)v_5)^2 + (|t|\mu^{-r} - \mu^{2+r}\mathrm{Re}(t)^2)v_5^2
\end{aligned}
$$

is positive when the inequality (4.3.12) holds.

If $u = 0$ and $0 < \mu \leq 1$. Then,

$$\begin{aligned}
&\mu^{-r}(1 + c_1\mu^r + c_3|t|\mu^r)v_1^2 + \mu^{-r}(1 + c_2|t|\mu^r + c_3\mu^r)v_2^2 \\
&+(c_1|t| + c_2 + c_3)v_3^2 + \mu^{-r}(2 + |t|)v_4^2 + \mu^{-r}(|t| + c_1\mu^r \\
&+c_2\mu^r)v_5^2 - 2c_1v_1v_2 - 2c_3\mathrm{Re}(t)v_2v_4 - 2\mu\mathrm{Re}(t)v_1v_5 \\
= \;& c_3|t|v_1^2 + \mu^{-r}(1 + c_2|t|)v_2^2 + (c_1|t| + c_2 + c_3)v_3^2 + 2\mu^{-r}v_4^2 + (c_1 + c_2)v_5^2 \\
&+c_1(v_1 - v_2)^2 + (\mu^{-r} - c_1)v_2^2 + c_3(v_2 - \mathrm{Re}(t)v_4)^2 + (\mu^{-r}|t| - c_3\mathrm{Re}(t)^2)v_4^2 \\
&+\mu^{-r}(v_1 - \mu^{1+r}\mathrm{Re}(t)v_5)^2 + (|t|\mu^{-r} - \mu^{2+r}\mathrm{Re}(t)^2)v_5^2
\end{aligned}$$

is positive when the inequalities (4.3.10) and (4.3.11) are satisfied.

Now if q, s and u are not equal to zero. Assume that $q = s = u = 1$. Then, by Lemma 4.5

$$z^T\psi_{(\mu,c_1,c_2,c_3)}\left(\begin{pmatrix}1\\1\\1\\t\end{pmatrix}\begin{pmatrix}1 & 1 & 1 & \bar{t}\end{pmatrix}\right)z$$

is positive for every $v = (v_1, v_2, v_3, v_4, v_5) \in \mathbb{R}^5$ and $t \in \mathbb{C}$ □

4.4 Complete (co)positivity of the linear maps

4.4.1 k-positivity and complete positivity

Here we show that, if a positive linear map ψ from $\mathcal{M}_n$ to $\mathcal{M}_m$ is k-positive where, $k \leq min\{n, m\}$, then the map is completely positive.

Proposition 4.7. *Let $\psi : \mathcal{M}_n \longrightarrow \mathcal{M}_m$ be a positive linear map and $k \leq min\{n, m\}$. Then the following are equivalent.*

(i). ψ is k-positive.

(ii). The block matrix $[\psi(E_{ij})]_{i,j=1}^{k} \geq 0$, where (E_{ij}) are matrix units in $\mathcal{M}_n$.

Proof. (i)$\Rightarrow$ (ii)
Let ψ be k-positive, then it is clear that the map $(\imath_k \otimes \psi) : \mathcal{M}_k \otimes \mathcal{M}_n \to \mathcal{M}_k \otimes \mathcal{M}_m$ is positive. Since the matrix $[E_{ij}]_{jl=1}^{k} \geq 0$,

$$(\imath_k \otimes \psi)[E_{ij}]_{jl=1}^{k} = [\psi(E_{ij})]_{i,j=1}^{k} \tag{4.4.1}$$

is positive.
(ii)$\Rightarrow$ (i)
Let the block matrix $[\psi(E_{ij})]_{i,j=1}^{k}$ be positive and let the standard basis for $\mathbb{C}^n$ be given by e_i so that the set $E_{ij} = e_i e_j$ is a basis for $\mathcal{M}_k$.
Now let $[E_{ij}]_{ij=1}^{k} = E_{ij} \otimes E_{ij} \in \mathcal{M}_k \otimes \mathcal{M}_m$, there exist $V_1, \ldots, V_r \in \mathcal{M}_{nm}$ so that;

$$\begin{aligned}
[\psi(E_{ij})] &= V_r^*[E_{ij}]V_r \\
&= (I_k \otimes V^*)(E_{ij} \otimes E_{ij})(I_k \otimes V) \\
&= (E_{ij} \otimes V^* E_{ij})(I_k \otimes V) \\
&= E_{ij} \otimes V^* E_{ij} V \\
&= \sum_{ij=1}^{k} (E_{ij} \otimes \psi([E_{ij}])) \\
&= (I_k \otimes \psi)(E_{ij} \otimes [E_{ij}]) \\
&= (E_{ij} \otimes \psi([E_{ij}])).
\end{aligned}$$

□

Proposition 4.8. *Let $\psi : \mathcal{M}_n \longrightarrow \mathcal{M}_m$ be a positive linear map and $k \leq min\{n, m\}$. Then the following are equivalent.*

(i.) ψ is k-copositive.

(ii.) The block matrix $[\psi(E_{ji})]_{i,j=1}^{k}$ is positive, where (E_{ij}) are matrix units in $\mathcal{M}_n$.

Proof. (i)$\Rightarrow$ (ii)

Let ψ be k-copositive, then the map $(\imath_k \otimes \psi) : \mathcal{M}_k \otimes \mathcal{M}_n \to \mathcal{M}_k \otimes \mathcal{M}_m$ is positive. Since the matrix $[E_{ij}]_{i,j=1}^{k}$ is positive its transpose matrix $[E_{ji}]_{j,l=1}^{k}$ is positive in $\mathcal{M}_n$,

$$(\imath_k \otimes \psi)[E_{ji}]_{j,l=1}^{k} = [\psi(E_{ji})]_{j,i=1}^{k}$$

is positive.

(ii)$\Rightarrow$ (i)

We show that the map $(I_k \otimes \psi)$ is positive if the block matrix $[\psi(E_{ji})]_{j,i=1}^{k}$ is positive. Let the block matrix $[\psi(E_{ji})]_{ij,i=1}^{k}$ be positive and let the standard basis for $\mathbb{C}^n$ be given by e_i so that the set $E_{ji} = (e_i e_j)^T$ is a basis for $\mathcal{M}_k$.

Now let $[E_{ji}]_{j,i=1}^{k} = E_{ij} \otimes E_{ji} \in \mathcal{M}_k \otimes \mathcal{M}_m$, there exist $V_1, \ldots, V_r \in \mathcal{M}_{nm}$ so that;

$$\begin{aligned}
[\psi(E_{ji})] &= V_r^*[E_{ji}]V_r \\
&= (I_k \otimes V^*)(E_{ji} \otimes E_{ij})(I_k \otimes V) \\
&= (E_{ij} \otimes V^* E_{ji})(I_k \otimes V) \\
&= E_{ij} \otimes V^* E_{ji} V \\
&= \sum_{ij=1}^{k} (E_{ij} \otimes \psi([E_{ji}])) \\
&= (I_k \otimes \psi)(E_{ij} \otimes [E_{ji}]).
\end{aligned}$$

$\square$

4.4.2 Characterization of the structure of the Choi matrices for 2−positive maps

Let the $\psi : \mathcal{M}_n \longrightarrow \mathcal{M}_{n+1}$ be a linear positive map where $n \geq 1, 2, 3, \dots$. We define the Choi matrix of these linear maps as 2×2 block matrix with (double line) partitions of the form;

$$C_\psi = \left(\begin{array}{c|ccccc||c|ccccc}
a_{11} & c_{11} & c_{12} & \dots & \dots & c_{1m} & 0 & y_{11} & y_{12} & \dots & \dots & y_{1m} \\ \hline
\bar{c}_{11} & b_{11} & 0 & \dots & \dots & 0 & \bar{z}_{11} & t_{11} & t_{12} & \dots & \dots & t_{1m} \\
\bar{c}_{21} & 0 & b_{22} & \ddots & & \vdots & \bar{z}_{21} & t_{21} & t_{22} & \dots & \dots & t_{21m} \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & & \ddots & \ddots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\bar{c}_{m1} & 0 & \dots & \dots & 0 & b_{mm} & \bar{z}_{m1} & t_{m1} & t_{m2} & \dots & \dots & t_{mm} \\ \hline\hline
0 & 0 & z_{11} & z_{12} & \dots & z_{1k} & d_{11} & f_{11} & f_{12} & \dots & \dots & f_{1m} \\ \hline
\bar{y}_{11} & \bar{t}_{11} & \bar{t}_{12} & \dots & \dots & \bar{t}_{1m} & \bar{f}_{11} & u_{11} & u_{12} & \dots & \dots & u_{1m} \\
\bar{y}_{21} & \bar{t}_{21} & \bar{t}_{22} & \dots & \dots & \bar{t}_{2m} & \bar{f}_{21} & u_{21} & u_{22} & \dots & \dots & u_{2m} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\bar{y}_{m1} & \bar{t}_{m1} & \bar{t}_{m2} & \dots & \dots & \bar{t}_{mm} & \bar{f}_{m1} & u_{m1} & u_{m2} & \dots & \dots & u_{mm}
\end{array}\right) \tag{4.4.2}$$

which we represent as:

$$C_\psi = \left(\begin{array}{cc|cc}
a & C_{1\times m} & 0 & Y_{1\times m} \\
C^*_{m\times 1} & B_{m\times m} & Z^*_{m\times 1} & T_{m\times m} \\ \hline
0 & Z_{1\times m} & d & F_{1\times m} \\
Y^*_{m\times 1} & T^*_{m\times m} & F^*_{m\times 1} & U_{m\times m}
\end{array}\right) \tag{4.4.3}$$

where a, d are positive real numbers while B, U and T are positive semidefinite matrices in $\mathcal{M}_m$ and C, Y, Z are vectors in $\mathbb{C}^m$. By $\bar{c}_{ij}$ we denote the conjugate of $c_{ij} \in \mathbb{C}$.

Next we wish to characterize complete positivity and complete copositivity with respect to the Choi matrix (4.4.3). We begin by introducing a lemma that will aid in understanding the main propositions.

Lemma 4.9. *Let A and B be positive diagonal matrices of order n and k respectively. Then $M = \begin{pmatrix} A & C \\ C^* & B \end{pmatrix}$ is a positive matrix of order $n+k$ satisfying the matrix inequality $C^*AC \leq (\det A)B$.*

Proof. Let $M = \begin{pmatrix} A & c \\ c^* & b \end{pmatrix}$ where $A \in M_{n-1}$, $c \in \mathbb{C}^{n-1}$ and $b \in \mathbb{R}$. Since A is a positive diagonal matrix and by Theorem. 3.8 and Theorem. (3.6,

$$\det M = \det A. \det(b - c^* A^{-1} c) \leq \det A. \det(b - c^* \frac{A}{\det A} c)$$

but $\det M$ is positive therefore $b - c^* \frac{A}{\det A} c \geq 0$. Thus $c^*Ac \leq (\det A)b$.

Let $M = \begin{pmatrix} A & C \\ C^* & B \end{pmatrix}$ where $A \in M_{n-2}$, $C \in M_{2,n-2}$ and $B \in M_2$. By Theorem. 3.8, Theorem. 3.6 and Theorem. 3.9.

$$\det M = \det A. \det(B - C^* A^{-1} C) \leq \det A. \det(B - C^* \frac{A}{\det A} C) \geq 0$$

therefore $C^*AC \leq (\det A)B$.

Next let $n = k$. Because A is invertible. By Theorem. 3.6 and block positivity of matrices, $B - C^*A^{-1}C \geq 0$. So,

$$B - C^* A^{-1} C \leq B - C^* \frac{A}{\det A} C \geq 0$$

Now let $n < k$, writing M in form $\begin{pmatrix} A_n & C_{n\times k} \\ C^*_{k\times n} & B_k \end{pmatrix}$. A is a diagonal matrix, By

Theorem. 3.8, Theorem. 3.6 and Theorem. 3.9.

$$\begin{aligned}\det M &= (\det A_n).\det(B_k - C^*_{k\times n}A_n^{-1}C_{n\times k})\\ &\leq (\det A_n).\det(B_k - C^*_{k\times n}\frac{A_n}{\det A_n}C_{n\times k}) \geq 0\end{aligned}$$

which holds if and only if $C^*_{k\times n}A_nC_{n\times k} \leq (\det A)B_k$. □

Recall that ψ is completely positive if and only if $[\psi(x_{ij})]^k$ is block-positive. We describe the conditions for complete positivity(complete copositivity) of this map.

Proposition 4.10. *Let $\psi : \mathcal{M}_n \longrightarrow \mathcal{M}_{n+1}$ be a 2-positive map with the Choi matrix of the form 4.4.3. Then ψ is completely positive if the following conditions holds.*

(i). $Z = 0$.

(ii). $C^*C \leq aB$.

(iii). $F^*F \geq dU$.

(iv) $Y^*Y \leq aU$.

(v) if B is invertible, then $TB^{-1}T^* \leq U$.

Proof. Let L_1 be a linear subspace generated by the vectors e_1 and let L_2 be the subspace spanned by $e_2, \ldots, e_q$ so that $\mathbb{C}^q = L_1 \oplus L_2$. A vector $v \in \mathbb{C}^q$ can therefore be uniquely decomposed to $v = v^1 + v^2$ where $v^i \in L_i, i = 1, 2$. The Choi matrix ((4.4.3) is represented as operators. $B, T, U : L_2 \longrightarrow L_2$, $C, Y, Z : L_2 \longrightarrow L_1$, and $A, D : L_1 \longrightarrow L_1$. For any $v_1, v_2 \in \mathbb{C}^q$ the positivity of the Choi matrices is given by

$$\langle \begin{pmatrix} v_1 \\ \hline v_2 \end{pmatrix}, \left(\begin{array}{cc|cc} a & C_{1\times m} & 0 & Y_{1\times m} \\ C^*_{m\times 1} & B_{m\times m} & Z^*_{m\times 1} & T_{m\times m} \\ \hline 0 & Z_{1\times m} & d & F_{1\times m} \\ Y^*_{m\times 1} & T^*_{m\times m} & F^*_{m\times 1} & U_{m\times m} \end{array}\right) \begin{pmatrix} v_1 \\ \hline v_2 \end{pmatrix} \rangle \tag{4.4.4}$$

which generates the inequality,

$$\langle v_1, \left(\begin{smallmatrix} a & C \\ C^* & B \end{smallmatrix}\right) v_1\rangle + \langle v_2, \left(\begin{smallmatrix} d & F \\ F^* & U \end{smallmatrix}\right) v_2\rangle + \langle v_1, \left(\begin{smallmatrix} 0 & Y \\ Z^* & T \end{smallmatrix}\right) v_2\rangle + \langle v_2, \left(\begin{smallmatrix} 0 & Z \\ Y^* & T^* \end{smallmatrix}\right) v_1\rangle \geq 0. \tag{4.4.5}$$

which is equivalent to

$$\begin{aligned} &\langle v_1^{(1)}, av_1^{(1)}\rangle + \langle v_1^{(2)}, Bv_1^{(2)}\rangle + \langle v_2^{(1)}, dv_2^{(1)}\rangle + \langle v_2^{(2)}, Uv_2^{(2)}\rangle + 2Re\langle v_1^{(1)}, Cv_1^{(2)}\rangle \\ &+2Re\langle v_2^{(1)}, Fv_2^{(2)}\rangle + 2Re\langle v_1^{(1)}, Yv_2^{(1)}\rangle + 2Re\langle v_2^{(1)}, Zv_1^{(2)}\rangle + 2Re\langle v_1^{(2)}, Tv_2^{(2)}\rangle \geq 0 \end{aligned}$$

where $v_j = v_j^{(1)} + v_j^{(2)}$ for $j = 1, 2$, and $v_1^{(1)}, v_2^{(1)} \in L_1$ and $v_1^{(2)}, v_2^{(2)} \in L_2$.

Assume that $v_1^{(1)} = v_2^{(2)} = 0$ with $v_1^{(2)}$ an arbitrary element in L_2. This gives

$$\langle v_1^{(2)}, Bv_1^{(2)}\rangle + 2Re\langle v_2^{(1)}, Zv_1^{(2)}\rangle + \langle v_2^{(1)}, av_2^{(1)}\rangle \geq 0. \tag{4.4.6}$$

Letting $v_2^{(1)} = -\alpha Zv_1^{(2)}$ for some $\alpha \geq 0$,

$$\langle v_1^{(2)}, Bv_1^{(2)}\rangle + 2Re\langle -\alpha Zv_1^{(2)}, Zv_1^{(2)}\rangle + \langle -\alpha Zv_1^{(2)}, -d\alpha Zv_1^{(2)}\rangle \geq .0 \tag{4.4.7}$$

This simplify to

$$\langle v_1^{(2)}, Bv_1^{(2)}\rangle - \alpha||Zv_1^{(2)}||^2 + \alpha^2||Z||^2\langle v_1^{(2)}, dv_1^{(2)}\rangle \geq 0. \tag{4.4.8}$$

This holds for any $v_2^{(1)} \in L_2$ and $\alpha > 0$ only for $Z = 0$.

Next, assume that ψ is a 2-positive linear map. The Choi matrix, (4.4.3) can be represented as;

$$C_\psi = \left(\begin{array}{c|c} \psi(E_{11}) & \psi(E_{12}) \\ \hline \psi(E_{21}) & \psi(E_{22}) \end{array}\right) \geq 0. \tag{4.4.9}$$

By Remark 3.7. Let $a = 0$, then

$$\psi(E_{11}) = \begin{pmatrix} a & C \\ C^* & B \end{pmatrix} \geq 0$$

provided $C = 0$. However, if $a \neq 0$. Then $C^*C \leq aB$. It is clear that

$$\psi(E_{22}) = \begin{pmatrix} d & F \\ F^* & U \end{pmatrix}.$$

$\psi(E_{22}$ is positive whenever $F^*F \leq dU$.

By Remark 3.7. Let $a = 0$, then

$$\psi(E_{12}) = \begin{pmatrix} 0 & Y \\ Z^* & T \end{pmatrix} \geq 0$$

if $Z = 0$. It is clear that the submatrices $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ are positive for every $a, d \in \mathbb{R}^+$.

Finally, $T^*B^{-1}T \leq U$, so

$$\begin{pmatrix} B & T \\ T^* & U \end{pmatrix} \in \mathcal{M}_2 \otimes \mathcal{M}_m$$

is positive. □

Remark 4.11. The transposition in this case imply the Partial Positive transpose of the Choi matrix, which we denote as $C_\psi^\Gamma \in \mathcal{M}_n(\mathcal{M}_{n+1})$. That is,

$$C_\psi^\Gamma = \left(\begin{array}{cc|cc} a & C^*_{1\times m} & 0 & Z^*_{1\times m} \\ C_{m\times 1} & B_{m\times m} & Y_{m\times 1} & T^*_{m\times m} \\ \hline 0 & Y^*_{1\times m} & d & F_{1\times m} \\ Z_{m\times 1} & T_{m\times m} & F^*_{m\times 1} & U_{m\times m} \end{array}\right) \in \mathcal{M}_n(\mathcal{M}_{n+1}). \tag{4.4.10}$$

Proposition 4.12. *Let $\psi : \mathcal{M}_n \longrightarrow \mathcal{M}_{n+1}$ be a 2-copositive map with the Choi matrix* *of the form 4.4.3. Then ψ is completely copositive if the following conditions holds.*

(i). $Y = 0$.

(ii). $CC^* \leq aB$.

(iii). $FF^* \leq dU$.

(iv) $ZZ^* \leq aU$.

(v) if B is invertible, then $T^* B^{-1} T = U$.

Proof. Let L_1 be a linear subspace generated by the vectors e_1 and let L_2 be the subspace spanned by $e_2, \ldots, e_q$ so that $\mathbb{C}^q = L_1 \oplus L_2$. A vector $v \in \mathbb{C}^q$ can therefore be uniquely decomposed to $v = v^1 + v^2$ where $v^i \in L_i, i = 1, 2$. The Choi matrices ((4.4.3) are interpreted as operators. $B, T, U : L_2 \longrightarrow L_2$, $C, Y, Z : L_2 \longrightarrow L_1$, and $A, D : L_1 \longrightarrow L_1$. For any $v_1, v_2 \in \mathbb{C}^q$ the positivity of the Choi matrices is given by

$$\langle \left(\begin{array}{c} v_1 \\ \hline v_2 \end{array} \right), \left(\begin{array}{cc|cc} a & C^*_{1\times m} & 0 & Z^*_{1\times m} \\ C_{m\times 1} & B_{m\times m} & Y_{m\times 1} & T^*_{m\times m} \\ \hline 0 & Y^*_{1\times m} & d & F_{1\times m} \\ Z_{m\times 1} & T_{m\times m} & F^*_{m\times 1} & U_{m\times m} \end{array} \right) \left(\begin{array}{c} v_1 \\ \hline v_2 \end{array} \right) \rangle \tag{4.4.11}$$

which generates the inequality,

$$\langle v_1, \left(\begin{smallmatrix} a & C^* \\ C & B \end{smallmatrix} \right) v_1 \rangle + \langle v_2, \left(\begin{smallmatrix} d & F^* \\ F & U \end{smallmatrix} \right) v_2 \rangle + \langle v_1, \left(\begin{smallmatrix} 0 & Z^* \\ Y & T^* \end{smallmatrix} \right) v_2 \rangle + \langle v_2, \left(\begin{smallmatrix} 0 & Y^* \\ Z & T \end{smallmatrix} \right) v_1 \rangle \geq 0. \tag{4.4.12}$$

which is equivalent to

$$\langle v_1^{(1)}, a v_1^{(1)} \rangle + \langle v_1^{(2)}, B v_1^{(2)} \rangle + \langle v_2^{(1)}, d v_2^{(1)} \rangle + \langle v_2^{(2)}, U v_2^{(2)} \rangle + 2Re \langle v_1^{(1)}, C v_1^{(2)} \rangle$$
$$+ 2Re \langle v_2^{(1)}, F v_2^{(2)} \rangle + 2Re \langle v_1^{(1)}, Z v_2^{(1)} \rangle + 2Re \langle v_2^{(1)}, Y v_1^{(2)} \rangle + 2Re \langle v_1^{(2)}, T v_2^{(2)} \rangle \geq 0$$

where $v_j = v_j^{(1)} + v_j^{(2)}$ for $j = 1, 2$, and $v_1^{(1)}, v_2^{(1)} \in L_1$ and $v_1^{(2)}, v_2^{(2)} \in L_2$.

Assume that $v_1^{(1)} = v_2^{(2)} = 0$ and $v_1^{(2)}$ an arbitrary element in L_2. This gives

$$\langle v_1^{(2)}, Bv_1^{(2)} \rangle + 2Re\langle v_2^{(1)}, Yv_1^{(2)} \rangle + \langle v_2^{(1)}, Dv_2^{(1)} \rangle \geq 0. \tag{4.4.13}$$

Letting $v_2^{(1)} = -\alpha Y v_1^{(2)}$ for some $\alpha \geq 0$

$$\langle v_1^{(2)}, Bv_1^{(2)} \rangle + 2Re\langle -\alpha Y v_1^{(2)}, Yv_1^{(2)} \rangle + \langle -\alpha Y v_1^{(2)}, -D\alpha Y v_1^{(2)} \rangle \geq .0 \tag{4.4.14}$$

This simplify to

$$\langle v_1^{(2)}, Bv_1^{(2)} \rangle - \alpha ||Yv_1^{(2)}||^2 + \alpha^2 ||Y||^2 \langle v_1^{(2)}, Dv_1^{(2)} \rangle \geq 0 \tag{4.4.15}$$

which holds for any $v_2^{(1)} \in L_2$ and $\alpha > 0$ only for $Y = 0$.
Assume that ψ is a 2-copositive linear map. The partial transposition of Choi matrix, (4.4.3) can be represented as;

$$C_\psi^\Gamma = \left(\begin{array}{c|c} \psi(E_{11}) & \psi(E_{21}) \\ \hline \psi(E_{12}) & \psi(E_{22}) \end{array}\right) \geq 0. \tag{4.4.16}$$

By Lemma 4.9, the positivity of the Choi matrix imply that,

$$\psi(E_{11}) = \begin{pmatrix} a & C^* \\ C & B \end{pmatrix}$$

Let $a = 0$, then $\psi(E_{11}) \geq 0$ provided $C = 0$. However, if $a \neq 0$, then $CC^* \leq aB$.

$$\psi(E_{22}) = \begin{pmatrix} d & F^* \\ F & U \end{pmatrix}$$

It is clear that $\psi(E_{22}) \geq 0$ whenever $FF^* \leq dU$. By Remark 3.7. Let $a = 0$, then

$$\psi(E_{12}) = \begin{pmatrix} 0 & Z^* \\ Y & T \end{pmatrix} \geq 0$$

if $Y = 0$. It is clearly that the submatrices $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ are positive for every $a, d \in \mathbb{R}^+$.

$TB^{-1}T^* \leq U$, so $\begin{pmatrix} B & T \\ T^* & U \end{pmatrix} \in \mathcal{M}_2 \otimes \mathcal{M}_m$ is positive. □

4.4.3 Complete (co)positivity of linear maps $\psi_{(\mu,c_1)}$

Next we establish the conditions under which the map $\psi_{(\mu,c_1)}$ is completely positive (respectively completely copositive).

Proposition 4.13. *The linear map $\psi_{(\mu,c_1)} : \mathcal{M}_2 \longrightarrow \mathcal{M}_3$ is completely positive.*

Proof. The Choi matrix $C_{\psi_{(\mu,c_1)}}$ is;

$$\left(\begin{array}{c|cc||c|cc} \mu^{-r} & 0 & 0 & 0 & 0 & -\mu \\ \hline 0 & \mu^{-r} & 0 & 0 & 0 & 0 \\ 0 & 0 & c_1 & 0 & 0 & 0 \\ \hline\hline 0 & 0 & 0 & c_1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \mu^{-r} & 0 \\ -\mu & 0 & 0 & 0 & 0 & \mu^{-r} \end{array}\right) \tag{4.4.17}$$

$a = \mu^{-r}$ and which is defined when $\mu > 0$. $c_1 \geq 0$ since C is a zero vector as $aB \geq 0$.

$$\begin{aligned} aU - Y^*Y &= \mu^{-r}\begin{pmatrix} \mu^{-r} & 0 \\ 0 & \mu^{-r} \end{pmatrix} - \begin{pmatrix} 0 \\ -\mu \end{pmatrix}\begin{pmatrix} 0 & -\mu \end{pmatrix} \\ &= \begin{pmatrix} \mu^{-2r} & 0 \\ 0 & \mu^{-2r} - \mu^2 \end{pmatrix} \geq 0. \end{aligned}$$

This holds when, $\mu^{-2r} - \mu^2 \geq 0$ for all $r \geq 0$.

Z is zero vector therefore,

$$dB - Z^*Z = c_1 \begin{pmatrix} \mu^{-r} & 0 \\ 0 & c_1 \end{pmatrix} \geq 0.$$

Finally, the matrix $T = 0$ implying $U - TB^{-1}T^* = U \geq 0$. $\square$

Proposition 4.14. *The linear map $\psi_{(\mu,c_1)} : \mathcal{M}_2 \longrightarrow \mathcal{M}_3$ is completely copositive.*

Proof. Considering the positive semidefinite matrix X, the Choi matrix of $\psi_{(\mu,c_1)}(X^T)$ is

$$C^{\Gamma}_{\psi_{(\mu,c_1)}} = \left(\begin{array}{c|cc||c|cc} \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & c_1 & -\mu & \cdot & \cdot \\ \hline\hline \cdot & \cdot & -\mu & c_1 & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} \end{array}\right). \tag{4.4.18}$$

linear map. The Choi matrix $C_{\psi_{((\mu,c_1)}} \in \mathcal{M}_2(\mathcal{M}_3)$ is of the form (4.4.16). $a = \mu^{-r}$. It is positive for all $\mu > 0$. $c_1 \geq 0$ and C is a zero matrix as $aB \geq 0$.

$$\begin{aligned} aU - Z^*Z &= \mu^{-r}\begin{pmatrix} \mu^{-r} & 0 \\ 0 & \mu^{-r} \end{pmatrix} - \begin{pmatrix} 0 \\ -\mu \end{pmatrix}\begin{pmatrix} 0 & -\mu \end{pmatrix} \\ &= \begin{pmatrix} \mu^{-2r} & 0 \\ 0 & \mu^{-2r} - \mu^2 \end{pmatrix} \geq 0 \end{aligned}$$

since $\mu^{-2r} - \mu^2 \geq 0$ for all $r \geq 0$.

Since Y is zero vector,

$$dB - Y^*Y = c_1 \begin{pmatrix} \mu^{-r} & 0 \\ 0 & c_1 \end{pmatrix} \geq 0.$$

Finally, $U - TB^{-1}T^* = U \geq 0$ since the matrix $T = 0$. Therefore the matrix $\left(\begin{smallmatrix} B & T \\ T^* & U \end{smallmatrix}\right)$ is positive. □

4.4.4 Complete (co)positivity of linear maps $\psi_{(\mu,c_1,c_2)}$

Proposition 4.15. *Let $\psi_{(\mu,c_1,c_2)}$ be a linear map. Then the following are equivalent:*

(i) $\psi_{(\mu,c_1,c_2)}$ is completely positive,

(ii) $\psi_{(\mu,c_1,c_2)}$ is 2-positive and,

(iii) $\psi_{(\mu,c_1,c_2)}$ positive.

Proof. *(ii)* $\Rightarrow$ *(iii)*.

Assume $\psi_{(\mu,c_1,c_2)}$ is 2-positive. Applying Theorem 2.7, consider $P = [a_i a_j]$ be a positive element in $\mathcal{M}_2(\mathbb{C}) \otimes \mathcal{M}_4(\mathbb{C})$ where $a_i = (1,0,0,0,0,1,1,1)^T$, we have that

$$
\mathcal{I}_2 \otimes \psi_{(\mu,c_1,c_2)}(P) = \left(\begin{array}{c|ccc||c|ccc}
\mu^{-r} & . & . & . & . & -c_1 & . & -\mu \\
\hline
. & c_2 & . & . & . & . & . & . \\
. & . & \mu^{-r} & . & . & . & . & . \\
. & . & . & c_1 & . & . & . & . \\
\hline\hline
. & . & . & . & c_1 & . & . & . \\
\hline
-c_1 & . & . & . & . & \mu^{-r} & -c_2 & . \\
. & . & . & . & . & -c_2 & \mu^{-r} & . \\
-\mu & . & . & . & . & . & . & \mu^{-r}
\end{array}\right) \tag{4.4.19}
$$

in $\mathcal{M}_3(\mathcal{M}_4(\mathbb{C}))$ is positive semidefinite. By direct calculation of minors one can check that the conditions in Lemma 4.3 are necessary condition for 2-positivity of $\psi_{(\mu,c_1,c_2)}$.

$(iii) \Rightarrow (i)$.

Let $\psi_{(\mu,c_1,c_2)}$ be a positive map. By computing the choi matrix

$$
C_{\psi_{(\mu,c_1,c_2)}} = \left(\begin{array}{c|ccccc||c|ccccc}
\mu^{-r} & 0 & 0 & 0 & 0 & -c_1 & 0 & 0 & 0 & 0 & 0 & -\mu \\
\hline
0 & c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-c_1 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 & 0 & 0 & 0 & -c_2 & 0 \\
\hline\hline
0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -c_2 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 \\
-\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r}
\end{array}\right). \tag{4.4.20}
$$

in $\mathcal{M}_3(\mathcal{M}_4(\mathbb{C}))$. Since $a \geq 0$,

$$aB - C^*C = \begin{pmatrix} c_2\mu^{-r} & 0 & 0 & 0 & 0 \\ 0 & \mu^{-2r} & 0 & 0 & 0 \\ 0 & 0 & c_1\mu^{-r} & 0 & 0 \\ 0 & 0 & 0 & c_1\mu^{-r} & 0 \\ 0 & 0 & 0 & 0 & \mu^{-2r} - c_1^2 \end{pmatrix}$$

is positive when

$$\mu^{-2r} \geq c_1^2. \tag{4.4.21}$$

$$aU - Y^*Y = \begin{pmatrix} c_2\mu^{-r} & 0 & 0 & 0 & 0 \\ 0 & c_2\mu^{-r} & 0 & 0 & 0 \\ 0 & 0 & c_1c_2\mu^{-r} & 0 & 0 \\ 0 & 0 & 0 & c_2\mu^{-r} & 0 \\ 0 & 0 & 0 & 0 & \mu^{-2r} - \mu^2 \end{pmatrix}.$$

This is positive when

$$\mu^{-2r} \geq \mu^2. \tag{4.4.22}$$

$$U - T^*B^{-1}T = \begin{pmatrix} c_2 & 0 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 & 0 \\ 0 & 0 & c_1 & 0 & 0 \\ 0 & 0 & 0 & \mu^{-r} - c_2^2\mu^r & 0 \\ 0 & 0 & 0 & 0 & \mu^{-r} \end{pmatrix} \geq 0.$$

This is positive when

$$\mu^{-r} \geq c_2. \tag{4.4.23}$$

Since (iii) is satisfied, the inequalities (4.4.21) and (4.4.23) holds, and consequently

$C_{\psi_{(\mu,c_1,c_2)}}$ is positive definite. Hence, complete positivity of $\psi_{(\mu,c_1,c_2)}$ follows.
$(i) \Rightarrow (ii)$ Let $\psi_{(\mu,c_1,c_2)}$ be a completely positive map. From the matrices (4.4.19) and (4.4.20) we have that,

$$\begin{vmatrix} \mu^{-r} & -c_1 & . & \mu \\ -c_1 & \mu^{-r} & -c_2 & . \\ . & -c_2 & \mu^{-r} & . \\ \mu & . & . & \mu \end{vmatrix} \geq 0 \tag{4.4.24}$$

Thus $\psi_{(\mu,c_1,c_2)}$ is 2-positive. □

Proposition 4.16. *Let $\psi_{(\mu,c_1,c_2)}$ be a map. Then following conditions are equivalent:*

(i) $\psi_{(\mu,c_1,c_2)}$ *is 2-copositive,*

(ii) $\psi_{(\mu,c_1,c_2)}$ *is completely copositive.*

(iii) $c_2 \geq c_1, \quad c_1\mu^{-r} \geq c_2^2 \quad$ *and* $\quad c_1 \geq \mu$.

Proof. $(iii) \Rightarrow (i)$
Assume the set of inequalities

$$c_2 \geq c_1, \quad c_1\mu^{-r} \geq c_2^2 \quad \text{and} \quad c_1 \geq \mu \tag{4.4.25}$$

hold. Consider P as in Proposition 4.15. We have that the 2-copositivity matrix,

$$\tau_2 \otimes \psi_{(\mu,c_1,c_2)}(P) = \left(\begin{array}{c|ccc||c|ccc}
\mu^{-r} & . & . & . & . & . & . & . \\
\hline
. & c_2 & . & . & -c_1 & . & . & . \\
. & . & \mu^{-r} & . & . & . & . & . \\
. & . & . & c_1 & -\mu & . & . & . \\
\hline\hline
. & -c_1 & . & -\mu & c_1 & . & . & -c_2 \\
\hline
. & . & . & . & . & \mu^{-r} & . & . \\
. & . & . & . & . & . & \mu^{-r} & . \\
. & . & . & . & -c_2 & . & . & \mu^{-r}
\end{array}\right) \tag{4.4.26}$$

in $\mathcal{M}_3(\mathcal{M}_4(\mathbb{C}))$ is positive semidefinite with all the principal minors positive semidefinite.

$(i) \Rightarrow (ii)$

By computation the choi matrix,

$$
C^{\Gamma}_{\psi_{(\mu,c_1,c_2)}} = \left(\begin{array}{c|ccccc||c|ccccc}
\mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & c_2 & 0 & 0 & -c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 \\
0 & -c_1 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline\hline
0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 & 0 & -c_2 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -c_2 & 0 & 0 & c_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r}
\end{array}\right) \tag{4.4.27}
$$

in $\mathcal{M}_3(\mathcal{M}_4(\mathbb{C}))$.

Assume $\psi_{((\mu,c_1,c_2)}$ is a 2-copositive linear map. Since $a \geq 0$ and C is a zero matrix,

$$
aB - C^*C \;=\; \mu^{-r} \begin{pmatrix} c_2 & 0 & 0 & -c_1 & 0 \\ 0 & \mu^{-r} & 0 & 0 & 0 \\ 0 & 0 & c_1 & 0 & 0 \\ -c_1 & 0 & 0 & c_1 & 0 \\ 0 & 0 & 0 & 0 & \mu^{-r} \end{pmatrix}
$$

is positive when $c_2 - c_1 \geq 0$.

Since Z is a zero vector, and U is a diagonal matrix with all positive entries,

$aU - Z^*Z \geq 0.$

$$dU - F^*F = \begin{pmatrix} \mu^{-r}c_2 & 0 & 0 & 0 & 0 \\ 0 & \mu^{-r}c_1 & 0 & 0 & 0 \\ 0 & 0 & \mu^{-r}c_1 - c_2^2 & 0 & 0 \\ 0 & 0 & 0 & \mu^{-2r} & 0 \\ 0 & 0 & 0 & 0 & \mu^{-2r} \end{pmatrix} \geq 0.$$

is positive when $\mu^{-r}c_1 \geq c_2^2$.

Finally,

$$U - T^*B^{-1}T = \begin{pmatrix} c_2 & -c_2 & 0 & 0 & 0 \\ -c_2 & c_2 & 0 & 0 & 0 \\ 0 & 0 & c_1 - \frac{\mu^2}{c_1} & 0 & 0 \\ 0 & 0 & 0 & \mu^{-r} & 0 \\ 0 & 0 & 0 & 0 & \mu^{-r} \end{pmatrix}$$

is positive when, $c_1 \geq \mu$.

Since set of inequalities in (4.4.25) are satisfied, $C^{\Gamma}_{\psi_{(\mu,c_1,c_2)}}$ is positive semidefinite.

$(ii) \Rightarrow (i)$ It is clear that if $C^{\Gamma}_{\psi_{(\mu,c_1,c_2)}}$ is positive semidefinite the the set of inequalities (4.4.25) hold. $\square$

4.4.5 Completely (co)positivity of $\psi_{((\mu,c_1,c_2,c_3)}$

Proposition 4.17. *Let $\psi_{((\mu,c_1,c_2,c_3)}$ be a positive map. Then following conditions are equivalent:*

(i) $\psi_{(\mu,c_1,c_2,c_3)}$ *is 2-positive.*

(ii) $\psi_{(\mu,c_1,c_2,c_3)}$ *is completely positive.*

Proof. $(i) \Rightarrow (iii)$.

Assume $\psi_{(\mu,c_1,c_2,c_3)}$ is 2-positive. The Choi matrix $C_{\psi_{(\mu,c_1,c_2,c_3)}}$ is;

$$\left(\begin{array}{c|ccccccccc||c|ccccccccc}
\mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & -c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & -c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\mu \\ \hline
\cdot & c_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & -c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & -c_3 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & -c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & -c_3 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_3 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_3 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -c_3 & \cdot & \cdot & \cdot & \cdot & \cdot & -c_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot \\
-\mu & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r}
\end{array}\right). \tag{4.4.28}$$

Since $a \geq 0$,

$$aB - C^*C = \begin{pmatrix}
c_3\mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & c_2\mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu^{-2r} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_1\mu^{-r} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_1\mu^{-r} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mu^{-2r}-c_1^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_3\mu^{-r} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-2r} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_2\mu^{-r}
\end{pmatrix}.$$

The inequality holds when

$$\mu^{-r} > c_1. \tag{4.4.29}$$

$$dU = c_2 \begin{pmatrix} c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 \\ 0 & 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 \\ 0 & -c_3 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} \end{pmatrix}.$$

The inequality $dU \geq 0$ holds when

$$\mu^{-r} > c_3. \tag{4.4.30}$$

$$aU - Y^*Y = \begin{pmatrix} c_1\mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu^{-2r} - c_2^2 & 0 & 0 & 0 & 0 & 0 & -c_3\mu^{-r} & 0 \\ 0 & 0 & \mu^{-2r} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_3\mu^{-r} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_3\mu^{-r} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_2\mu^{-r} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_1\mu^{-r} & 0 & 0 \\ 0 & -c_3\mu^{-r} & 0 & 0 & 0 & 0 & 0 & \mu^{-2r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-2r} - \mu^2 \end{pmatrix}$$

is positive whenever

$$\mu^{-2r} \geq c_2^2 + c_3^2. \tag{4.4.31}$$

From positivity of $\psi_{(\mu,c_1,c_2,c_3)}$ this holds since $\mu^{-r} \geq c_2$ and $\mu^{-r} > c_3$. $U - T^*BT$

$$= \begin{pmatrix} c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 \\ 0 & 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 \\ 0 & -c_3 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -c_2 & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} \frac{1}{c_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{c_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu^{r} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{c_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{c_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu^{r} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{c_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{c_2} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -c_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -c_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 \\ 0 & 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_1 - c_2^2\mu^r - c_3^2\mu^r & 0 & 0 \\ 0 & -c_3 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} \end{pmatrix}.$$

All the principal minors of $U - TB^{-1}T$ are positive whenever $\mu^{-r} \geq c_3$ and

$$c_1\mu^{-r} - (c_2^2 + c_3^2) > 0. \tag{4.4.32}$$

From positivity of $\psi_{(\mu,c_1,c_2,c_3)}$ this holds since $\mu^{-r} > c_2$ and $c_1\mu^{-r} \geq c_3^2$. Thus $\psi_{(\mu,c_1,c_2,c_3)}$ is completely positive.

$(i) \Rightarrow (ii)$.

Assume $\psi_{(\mu,c_1,c_2,c_3)}$ is completely positive. Since a completely positive linear map is positive, consider a rank one matrix $P = [x_i x_j]$ a positive element in $\mathcal{M}_2(\mathcal{M}_5(\mathbb{C}))$ where $x_i = (1,1,0,0,1,1,0,0,1,1)^T$,. We have that

$$\mathcal{I}_2 \otimes \psi_{(\mu,c_1,c_2,c_3)}(P) = \left(\begin{array}{ccccc|ccccc} \mu^{-r} & . & . & . & . & . & -c_1 & -c_2 & . & -\mu \\ . & c_3 & . & . & . & . & . & . & . & . \\ . & . & c_2 & . & . & . & . & . & . & . \\ . & . & . & \mu^{-r} & . & . & . & . & . & . \\ . & . & . & . & c_1 & . & . & . & . & . \\ \hline . & . & . & . & . & c_1 & . & . & . & . \\ -c_1 & . & . & . & . & . & \mu^{-r} & -c_2 & -c_3 & . \\ -c_2 & . & . & . & . & . & -c_2 & \mu^{-r} & -c_3 & . \\ . & . & . & . & . & . & -c_3 & -c_3 & \mu^{-r} & . \\ -\mu & . & . & . & . & . & . & . & . & \mu^{-r} \end{array}\right) \tag{4.4.33}$$

in $\mathcal{M}_2(\mathcal{M}_5(\mathbb{C}))$. From the matrices (4.4.28) and (4.4.33),

$$\begin{vmatrix} \mu^{-r} & -c_1 & -c_2 & . & \mu \\ -c_1 & \mu^{-r} & -c_2 & -c_3 & . \\ -c_2 & -c_2 & \mu^{-r} & -c_3 & . \\ . & -c_3 & -c_3 & \mu^{-r} & . \\ \mu & . & . & . & \mu \end{vmatrix} \geq 0 \tag{4.4.34}$$

provided

$$\mu^{-r} > c_1, \quad \mu^{-r} > c_2 \quad \text{and} \quad \mu^{-r} \geq c_3. \tag{4.4.35}$$

Thus $\psi_{(\mu,c_1,c_2,c_3)}$ is 2-positive. $\square$

Proposition 4.18. *Let $\psi_{((\mu,c_1,c_2,c_3)}$ be a positive map. Then following conditions are equivalent:*

(i) *$\psi_{(\mu,c_1,c_2,c_3)}$ is completely copositive.*

(ii) *$\psi_{(\mu,c_1,c_2,c_3)}$ is 2-copositive.*

Proof. *(i)* $\Rightarrow$ *(ii)*. Assume $\psi_{(\mu,c_1,c_2,c_3)}$ is completely copositive. Since a completely copositive map is positive, consider a rank one matrix P an element in $\mathcal{M}_2(\mathcal{M}_5(\mathbb{C}))$ where $x_i = (1,1,0,0,1,1,0,0,1,1)^T$, we have that,

$$\tau_2 \otimes \psi_{(\mu,c_1,c_2,c_3)}(P) = \left(\begin{array}{ccccc|ccccc} \mu^{-r} & . & . & . & . & . & . & . & . & . \\ . & c_3 & . & . & . & -c_1 & . & . & . & . \\ . & . & c_2 & . & . & -c_2 & . & . & . & . \\ . & . & . & \mu^{-r} & . & . & . & . & . & . \\ . & . & . & . & c_1 & -\mu & . & . & . & . \\ \hline . & -c_1 & -c_2 & . & -\mu & c_1 & . & . & . & . \\ . & . & . & . & . & . & \mu^{-r} & -c_2 & -c_3 & . \\ . & . & . & . & . & . & -c_2 & \mu^{-r} & -c_3 & . \\ . & . & . & . & . & . & -c_3 & -c_3 & \mu^{-r} & . \\ . & . & . & . & . & . & . & . & . & \mu^{-r} \end{array}\right)$$

in $\mathcal{M}_2(\mathcal{M}_5(\mathbb{C}))$. By computation of the minors, $\mathcal{I}_2 \otimes \psi_{(\mu,c_1,c_2,c_3)}(P)$ is positive semidefinite on condition that;

$$\mu^{-r} > c_1, \quad \mu^{-r} > c_2 \quad \mu^{-r} \geq 2c_3 \quad c_3 \geq c_1 \quad \textit{and} \quad c_1 \geq c_2 \tag{4.4.36}$$

holds. Thus $\psi_{(\mu,c_1,c_2,c_3)}$ is 2-copositive.

$(ii) \Rightarrow (i)$. Assume $\psi_{(\mu,c_1,c_2,c_3)}$ is 2-copositive. By computation the choi matrix is,

$$
C^{\Gamma}_{\psi_{(\mu,c_1,c_2,c_3)}} = \left(\begin{array}{c|ccccccccc||c|ccccccccc}
\mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & c_3 & \cdot & \cdot & \cdot & -c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\mu & \cdot & \cdot & \cdot & \cdot \\
\cdot & -c_1 & \cdot & \cdot & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_3 & \cdot & \cdot & \cdot & -c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -c_3 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline\hline
\cdot & \cdot & -c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -c_2 & \cdot & \cdot & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & -c_3 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_3 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & -\mu & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_3 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -c_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -c_3 & \cdot & \cdot & \cdot & c_1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r}
\end{array}\right).
$$

Since $a \geq 0$ and $C = 0$.

$$aB - CC^* = \mu^{-r}\begin{pmatrix} c_3 & 0 & 0 & 0 & -c_1 & 0 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 \\ -c_1 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_2 \end{pmatrix}$$

The inequality holds when $c_3 \geq c_1$.
Since F is a zero matrix, $dU - FF^*$ is positive when the inequality $c_1\mu^{-r} > c_3^2$ holds.

$$aU - ZZ^* = \begin{pmatrix} \mu^{-r}c_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu^{-2r} - c_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu^{-2r} & 0 & 0 & 0 & -c_3\mu^{-r} & 0 & 0 \\ 0 & 0 & 0 & \mu^{-r}c_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu^{-r}c_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu^{-r}c_2 & 0 & 0 & 0 \\ 0 & 0 & -c_3\mu^{-r} & 0 & 0 & 0 & \mu^{-r}c_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-2r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-2r} \end{pmatrix}.$$

The matrix is positive when the inequalities $\mu^{-2r} - c_1^2$ and $c_2\mu^{-r} > c_2^2$ holds.
Finally,

$U - TB^{-1}T^*$

$$= \begin{pmatrix} c_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu^{-r} & 0 & 0 & 0 & -c_3 & 0 & 0 \\ 0 & 0 & 0 & c_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 \\ 0 & 0 & -c_3 & 0 & 0 & 0 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -c_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} c_3 & 0 & 0 & 0 & -c_1 & 0 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 \\ -c_1 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -c_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T$$

$$= \begin{pmatrix} c_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu^{-r} & 0 & 0 & 0 & -c_3 & 0 & 0 \\ 0 & 0 & 0 & c_3 & 0 & 0 & \frac{-c_2\mu}{c_1+c_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 \\ 0 & 0 & -c_3 & 0 & 0 & 0 & \frac{c_1^2+c_1c_3-c_2^2}{c_1+c_3} & 0 & 0 \\ 0 & 0 & 0 & -c_3\mu^{1+r} & 0 & 0 & 0 & \mu^{-r}-c_3^2\mu^r & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} \end{pmatrix}.$$

The matrix $U - TB^{-1}T^*$ is positive provided the inequalities,

$$\mu^{-r} > c_3, \quad c_1 \geq c_2 \quad \text{and} \quad c_1\mu^{-r} > c_3^2$$

holds. Thus the set of inequalities 4.4.36 are satisfied and complete positivity of $\psi(\mu, c_1, c_2, c_3)$ follows. $\square$

4.5 Decomposability of positive maps

The result of Choi [15] shows that a positive map ψ from $\mathcal{M}_n$ to $\mathcal{M}_m$ is k-decomposable if there are positive maps ψ_1 and ψ_2 from $\mathcal{M}_n$ to $\mathcal{M}_m$, where ψ_1 is k-positive and ψ_2 is k-copositive and such that $\psi = \psi_1 + \psi_2$. We start with an example of a decomposable linear map $\psi_{(\alpha,\eta)}$ from $\mathcal{M}_2$ to $\mathcal{M}_2$ which is $2-$positive and completely positive with the Choi matrix in the form (4.4.3).

4.5.1 Decomposability by Choi matrices

Example 4.19. Let $-\frac{1}{2} \leq \eta \leq \frac{1}{2}$ be a real number and ψ_η be a linear map from $\mathcal{M}_2(\mathbb{C})$ to $\mathcal{M}_2(\mathbb{C})$ defined by

$$\psi_\eta(X) = \begin{pmatrix} \alpha_1 & \eta(x_1\bar{x}_2 + x_2\bar{x}_1) \\ \eta(x_2\bar{x}_1 + x_1\bar{x}_2) & \alpha_2 \end{pmatrix}$$

where $\alpha_1 = \bar{x}_1^* x_1$ and $\alpha_2 = \bar{x}_2^* x_2$.

The Choi matrix,

$$C_{\psi_\eta} = \left(\begin{array}{cc||cc} 1 & . & . & \eta \\ \hline . & . & \eta & . \\ \hline\hline . & \eta & . & . \\ \hline \eta & . & . & 1 \end{array}\right) \tag{4.5.1}$$

is completely positive. We observe that the Choi matrix $C_{\psi_\eta} = C_{\psi_\eta}^\Gamma$ and is of the form (4.4.3).

$$C_{\psi_{1\eta}} = \left(\begin{array}{c|c||c|c} 1-\rho & . & . & \eta \\ \hline . & . & . & . \\ \hline\hline . & . & . & . \\ \hline \eta & . & . & 1-\rho \end{array}\right) \text{ and } C_{\psi_{2\eta}} = \left(\begin{array}{c|c||c|c} \rho & . & . & . \\ \hline . & . & \eta & . \\ \hline\hline . & \eta & . & . \\ \hline . & . & . & \rho \end{array}\right).$$

The linear map is decomposable when $0 \leq \rho < 1$ and $(1-\rho)^2 > \eta^2$. $C_{\psi_1} \geq 0$ when $Z = 0$ and $C_{\psi_{2\eta}} \geq 0$ when $Y = 0$. Thus the Choi matrix is decomposable with

$$C_{\psi_\eta} = \left(\begin{array}{c|c||c|c} 1-\rho & . & . & \eta \\ \hline . & . & . & . \\ \hline\hline . & . & . & . \\ \hline \eta & . & . & 1-\rho \end{array}\right) + \left(\begin{array}{c|c||c|c} \rho & . & . & . \\ \hline . & . & \eta & . \\ \hline\hline . & \eta & . & . \\ \hline . & . & . & \rho \end{array}\right)$$

Next, we look at the decomposability of the positie maps $\psi_{(\mu,c_1,\ldots,c_{n-1})}$.

Decomposability of linear maps $\psi_{(\mu,c_1)}$

Proposition 4.20. *The positive map $\psi_{(\mu,c_1)}$ is decomposable.*

Proof. Assume $\psi_{(\mu,c_1)}$ is 2-positive. Let $c_\eta + c_\sigma = c_1$ and since there exist $\eta, \sigma \in \mathbb{R}^+$ such $\eta^{-r} + \sigma^{-r} = \mu^{-r}$ (a simple case is by Pythagoras Theorem), then

$$C_{\psi_{1(\mu,c_1)}} = \left(\begin{array}{ccc|ccc} \eta^{-r} & . & . & . & . & -\varepsilon\mu \\ . & \eta^{-r} & . & . & . & . \\ . & . & c_\eta & . & . & . \\ \hline . & . & . & c_\eta & . & . \\ . & . & . & . & \eta^{-r} & . \\ -\varepsilon\mu & . & . & . & . & \eta^{-r} \end{array}\right) \tag{4.5.2}$$

is positive for $\varepsilon = 1$ and

$$C_{\psi_{2(\mu,c_1)}} = \left(\begin{array}{ccc|ccc} \sigma^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \sigma^{-r} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & c_\sigma & -\mu(1-\varepsilon) & \cdot & \cdot \\ \hline \cdot & \cdot & -\mu(1-\varepsilon) & c_\sigma & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \sigma^{-r} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \sigma^{-r} \end{array}\right) \qquad (4.5.3)$$

is positive for $\varepsilon = 0$. The minors of C_{ψ_1} are C_{ψ_2} are positive from Proposition 4.13 and Proposition 4.14 respectively. Thus $\psi_{(\mu,c_1)}$ is decomposable if the Choi matrix $C_{\psi_{(\mu,c_1)}}$ is a sum of $C_{\psi_{1(\mu,c_1)}}$ and $C_{\psi_{2(\mu,c_1)}}$. $\square$

Example 4.21. Let $c_1 \geq 1$, $r \in \mathbb{N}$, $0 < \mu \geq 1$. We have that;

$$C_{\psi_{1(\mu,c_1)}} = \left(\begin{array}{c|cc||c|cc} \mu & \cdot & \cdot & \cdot & \cdot & -\varepsilon\mu \\ \hline \cdot & \mu & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & c_1-\mu & \cdot & \cdot & \cdot \\ \hline\hline \cdot & \cdot & \cdot & c_1-\mu & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \mu & \cdot \\ -\varepsilon\mu & \cdot & \cdot & \cdot & \cdot & \mu \end{array}\right)$$

By Proposition 4.10 we show that C_{ψ_1} is positive,

Z is a zero vector. Since C is a zero vector,

$$aB = \mu \begin{pmatrix} \mu & 0 \\ 0 & c_1-\mu \end{pmatrix} = \begin{pmatrix} \mu^2 & 0 \\ 0 & \mu(c_1-\mu) \end{pmatrix} \geq 0.$$

Since $c_1 - \mu \geq 0$, $dU \geq 0$

$$\begin{aligned} aU - Y^*Y &= \mu \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} - \begin{pmatrix} 0 \\ -\varepsilon\mu \end{pmatrix} \begin{pmatrix} 0 & -\varepsilon\mu \end{pmatrix} \\ &= \begin{pmatrix} \mu^2 & 0 \\ 0 & \mu - \varepsilon^2\mu^2 \end{pmatrix} \geq 0 \end{aligned}$$

Finally, since $T = 0$, $U - TB^{-1}T^* = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}$ is positive.

Next we show that C_{ψ_2} is positive. Applying Theorem 4.12 for;

$$C_{\psi_{2(\mu,c_1)}} = \left(\begin{array}{c|cc||c|cc} \mu^{-r}-\mu & . & . & . & . & . \\ \hline . & \mu^{-r}-\mu & . & . & . & . \\ . & . & \mu & -(1-\varepsilon)\mu & . & . \\ \hline\hline . & . & -(1-\varepsilon)\mu & \mu & . & . \\ \hline . & . & . & . & \mu^{-r}-\mu & . \\ . & . & . & . & . & \mu^{-r}-\mu \end{array}\right).$$

Y and C is a zero vectors,

$aB - C^*C = (\mu^{-r} - \mu) \begin{pmatrix} \mu^{-r} - \mu & 0 \\ 0 & \mu^{-r} - \mu \end{pmatrix} \geq 0.$

Similarly $d > 0$. so $dU \geq 0$ w.ith U positive diagonal matrix.

$aU - Z^*Z$

$$\begin{aligned} &= (\mu^{-r} - \mu) \begin{pmatrix} \mu^{-r} - \mu & 0 \\ 0 & \mu^{-r} - \mu \end{pmatrix} - \begin{pmatrix} 0 \\ -(1-\varepsilon)\mu \end{pmatrix} \begin{pmatrix} 0 & -(1-\varepsilon)\mu \end{pmatrix} \\ &= \begin{pmatrix} (\mu^{-r} - \mu)^2 & 0 \\ 0 & (\mu^{-r} - \mu)^2 - (1-\varepsilon)^2\mu^2 \end{pmatrix} \geq 0 \end{aligned}$$

since $T = 0$,

$$U - TB^{-1}T^* = \begin{pmatrix} \mu^{-r} - \mu & 0 \\ 0 & \mu^{-r} - \mu \end{pmatrix}$$

is positive. The positive map $\psi_{1(\mu,c_1)}$ is decomposable such that $C_{\psi_{(\mu,c_1)}} = C_{\psi_{1(\mu,c_1)}} + C_{\psi_{2(\mu,c_1)}}$. Hence $\psi_{(\mu,c_1)}$ is decomposable when $0 < \mu < 1$ and $c_1 \geq 1$.

Remark 4.22. The decomposition of the map $\psi_{(\mu,c_1)}$ is not unique as shown by the decomposition below. This is one of the reasons decomposition of positive maps even in low dimensions is such complicated to be expressed with a unique algorithm.

Example 4.23. Let $r = 1$ and define $\psi_{(\frac{1}{2},2)} : \mathcal{M}_2(\mathbb{C}) \longrightarrow \mathcal{M}_3(\mathbb{C})$. Then the Choi matrix $C_{\psi_{(\frac{1}{2},2)}}$ is decomposable with,

$$C_{\psi_{(\frac{1}{2},2)}} = \left(\begin{array}{c|cc||c|cc} \frac{1}{2} & \cdot & \cdot & \cdot & \cdot & -\frac{1}{3} \\ \hline \cdot & \frac{1}{2} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \frac{3}{2} & \cdot & \cdot & \cdot \\ \hline\hline \cdot & \cdot & \cdot & \frac{2}{3} & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \frac{1}{2} & \cdot \\ -\frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \frac{1}{2} \end{array}\right)$$

Since C is a zero vector,

$$aB - C^*C = \frac{1}{2}\begin{pmatrix} \frac{1}{2} & \cdot \\ 0 & \frac{1}{2} \end{pmatrix} > 0.$$

$d > 0$ is a zero square matrix, so $dU = \frac{1}{2}\begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} > 0$.

$$\begin{aligned} aU - Y^*Y &= \frac{1}{2}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} - \begin{pmatrix} 0 \\ -\frac{1}{3} \end{pmatrix}\begin{pmatrix} 0 & -\frac{1}{3} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{5}{36} \end{pmatrix} > 0 \end{aligned}$$

Finally, $\begin{pmatrix} B & T \\ T^* & U \end{pmatrix}$ is positive since

$$U - T^*B^{-1}T = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} > 0.$$

Thus C_{ψ_1} is positive.

$$\left(\begin{array}{c|cc||c|cc} \frac{3}{2} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \frac{3}{2} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \frac{1}{2} & -\frac{1}{6} & \cdot & \cdot \\ \hline\hline \cdot & \cdot & -\frac{1}{6} & \frac{1}{2} & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \frac{3}{2} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \frac{3}{2} \end{array}\right)$$

Y and C are zero vectors.

$$aB - C^*C = \frac{3}{2}\begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} > 0.$$

$d > 0$ is a zero square matrix, so $dU = \frac{1}{2}\begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix} > 0.$

$$\begin{aligned} aU - Z^*Z &= \frac{3}{2}\begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix} - \begin{pmatrix} 0 \\ -\frac{1}{6} \end{pmatrix}\begin{pmatrix} 0 & -\frac{1}{6} \end{pmatrix} \\ &= \begin{pmatrix} \frac{9}{4} & 0 \\ 0 & \frac{53}{36} \end{pmatrix} > 0. \end{aligned}$$

Finally, $\begin{pmatrix} B & T \\ T^* & U \end{pmatrix}$ is positive as

$$U - T^*B^{-1}T = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} > 0.$$

Thus C_{ψ_2} is positive. This shows that the decomposition of $\psi_{1(\mu,c_1)}$ not unique.

Decomposability of linear maps $\psi_{(\mu,c_1,c_2)}$

Proposition 4.24. *The linear map $\psi_{(\mu,c_1,c_2)}$ is decomposable.*

Proof. Let $\eta^{-r} + \sigma^{-r} = \mu^{-r}$ and $c_{i\eta} + c_{i\sigma} = c_i$ for $i = 1, 2$. Consider the decomposition

$C_{\psi_{(\mu,c_1,c_2)}} = C_{\psi_{1(\eta,c_{1\eta},c_{2\eta})}} + C_{\psi_{2(\sigma,c_{1\sigma},c_{2\sigma})}}$, where

$$C_{1\psi_{(\eta,c_1,c_2)}} = \left(\begin{array}{cccc|cccc|cccc}
\eta^{-r} & . & . & . & . & -c_1 & . & . & . & . & . & -\mu \\
. & c_{2\eta} & . & . & c_{1\eta} & . & . & . & . & . & . & . \\
. & . & \eta^{-r} & . & . & . & . & . & . & . & . & . \\
. & . & . & c_{1\eta} & . & . & . & . & . & . & . & . \\
\hline
. & c_{1\eta} & . & . & c_{1\eta} & . & . & . & . & . & . & . \\
-c_1 & . & . & . & . & \eta^{-r} & . & . & . & . & -c_2 & . \\
. & . & . & . & . & . & \eta^{-r} & . & . & c_{1\eta} & . & . \\
. & . & . & . & . & . & . & c_{2\eta} & . & . & . & . \\
\hline
. & . & . & . & . & . & . & . & c_{2\eta} & . & . & . \\
. & . & . & . & . & . & c_{1\eta} & . & . & c_{1\eta} & . & . \\
. & . & . & , & . & -c_2 & . & . & . & . & \eta^{-r} & . \\
-\mu & . & . & . & . & . & . & . & . & . & . & \eta^{-r}
\end{array}\right).$$

By computation of minors show that $C_{1\psi_{2(\sigma,c_1,c_2)}}$ is positive semidefinite when;

$$\begin{aligned} c_{2\eta} &\geq c_{1\eta} \\ \eta^{-r} &\geq c_{1\eta} \end{aligned}$$

while,

$$C_{2\psi_{2(\sigma,c_{1\sigma},c_{2\sigma})}} = \left(\begin{array}{cccc|cccc|cccc}
\sigma^{-r} & . & . & . & . & . & . & . & . & . & . & . \\
. & c_{2\sigma} & . & . & -c_{1\eta} & . & . & . & . & . & . & . \\
. & . & \sigma^{-r} & . & . & . & . & . & . & . & . & . \\
. & . & . & c_{1\sigma} & . & . & . & . & . & . & . & . \\
\hline
. & -c_{1\eta} & . & . & c_{1\sigma} & . & . & . & . & . & . & . \\
. & . & . & . & . & \sigma^{-r} & . & . & . & . & . & . \\
. & . & . & . & . & . & \sigma^{-r} & . & . & -c_{1\eta} & . & . \\
. & . & . & . & . & . & . & c_{2\sigma} & . & . & . & . \\
\hline
. & . & . & . & . & . & . & . & c_{2\sigma} & . & . & . \\
. & . & . & . & . & . & -c_{1\eta} & . & . & c_{1\sigma} & . & . \\
. & . & . & , & . & . & . & . & . & . & \sigma^{-r} & . \\
. & . & . & . & . & . & . & . & . & . & . & \sigma^{-r}
\end{array}\right)$$

is completely copositive when;

$$\begin{aligned}
c_{1\sigma}c_{2\sigma} &\geq c_{1\eta}^2 \\
c_{1\sigma}\sigma^{-r} &\geq c_{1\eta}^2.
\end{aligned}$$

Hence $\psi_{\mu,c_1,c_2} = \psi_{1(\eta,c_{1\eta},c_{2\eta})} + \psi_{2(\sigma,c_{1\sigma},c_{2\sigma})}$ with,

$$\psi_{1(\eta,c_{1\eta},c_{2\eta})} = \begin{pmatrix}
P_1^\eta & -c_1x_1\bar{x}_2 + c_{1\eta}x_2\bar{x}_1 & 0 & -\mu x_1\bar{x}_3 \\
-c_1x_2\bar{x}_1 + c_{1\eta}x_1\bar{x}_2 & P_2^\eta & -c_2x_2\bar{x}_3 + c_{1\eta}x_3\bar{x}_2 & 0 \\
0 & -c_2x_3\bar{x}_2 + c_{1\eta}x_2\bar{x}_3 & P_3^\eta & 0 \\
-\mu x_3\bar{x}_1 & 0 & 0 & P_4^\eta
\end{pmatrix}$$

and

$$\psi_{2(\sigma,c_{1\sigma},c_{2\sigma})} = \begin{pmatrix} P_1^{\xi} & -c_{1\eta}x_2\bar{x}_1 & 0 & 0 \\ -c_{1\eta}x_1\bar{x}_2 & P_2^{\xi} & -c_{1\eta}x_3\bar{x}_2 & 0 \\ 0 & -c_{1\eta}x_2\bar{x}_3 & P_3^{\xi} & 0 \\ 0 & 0 & 0 & P_4^{\xi} \end{pmatrix}.$$

Therefore $\psi_{(\mu,c_1,c_2)}$ is decomposable. □

Example 4.25. The linear map $\psi_{(\frac{1}{5},\frac{3}{4},\frac{3}{4})}$ with $r = 2$ whose Choi matrix is

$$C_{\psi_{(\frac{1}{5},\frac{3}{4},\frac{3}{4})}} = \left(\begin{array}{cccc|cccc|cccc} 25 & 0 & 0 & 0 & 0 & -\frac{3}{4} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{5} \\ 0 & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{3}{4} & 0 & 0 & 0 & 0 & 25 & 0 & 0 & 0 & 0 & -\frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 25 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{3}{4} & 0 & 0 & 0 & 0 & 25 & 0 \\ -\frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 25 \end{array}\right).$$

The Choi matrix $C_{\psi_{(\frac{1}{5},\frac{1}{2},2)}}$ is a positive semidefinite matrix with eigenvalues

$$\{26.0702, 25.1402, 25., 25., 24.8598, 23.9298, 0.75, 0.75, 0.75, 0.75, 0.75, 0.75\}.$$

The linear map $\psi_{(\frac{1}{5},\frac{3}{4},\frac{3}{4})}$ when $r = 2$ is decomposable to;

$$C_{\psi_{1(\frac{1}{3},\frac{1}{4},\frac{1}{4})}} = \left(\begin{array}{cccc|cccc|cccc} 9 & 0 & 0 & 0 & 0 & -\frac{3}{4} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{5} \\ 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{3}{4} & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 & -\frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{3}{4} & 0 & 0 & 0 & 0 & 9 & 0 \\ -\frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \end{array}\right) \quad \text{and} \quad C_{\psi_{2(\frac{1}{4},\frac{1}{2},\frac{1}{2})}} = \left(\begin{array}{cccc|cccc|cccc} 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & -\frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 \end{array}\right).$$

The Choi matrices $C_{\psi_{1(\frac{1}{3},\frac{1}{4},\frac{1}{4})}}$ and $C_{\psi_{2(\frac{1}{4},\frac{1}{2},\frac{1}{2})}}$ are positive semidefinite matrix with eigenvalues, $\{10.0702, 9.14016, 9.00714, 9., 8.85984, 7.92979, 0.5, 0.25, 0.25, 0.25, 0.242863, 0.\}$ and $\{16.004, 16., 16., 16., 16., 16., 0.75, 0.5, 0.5, 0.5, 0.495969, 0.25\}$ respectively. Thus, $\psi_{(\frac{1}{5},\frac{3}{4},\frac{3}{4})}$ is decomposable.

Decomposability of linear maps $\psi_{(\mu,c_1,c_2,c_3)}$

Proposition 4.26. *The linear map $\psi_{(\mu,c_1,c_2,c_3)}$ is decomposable.*

Proof. Let $\eta^{-r}+\sigma^{-r} = \mu^{-r}$ and $c_{i\eta}+c_{i\sigma} = c_i$ for $i = 1, 2, 3$. Consider the decomposition

$$C_{\psi_{(\mu,c_1,c_2,c_3)}} = C_{\psi_{1(\eta,c_{1\eta},c_{2\eta},c_{3\eta})}} + C_{\psi_{2(\sigma,c_{1\sigma},c_{2\sigma},c_{3\sigma})}}$$

By direct calculation of minors, the Choi matrix (4.5.4) is positive semidefinite when;

$$\begin{aligned} c_{3\eta} &\geq c_{1\eta}. \\ \eta^{-r} &\geq c_{1\eta}. \\ c_{2\eta}\eta^{-r} &\geq c_{2\eta}^2. \end{aligned}$$

while, Choi matrix (4.5.5) is positive semidefinite when;

$$\begin{aligned} c_{1\sigma}c_{3\sigma} &\geq c_{1\eta}^2. \\ c_{1\sigma}c_{3\sigma} &\geq c_{1\eta}^2. \\ c_{1\sigma}\sigma^{-r} &\geq c_{1\eta}^2. \\ c_{2\sigma}\sigma^{-r} &\geq c_{2\eta}^2. \\ c_{2\sigma} &\geq c_{2\eta}. \end{aligned}$$

Hence $\psi(\mu, c_1, c_2, c_3) = C_{\psi_{1(\eta,c_{1\eta},c_{2\eta},c_{3\eta})}} + C_{\psi_{2(\sigma,c_{1\sigma},c_{2\sigma},c_{3\sigma})}}$. □

$$
C_{\psi_{1(\eta,c_{1\eta},c_{2\eta},c_{3\eta})}}
$$

$$
= \left(\begin{array}{ccccc|ccccc|ccccc|ccccc}
\eta^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & -c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & -c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\mu \\
\cdot & c_{3\eta} & \cdot & \cdot & \cdot & c_{1\eta} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & c_{2\eta} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_{1\eta} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \eta^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & c_{1\eta} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & c_{1\eta} & \cdot & \cdot & \cdot & c_{1\eta} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \eta^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & -c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & -c_3 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_{3\eta} & \cdot & \cdot & \cdot & c_{1\eta} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \eta^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_{2\eta} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_{2\eta} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & c_{2\eta} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_{2\eta} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_{1\eta} & \cdot & \cdot & \cdot & c_{1\eta} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & -c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \eta^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & -c_3 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \eta^{-r} & \cdot & \cdot & \cdot & c_{1\eta} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_{3\eta} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & ; & \cdot & \cdot & \cdot & \cdot & \cdot & c_{3\eta} & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_{2\eta} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_{2\eta} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_{1\eta} & \cdot & \cdot & \cdot & c_{1\eta} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -c_3 & \cdot & \cdot & \cdot & \cdot & \cdot & -c_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \eta^{-r} & \cdot \\
\mu & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \eta^{-r}
\end{array}\right) \qquad (4.5.4)
$$

$$C_{\psi_{2(\sigma,c_{1\sigma},c_{2\sigma},c_{3\sigma})}}$$

$$= \left(\begin{array}{ccccc|ccccc|ccccc|ccccc}
\sigma^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & c_{3\sigma} & \cdot & \cdot & \cdot & -c_{1\eta} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & c_{2\sigma} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -c_{2\eta} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \sigma^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & c_{1\sigma} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & -c_{1\eta} & \cdot & \cdot & \cdot & c_{1\sigma} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \sigma^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_{3\sigma} & \cdot & \cdot & \cdot & -c_{1\eta} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \sigma^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -c_{2\eta} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_{2\sigma} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & -c_{2\eta} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_{2\sigma} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -c_{1\eta} & \cdot & \cdot & \cdot & c_{1\sigma} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \sigma^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \sigma^{-r} & \cdot & \cdot & \cdot & -c_{1\eta} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_{3\sigma} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & ; & \cdot & \cdot & \cdot & \cdot & \cdot & c_{3\sigma} & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -c_{2\eta} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_{2\sigma} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -c_{1\eta} & \cdot & \cdot & \cdot & c_{1\sigma} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \sigma^{-r} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \sigma^{-r}
\end{array}\right) \tag{4.5.5}$$

4.5.2 Størmer's decomposability criteria

A positive a map from $\mathcal{M}_n$ to $\mathcal{M}_m$ is decomposable if for every natural number k, there exist block matrix $[X_{ij}] \in \mathcal{M}_k(\mathcal{M}_n(\mathbb{C}))^+$, such that both $[X_{ij}]$ and $[X_{ji}]$ are completely positive in $\mathcal{M}_k(\mathcal{M}_n(\mathbb{C}))^+$. If the matrix $[\psi([X_{ij}])]$ is in $\mathcal{M}_k(\mathcal{M}_m)^+$,then there are linear maps ψ_1, ψ_2, such that ψ_1 is completely positive and ψ_2 is completely copositive, with $\psi = \psi_1 + \psi_2$ ([92], Theorem 1.1). We observe that $[X_{ij}]$ is a block matrix with several entries. In addition, existence of one such matrix is enough to show decomposability. For these two reasons we proceed to prove by example that the maps $\psi_{(\mu,c_1)}$, $\psi_{(\mu,c_1,c_2)}$ and $\psi_{(\mu,c_1,c_2,c_3)}$ are decomposable.

Theorem 4.27. *The positive map $\psi_{(\mu,c_1)}$ is decomposable.*

Proof. Let $[x_{ij}]$ in $\mathcal{M}_2(\mathcal{M}_3(\mathbb{C}))$ be the matrix

$$[X_{ij}] = \left(\begin{array}{cc|cc} 2 & . & . & 2 \\ . & 4 & . & . \\ \hline . & . & 4 & . \\ 2 & . & . & 2 \end{array}\right) \geq 0$$

Since the eigenvalues are $\{0, 4, 4, 4\}$. We know that both $[X_{ij}]$ and $[x_{ij}]$ belong to $\mathcal{M}_2(\mathcal{M}_2(\mathbb{C}))^+$.

$$[\psi_{(\mu,c_1)}([x_{ij}])] \quad = \left(\begin{array}{cc|cc|cc} 2\mu^{-r}+4c_1 & . & . & . & . & -2\mu \\ . & 4\mu^{-r}+2c_1 & . & . & . & . \\ \hline . & . & 6\mu^{-r} & . & . & . \\ . & . & . & 6\mu^{-r} & . & . \\ \hline . & . & . & . & 2(\mu^{-r}+c_1) & . \\ -2\mu & . & . & . & . & 4(\mu^{-r}+c_1) \end{array}\right)$$

in $\mathcal{M}_2(\mathcal{M}_3(\mathbb{C}))$. By computation, of the minor we have that

$$\begin{aligned}\begin{vmatrix} 2\mu^{-r}+4c_1 & -2\mu \\ -2\mu & 4(\mu^{-r}+c_1) \end{vmatrix} &= 4(\mu^{-r}+c_1)(2\mu^{-r}+4c_1)-4\mu^2 \\ &= 4(2\mu^{-2r}-\mu^2)+24c_1\mu^{-r}+16c_1^2. \\ &> 0.\end{aligned}$$

Therefore $[\psi_{(c_1,\mu)}(x_{ij})]$ is positive. Thus the map $\psi_{(\mu,c_1)}$ is decomposable. □

Proposition 4.28. *Let $\psi_{(\mu,c_1,c_2)}$ be a positive map. Then $\psi_{(\mu,c_1,c_2)}$ is decomposable.*

Proof. Let

$$[X_{ij}] = \left(\begin{array}{ccc|ccc|ccc} 2 & . & . & . & 2 & . & . & . & 2 \\ . & 4 & . & . & . & . & . & . & . \\ . & . & 1 & . & . & . & . & . & . \\ \hline . & . & . & 1 & . & . & 1 & . & . \\ 2 & . & . & . & 2 & . & . & . & 2 \\ . & . & . & . & . & 4 & . & . & . \\ \hline . & . & . & 1 & . & . & 4 & . & . \\ . & . & . & . & . & . & . & 1 & . \\ 2 & . & . & . & 2 & . & . & . & 2 \end{array}\right)$$

be a matrix in $\mathcal{M}_3(\mathcal{M}_3)$. Since the eigenvalues are $\{6,4,4,4,1,1,1,0,0\}$. We can observe that both $[X_{ij}]$ and $[x_{ji}]$ belong to $\mathcal{M}_4(\mathcal{M}_3(\mathbb{C}))^+$.

The block matrix

$$
[\psi_{(\mu,c_1,c_2)}([X_{ij}])] = \left(\begin{array}{ccc|ccc|ccc|ccc}
2\mu^{-r}+\omega & . & . & . & -2\mu & . & . & . & . & . & . & -2\mu \\
. & 4\mu^{-r}+\beta & . & . & . & . & . & . & . & . & . & . \\
. & . & \mu^{-r}+\kappa & . & . & . & . & . & . & . & . & . \\
\hline
. & . & . & \mu^{-r}+\kappa & . & . & . & . & . & . & . & . \\
-2\mu & . & . & . & 2\mu^{-r}+\omega & . & . & . & -2\mu & . & . & . \\
. & . & . & . & . & 4\mu^{-r}+\beta & . & . & . & . & . & . \\
\hline
. & . & . & . & . & . & 7\mu^{-r} & . & . & . & . & . \\
. & . & . & . & . & . & . & 7\mu^{-r} & . & . & . & . \\
. & . & . & . & -2\mu & . & . & . & 7\mu^{-r} & . & . & . \\
\hline
. & . & . & . & . & . & . & . & . & 4\mu^{-r}+\beta & . & . \\
. & . & . & . & . & . & . & . & . & . & 2\mu^{-r}+\omega & . \\
-2\mu & . & . & . & . & . & . & . & . & . & . & \mu^{-r}+\kappa
\end{array}\right). \tag{4.5.6}
$$

Where $\omega = c_1 + 4c_2$, $\beta = 2c_1 + c_2$ and $\kappa = 4c_1 + 2c$.

The positive definiteness of $[\psi_{(\mu,c_1,c_2)}(x_{ij})]$ is observed from the

$$\begin{vmatrix} 2\mu^{-r}+\omega & -2\mu & 0 & -2\mu \\ -2\mu & 2\mu^{-r}+\omega & -2\mu & 0 \\ 0 & -2\mu & 7\mu^{-r} & 0 \\ -2\mu & 0 & 0 & \mu^{-r}+\kappa \end{vmatrix} = 7\mu^{-r}(2\mu^{-r}+\omega)^2(\mu^{-r}+\kappa) - 16\mu^4 \geq 0,$$

$$\begin{vmatrix} 2\mu^{-r}+\omega & -2\mu \\ -2\mu & 2\mu^{-r}+\omega \end{vmatrix} = (2\mu^{-r}+\omega)(2\mu^{-r}+\omega) - 4\mu^2 \geq 0,$$

and

$$\begin{vmatrix} 2\mu^{-r}+\omega & -2\mu & 0 \\ -2\mu & 2\mu^{-r}+\omega & -2\mu \\ 0 & -2\mu & 7\mu^{-r} \end{vmatrix} = 4\mu^{-r}(7\mu^{-2r} - 9\mu^2) + 4\omega(7\mu^{-2r} - \mu^2) + 7\mu^{-r}\omega \geq 0.$$

Therefore the map $\psi_{(\mu,c_1,c_2)}$ is decomposable. □

Proposition 4.29. *Let $\psi_{(\mu,c_1,c_2,c_3)}$ be a positive map. Then $\psi_{(\mu,c_1,c_2,c_3)}$ is decomposable.*

Proof. Let

$$[X_{ij}] = \left(\begin{array}{cccc|cccc|cccc|cccc}
2 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \frac{1}{2} \\
\cdot & 4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 4 & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 4 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\frac{1}{2} & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2
\end{array}\right) \in \mathcal{M}_4(\mathcal{M}_4)$$

self-adjoint matrix. It is clear that $[X_{ij}] = [x_{ji}]$ is positive with eigenvalues

$$\{4.34208, 4., 4., 4., 4., 2.07531, 1., 1., 1., 1., 1., 1., 1., 1., 1., 0.582611\}.$$

By computation, the block matrix, $[\psi_{(\mu,c_1,c_2,c_3)}([X_{ij}])] =$

$$\left(\begin{array}{cccc|cccc|cccc|cccc|cccc}
\Delta & \cdot & \cdot & \cdot & \cdot & -c_1 & \cdot & \cdot & \cdot & \cdot & -c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\frac{1}{2}\mu \\
\cdot & \Theta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \Pi & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \Lambda & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \Lambda & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-c_1 & \cdot & \cdot & \cdot & \cdot & \Delta & \cdot & \cdot & \cdot & \cdot & -c_2 & \cdot & \cdot & \cdot & \cdot & -c_3 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \Theta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \Pi & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \Pi & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \Lambda & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-c_2 & \cdot & \cdot & \cdot & \cdot & -c_2 & \cdot & \cdot & \cdot & \cdot & \Delta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \Theta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 8\mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 8\mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 8\mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -c_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 8\mu^{-r} & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \Delta & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \Theta & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \Pi & \cdot \\
-\frac{1}{2}\mu & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \Lambda
\end{array}\right).$$

$$\Delta = 2\mu^{-r} + \delta \quad \text{where} \quad \delta = c_1 + c_2 + 4c_3.$$

$$\Theta = 4\mu^{-r} + \gamma \quad \text{where} \quad \gamma = 2c_1 + c_2 + c_3.$$

$$\Pi = \mu^{-r} + \xi \quad \text{where} \quad \xi = 4c_1 + 2c_2 + c_3.$$

$$\Lambda = \mu^{-r} + \chi \quad \text{where} \quad \chi = c_1 + 4c_2 + c_3.$$

The positive definiteness of $[\psi_{(\mu,c_1,c_2,c_3)}(x_{ij})]$ is observed from;

$$\begin{pmatrix}
2\mu^{-r} + \delta & -c_1 & -c_2 & 0 & -\frac{1}{2}\mu \\
-c_1 & 2\mu^{-r} + \delta & -c_2 & -c_3 & 0 \\
-c_2 & -c_2 & 2\mu^{-r} + \delta & 0 & 0 \\
0 & -c_3 & 0 & 8\mu^{-r} & 0 \\
-\frac{1}{2}\mu & 0 & 0 & 0 & 2\mu^{-r} + \delta
\end{pmatrix}$$

whose determinant is

$$\begin{aligned} 8\mu^{-r}(2\mu^{-r}+\delta)^3 - \frac{1}{4}c_3^2\mu^2 &\geq 8\mu^{-r}(2\mu^{-r})^3 - \frac{1}{4}c_3^2\mu^2 \\ &= 64\mu^{-4r} - \frac{1}{4}c_3^2\mu^2 \\ &\geq 0. \end{aligned}$$

The determinant of the minor

$$\begin{pmatrix} 2\mu^{-r}+\delta & -c_1 & -c_2 & 0 \\ -c_1 & 2\mu^{-r}+\delta & -c_2 & -c_3 \\ -c_2 & -c_2 & 2\mu^{-r}+\delta & 0 \\ 0 & -c_3 & 0 & 8\mu^{-r} \end{pmatrix}$$

as

$8\mu^{-r}(2\mu^{-r}+\delta)^3 - (8c_2^2\mu^{-r}(2\mu^{-r}+\delta) - c_3^2(2\mu^{-r}+\delta)^2)$

$$\begin{aligned} &\geq 8\mu^{-r}(2\mu^{-r})^2 - [8c_2^2\mu^{-r} - c_3^2(2\mu^{-r})] \\ &= 32\mu^{-3r} - 4c_2^2 - c_3^2 \\ &\geq 0. \end{aligned}$$

$$\begin{vmatrix} 2\mu^{-r}+\delta & -c_1 & -c_2 \\ -c_1 & 2\mu^{-r}+\delta & -c_2 \\ -c_2 & -c_2 & 2\mu^{-r}+\delta \end{vmatrix} = (2\mu^{-r}+\delta)[(2\mu^{-r}+\delta)^2 - 2c_2^2 - c_1^2] - 2c_2^2c_1 \geq 0$$

and

$$\begin{vmatrix} 2\mu^{-r}+\delta & -c_1 \\ -c_1 & 2\mu^{-r}+\delta \end{vmatrix} = (2\mu^{-r}+\delta)^2 - c_1^2 \geq 0.$$

Therefore the map $\psi_{(\mu,c_1,c_2,c_3)}$ is decomposable. $\square$

4.6 Decomposability of linear map from $\mathcal{M}_3(\mathbb{C})$ to $\mathcal{M}_2(\mathcal{M}_2(\mathbb{C}))$

Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras, and $\psi : \mathcal{M}_n(\mathcal{A})^+ \longrightarrow \mathcal{M}_m(\mathcal{B})^+$ be a positive map. Define $\psi_n : \mathcal{M}_k \otimes \mathcal{M}_n(\mathcal{A}) \longrightarrow \mathcal{M}_n \otimes \mathcal{M}_m(\mathcal{B})$ by $\psi_n = \mathcal{I}_n \otimes \psi$.

If $A = [a_{ij}] \in \mathcal{M}_n(\mathcal{A})$, $B = [b_{ij}] \in \mathcal{M}_n(\mathcal{B})$, we define $A \otimes B = [a_{ij} \otimes b_{ij}] \in \mathcal{M}_n(A \otimes B)$.

Proposition 4.30. *Let $A \in \mathcal{M}_n(\mathcal{A})^+$, $B \in \mathcal{M}_n(\mathcal{B})^+$ and $A \otimes B \in \mathcal{M}_n(\mathcal{A} \otimes \mathcal{B})^+$. Then $\psi : \mathcal{M}_n(\mathcal{A})^+ \longrightarrow \mathcal{M}_m(\mathcal{B})^+$ is completely positive.*

Proof. Suppose $[a_{ij}] \in \mathcal{M}_n(\mathcal{A})^+$ for $i, j = 1, \ldots, n$ and $[[b_{ij}]]_{st} \in \mathcal{M}_k(\mathcal{M}_n(\mathcal{B}))^+$ for $i, j = 1, \ldots, k$. Let $T_k = 1$ be a matrix unit in $\mathcal{M}_k$. Then by Choi's definition of complete positivity;

$$\begin{aligned}
[\psi(A_{ij})] &= \sum_{i,j=1}^{n} [a_{ij}] \otimes \psi([a_{ij}]) \\
&= \sum_{i,j=1}^{k} \sum_{s,t=1}^{n} [T_k]_{st}[a_{ij}] \otimes [[b_{ij}]]_{st} \\
&= \sum_{i,j=1}^{k} \sum_{s,t=1}^{n} [T_k a_{ij}]_{st} \otimes [[b_{ij}]]_{st} \\
&= \sum_{i,j=1}^{k} \sum_{s,t=1}^{n} [[a_{ij}]]_{st} \otimes [[b_{ij}]]_{st} \\
&\geq 0
\end{aligned}$$

□

Let $X \in \mathcal{M}_3(\mathbb{C})$. Let $0 < \mu < 1$, $c_1, c_2 > 0$ and $r \in \mathbb{N}$. Then we define the family of positive maps ψ as follows:

$$\Phi_{(\mu, c_1, c_2)} : \mathcal{M}_3(\mathbb{C})^+ \longrightarrow \mathcal{M}_2(\mathcal{M}_2)\mathbb{C})^+.$$

$$X \mapsto \left(\begin{array}{cc|cc} P_1 & -c_1 x_1 \bar{x}_2 & 0 & -\mu x_1 \bar{x}_3 \\ -c_1 x_2 \bar{x}_1 & P_2 & -c_2 x_2 \bar{x}_3 & 0 \\ \hline 0 & -c_2 x_3 \bar{x}_2 & P_3 & 0 \\ -\mu x_3 \bar{x}_1 & 0 & 0 & P_4 \end{array}\right), \tag{4.6.1}$$

where

$$\begin{aligned} P_1 &= \mu^{-r}(\alpha_1 + c_1\alpha_2\mu^r + c_2\alpha_3\mu^r) \\ P_2 &= \mu^{-r}(\alpha_2 + c_1\alpha_3\mu^r + c_2\alpha_1\mu^r) \\ P_3 &= \mu^{-r}(\alpha_1 + \alpha_2 + \alpha_3) \\ P_4 &= \mu^{-r}(\alpha_3 + c_1\alpha_1\mu^r + c_2\alpha_2\mu^r) \end{aligned}$$

4.6.1 Complete (co)positivity of $\Phi_{(\mu,c_1,c_2)}$

The structure of the Choi matrix $C_{\Phi_{(\mu,c_1,c_2)}} \in \mathcal{M}_3(\mathcal{M}_2(\mathcal{M}_2))$ is visualized as a block matrix whose entries are 2×2 matrices within the 6×6 matrix.

Proposition 4.31. *Let $\psi_{(\mu,c_1,c_2)}$ a positive map. The following are equivalent:*

(i) $\Phi_{(\mu,c_1,c_2)}$ *is completely positive,*

(ii) $\Phi_{(\mu,c_1,c_2)}$ *is 2-positive and,*

(iii) $\mu^{-2r} > c_1^2 + c_2^2$.

Proof. (ii) $\Rightarrow$ (iii).
Assume $\Phi_{(\mu,c_1,c_2)}$ is 2-positive. Consider $P \in \mathcal{M}_3(\mathcal{M}_2(\mathcal{M}_2(\mathbb{C})))$

where $a_i = (1,0,0,0,0,1,1,1)^T$, we have that

$$
P = [a_i a_j] = \left(\begin{array}{cc|cc||cc|cc}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1
\end{array}\right) \tag{4.6.2}
$$

in $\mathcal{M}_3(\mathcal{M}_2 \otimes \mathcal{M}_2)$, where zeros are replaced by dots.

$$
\mathcal{I}_2 \otimes \Phi_{(\mu,c_1,c_2)}(X) = \left(\begin{array}{cc|cc||cc|cc}
\mu^{-r} & \cdot & \cdot & \cdot & \cdot & -c_1 & \cdot & -\mu \\
\cdot & c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot \\
\hline\hline
\cdot & \cdot & \cdot & \cdot & c_1 & \cdot & \cdot & \cdot \\
-c_1 & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & -c_2 & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & -c_2 & \mu^{-r} & \cdot \\
-\mu & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r}
\end{array}\right) \tag{4.6.3}
$$

in $\mathcal{M}_2(\mathcal{M}_2 \otimes \mathcal{M}_2)$ is positive semidefinite. Therefore,

$$
\begin{vmatrix}
\mu^{-r} & -c_1 & 0 & -\mu \\
-c_1 & \mu^{-r} & -c_2 & 0 \\
0 & c_2 & \mu^{-r} & 0 \\
-\mu & 0 & 0 & \mu^{-r}
\end{vmatrix} \geq 0 \tag{4.6.4}
$$

By calculation of minors, $\mu^{-r} > c_1$ and $\mu^{-2r} > c_1^2 + c_2^2$ are necessary conditions for

2-positivity of $\Phi_{(\mu,c_1,c_2)}$.

$(iii) \Rightarrow (i)$.

The Choi matrix is

$$C_{\Phi_{(\mu,c_1,c_2)}} = \left(\begin{array}{cc|cc||cc|cc||cc|cc}
\mu^{-r} & \cdot & \cdot & \cdot & \cdot & -c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & -\mu \\
\cdot & c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline\hline
\cdot & \cdot & \cdot & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-c_1 & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & -c_2 & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_2 & \cdot & \cdot & \cdot & \cdot \\
\hline\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_1 & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & -c_2 & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot \\
-\mu & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r}
\end{array}\right) \tag{4.6.5}$$

in $\mathcal{M}_3(\mathcal{M}_2 \otimes \mathcal{M}_2)$.

Recall that complete positivity of $\Phi_{(\mu,c_1,c_2)}$ is equivalent to positive definiteness [15] of $C_{\Phi_{(\mu,c_1,c_2)}}$. Since (iii) is satisfied, the inequality (4.6.4) holds, and consequently $C_{\Phi_{(\mu,c_1,c_2)}}$ is positive definite. Hence, complete positivity of $\psi_{(\mu,c_1,c_2)}$ follows.

$(i) \Rightarrow (ii)$.

It follows from Proposition 4.30 that 2-positivity of $\Phi_{(\mu,c_1,c_2)}$ implies complete positivity. □

The Partial Positive transposition is operated with respect to the blocks $\mathcal{M}_2$ as the entry elements of the matrix $\mathcal{M}_3(\mathcal{M}_2)$. This leads to the Choi matrix $C^{\Gamma}_{\Phi_{(\mu,c_1,c_2)}} \in \mathcal{M}_3(\mathcal{M}_2)$ with the structure given in the next proposition.

Proposition 4.32. *Let $\psi_{((\mu,c_1,c_2)}$ be a positive map. Then following conditions are* *equivalent:*

(i) $\Phi_{(\mu,c_1,c_2)}$ *is completely copositive,*

(ii) $\Phi_{(\mu,c_1,c_2)}$ *is 2-copositive and,*

(iii) $c_1\mu^{-r} \geq c_2^2$

Proof. *(ii)* $\Rightarrow$ *(iii)*.

Assume $\Phi_{(\mu,c_1,c_2)}$ is 2-copositive. Consider P as in Proposition 4.15, we have that

$$P^{\Gamma} = [a_i a_j]^{\Gamma} = \left(\begin{array}{cc|cc||cc|cc} 1 & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ \hline . & . & . & . & . & 1 & . & . \\ . & . & . & . & . & . & . & . \\ \hline\hline . & . & 1 & . & . & . & . & . \\ . & . & 1 & . & . & . & . & . \\ \hline . & . & . & . & . & . & 1 & 1 \\ . & . & . & . & . & . & 1 & 1 \end{array}\right) \tag{4.6.6}$$

in $\mathcal{M}_2(\mathcal{M}_2 \otimes \mathcal{M}_2)$. Therefore,

$$\tau_2 \otimes \Phi_{(\mu,c_1,c_2)}(P) = \left(\begin{array}{cc|cc||cc|cc} \mu^{-r} & . & . & . & . & -c_1 & . & . \\ . & c_2 & . & . & . & . & . & . \\ \hline . & . & \mu^{-r} & . & . & -\mu & . & . \\ . & . & . & c_1 & . & . & . & . \\ \hline\hline . & . & . & . & c_1 & . & . & -c_2 \\ -c_1 & . & -\mu & . & . & \mu^{-r} & . & . \\ \hline . & . & . & . & . & . & \mu^{-r} & . \\ . & . & . & . & -c_2 & . & . & \mu^{-r} \end{array}\right) \tag{4.6.7}$$

in $\mathcal{M}_2(\mathcal{M}_2 \otimes \mathcal{M}_2)$ is positive semidefinite with the minors positive when conditions in (iii) holds.

$(iii) \Rightarrow (i)$

The choi matrix,

$$
C^{\Gamma}_{\Phi_{(\mu,c_1,c_2)}} = \left(\begin{array}{cc|cc||cc|cc||cc|cc}
\mu^{-r} & \cdot & \cdot & \cdot & \cdot & -c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\mu & \cdot & \cdot \\
\cdot & \cdot & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline\hline
\cdot & \cdot & \cdot & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-c_1 & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_2 & -c_2 & \cdot & \cdot & \cdot \\
\hline\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -c_2 & c_2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & -\mu & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_1. & \cdot & \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r}
\end{array}\right) \tag{4.6.8}
$$

in $\mathcal{M}_3(\mathcal{M}_2 \otimes \mathcal{M}_3)$. Since (iii) is satisfied, by calculation of the minor, $C^{\Gamma}_{\Phi_{(\mu,c_1,c_2)}}$ is positive semidefinite when $\mu^{-r} \geq c_1$ holds. Hence, complete copositivity follows.

$(i) \Rightarrow (ii)$ follows from Proposition 4.30 that 2-positivity of $\Phi_{(\mu,c_1,c_2)}$ implies complete positivity. $\square$

4.6.2 Decomposability of $\Phi_{(\mu,c_1,c_2)}$

Proposition 4.33. *The linear map $\Phi_{(\mu,c_1,c_2)}$ is decomposable.*

Proof. Let $\Phi_{(\mu,a_1,a_2)}$ a 2-positive map and $\Phi_{(\mu,b_1,b_2)}$ a 2-copositive map as in Proposition 4.15 and Proposition 4.32 respectively. Considering the matrix $[\Phi_{(\mu,c_1,c_2)}(X)] \in \mathcal{M}_3(\mathcal{M}_2(\mathcal{M}_2(\mathbb{C})))$. Since there exist $\eta, \xi \in (0,1)$ such that $\eta^{-r} + \xi^{-r} = \mu^{-r}$. Then $C_{\Phi_{(\mu,c_1,c_2)}}$ is the matrix

$$
\left(\begin{array}{cc|cc||cc|cc||cc|cc}
\eta^{-r}+\xi^{-r} & . & . & . & . & -(a_1+b_1) & . & . & . & . & . & -\varepsilon\mu \\
. & (a_2+b_2) & . & . & . & . & . & . & . & . & . & . \\
\hline
. & . & \eta^{-r}+\xi^{-r} & . & . & . & . & . & . & -(1-\varepsilon)\mu & . & . \\
. & . & . & (a_1+b_1) & . & . & . & . & . & . & . & . \\
\hline\hline
. & . & . & . & (a_1+b_1) & . & . & . & . & . & . & . \\
-(a_1+b_1) & . & . & . & . & \eta^{-r}+\xi^{-r} & . & . & . & . & -(a_2+b_2) & . \\
\hline
. & . & . & . & . & . & \eta^{-r}+\xi^{-r} & . & . & . & . & . \\
. & . & . & . & . & . & . & (a_2+b_2) & . & . & . & . \\
\hline\hline
. & . & . & . & . & . & . & . & (a_2+b_2) & . & . & . \\
. & . & -(1-\varepsilon)\mu & . & . & . & . & . & . & (a_1+b_1). & . & \\
\hline
. & . & . & . & . & -(a_2+b_2) & . & . & . & . & \eta^{-r}+\xi^{-r} & . \\
-\varepsilon\mu & . & . & . & . & . & . & . & . & . & . & \eta^{-r}+\xi^{-r}
\end{array}\right)
$$

in $\mathcal{M}_3(\mathcal{M}_2(\mathcal{M}_2)\mathbb{C})$. The matrix given by the sum of the Choi matrices of completely positive map

$$\Phi_{(\eta,a_1,a_2)} = \left(\begin{array}{cc|cc} P_1^\eta & -a_1x_1\bar{x}_2 & 0 & -\varepsilon\mu x_1\bar{x}_3 \\ -a_1x_2\bar{x}_1 & P_2^\eta & -a_2x_2\bar{x}_3 & 0 \\ \hline 0 & -a_2x_3\bar{x}_2 & P_3^\eta & 0 \\ -\varepsilon\mu x_3\bar{x}_1 & 0 & 0 & P_4^\eta \end{array}\right)$$

and completely copositive

$$\Phi_{(\xi,b_1,b_2)} = \left(\begin{array}{cc|cc} P_1^\xi & -b_1x_2\bar{x}_1 & 0 & -(1-\varepsilon)\mu x_1\bar{x}_3 \\ -b_1x_1\bar{x}_2 & P_2^\xi & 0 & 0 \\ \hline 0 & 0 & P_3^\xi & 0 \\ -(1-\varepsilon)\mu x_3\bar{x}_1 & 0 & 0 & P_4^\xi \end{array}\right).$$

$\Phi_{(\mu,c_1,c_2)}$ is completely positive when $q = 0$ while $\Phi_{(\mu,c_1,c_2)}$ is completely copositive when $q = 1$.

By Proposition 4.15 and Proposition 4.32, $\Phi_{(\mu,c_1,c_2)}$ is positive semidefinite. $\square$

Note that the decomposition of these maps is not unique.

4.7 Merging of Completely positive maps

4.7.1 Decomposability of $\Psi_{(\mu,c_1,c_2)}$

Let $X \in \mathcal{M}_3(\mathbb{C})$ and $0 < \mu < 1$, $c_1, c_2 > 0$ and $r \in \mathbb{N}$. Then we define the positive maps Ψ as follows:

$$\Psi_{(\mu,c_1,c_2)} : \mathcal{M}_3(\mathbb{C}) \longrightarrow \mathcal{M}_4(\mathbb{C}).$$

$$X \mapsto \begin{pmatrix} 2P_1 & -c_1(x_1\bar{x}_2 + x_2\bar{x}_1) & 0 & -\mu(x_1\bar{x}_4 + x_4\bar{x}_1) \\ -c_1(x_2\bar{x}_1 + x_1\bar{x}_2) & 2P_2 & -c_2(x_2\bar{x}_3 + x_3\bar{x}_2) & 0 \\ 0 & -c_3(x_4\bar{x}_2 + x_2\bar{x}_4) & 2P_3 & 0 \\ -\mu(x_4\bar{x}_1 + x_1\bar{x}_4) & 0 & 0 & 2P_4 \end{pmatrix}, \quad (4.7.1)$$

where

$$\begin{aligned} P_1 &= \mu^{-r}(|x_1| + c_1|x_2|\mu^r + c_2|x_3|\mu^r) \\ P_2 &= \mu^{-r}(|x_2| + c_1|x_3|\mu^r + c_2|x_1|\mu^r) \\ P_3 &= \mu^{-r}(|x_1| + |x_2| + |x_3|) \\ P_4 &= \mu^{-r}(|x_3| + c_1|x_1|\mu^r + c_2|x_2|\mu^r) \end{aligned}$$

Proposition 4.34. *The linear map $\Psi_{(\mu,c_1,c_2)}$ is positive provided the conditions in Lemma 4.3 holds.*

Proof. Let $b_{ij} = x_i\bar{x}_j + x_j\bar{x}_i$. Then $\Psi_{(\mu,c_1,c_2)}$ reduce to

$$\Psi_{(\mu,c_1,c_2)}(X) = \begin{pmatrix} 2P_1 & -c_1b_{12} & 0 & -\mu b_{14} \\ -c_1b_{21} & 2P_2 & -c_2b_{23} & 0 \\ -c_2b_{31} & -c_2b_{32} & 2P_3 & 0 \\ -\mu b_{41} & 0 & 0 & 2P_4 \end{pmatrix}.$$

The proof is the same as in Proposition 4.4. □

Proposition 4.35. *The linear map $\Psi_{(\mu,c_1,c_2)}$ is 2-positive(2-copositive).*

Proof. Let $\Psi_{(\mu,c_1,c_2)}$ be positive. We have that

$\mathcal{I}_2 \otimes \Psi_{(\mu,c_1,c_2)}(X)$

$$= \left(\begin{array}{cccc|cccc} 2\mu^{-r} & . & . & . & . & -c_1 & -c_2 & -\mu \\ . & 2c_2 & . & . & -c_1 & . & . & . \\ . & . & 2\mu^{-r} & . & . & . & . & . \\ . & . & . & 2c_1 & -\mu & . & . & . \\ \hline . & -c_1 & . & -\mu & 2c_1 & . & . & . \\ -c_1 & . & . & . & . & 2\mu^{-r} & -2c_2 & . \\ -c_2 & . & . & . & . & -2c_2 & 2\mu^{-r} & . \\ -\mu & . & . & . & . & . & . & 2\mu^{-r} \end{array}\right). \tag{4.7.2}$$

By computation of the principal minors, the matrix is positive whenever the inequalities

$$\mu^{-r} > c_1, \quad \mu^{-r} > c_2, \quad c_1 \geq \mu \quad \textit{and} \quad c_2 \geq c_1$$

holds. Thus $\Psi_{(\mu,c_1,c_2)}$ is 2-positive,

□

Proposition 4.36. *The linear map $\Psi_{(\mu,c_1,c_2)}$ is completely positive(completely copositive).*

Proof. The computation of the Choi matrix of the linear map $\Psi_{(\mu,c_1,c_2)}$ gives $C_{\Psi_{(\mu,c_1,c_2)}}$

as

$$
\left(\begin{array}{c|ccccc||c|ccccc}
2\mu^{-r} & . & . & . & . & -c_1 & . & . & . & . & . & -\mu \\
\hline
. & 2c_2 & . & . & -c_1 & . & . & . & . & . & . & . \\
. & . & 2\mu^{-r} & . & . & . & . & . & . & . & . & . \\
. & . & . & 2c_1 & . & . & . & . & -\mu & . & . & . \\
. & -c_1 & . & . & 2c_1 & . & . & . & . & . & . & . \\
-c_1 & . & . & . & . & 2\mu^{-r} & . & . & . & . & -c_2 & . \\
\hline\hline
. & . & . & . & . & . & 2\mu^{-r} & . & . & -c_2 & . & . \\
\hline
. & . & . & . & . & . & . & 2c_2 & . & . & . & . \\
. & . & . & -\mu. & . & . & . & . & 2c_2 & . & . & . \\
. & . & . & . & . & . & -c_2 & . & . & 2c_1. & . & \\
. & . & . & . & . & -c_2 & . & . & . & . & 2\mu^{-r} & . \\
-\mu & . & . & . & . & . & . & . & . & . & . & 2\mu^{-r}
\end{array}\right)
$$

$$
\begin{aligned}
aB - C^*C &= 2\mu^{-r} \begin{pmatrix} 2c_2 & 0 & 0 & -c_1 & 0 \\ 0 & 2\mu^{-r} & 0 & 0 & 0 \\ 0 & 0 & 2c_1 & 0 & 0 \\ -c_1 & 0 & 0 & 2c_1 & 0 \\ 0 & 0 & 0 & 0 & 2\mu^{-r} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -c_1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -c_1 \end{pmatrix}^T \\
&= \begin{pmatrix} 4c_2\mu^{-r} & 0 & 0 & -2c_1\mu^{-r} & 0 \\ 0 & 4\mu^{-2r} & 0 & 0 & 0 \\ 0 & 0 & 4c_1\mu^{-r} & 0 & 0 \\ -2c_1\mu^{-r} & 0 & 0 & 4c_1\mu^{-r} & 0 \\ 0 & 0 & 0 & 0 & 4c_1\mu^{-r} - c_1^2 \end{pmatrix}.
\end{aligned}
$$

The matrix is positive when $2\mu^{-r} > c_1$ and $4c_2 > c_1$.

$$dU - F^*F = 2\mu^{-r}\begin{pmatrix} 2c_2 & 0 & 0 & 0 & 0 \\ 0 & 2c_2 & 0 & 0 & 0 \\ 0 & 0 & 2c_1 & 0 & 0 \\ 0 & 0 & 0 & 2\mu^{-r} & 0 \\ 0 & 0 & 0 & 0 & 2\mu^{-r} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ -c_2 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ -c_2 \\ 0 \\ 0 \end{pmatrix}^T$$

$$= \begin{pmatrix} 4c_2\mu^{-r} & 0 & 0 & 0 & 0 \\ 0 & 4c_2\mu^{-r} & 0 & 0 & 0 \\ 0 & 0 & 4c_1\mu^{-r} - c_2^2 & 0 & 0 \\ 0 & 0 & 0 & 4\mu^{-2r} & 0 \\ 0 & 0 & 0 & 0 & 4\mu^{-2r} \end{pmatrix}.$$

$dU \geq 0$ is positive when $4c_1\mu^{-r} > c_2^2$.

$$aU - Y^*Y = 2\mu^{-r}\begin{pmatrix} 2c_2 & 0 & 0 & 0 & 0 \\ 0 & 2c_2 & 0 & 0 & 0 \\ 0 & 0 & 2c_1 & 0 & 0 \\ 0 & 0 & 0 & 2\mu^{-r} & 0 \\ 0 & 0 & 0 & 0 & 2\mu^{-r} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\mu \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\mu \end{pmatrix}^T$$

$$= \begin{pmatrix} 4c_2\mu^{-r} & 0 & 0 & 0 & 0 \\ 0 & 4c_2\mu^{-r} & 0 & 0 & 0 \\ 0 & 0 & 4c_1\mu^{-r} & 0 & 0 \\ 0 & 0 & 0 & 4\mu^{-2r} & 0 \\ 0 & 0 & 0 & 0 & 4\mu^{-2r} - \mu^2 \end{pmatrix}.$$

is positive.

$$aU - Z^*Z = 2\mu^{-r}\begin{pmatrix} 2c_2 & 0 & 0 & 0 & 0 \\ 0 & 2c_2 & 0 & 0 & 0 \\ 0 & 0 & 2c_1 & 0 & 0 \\ 0 & 0 & 0 & 2\mu^{-r} & 0 \\ 0 & 0 & 0 & 0 & 2\mu^{-r} \end{pmatrix}$$

is positive.

$$U - TB^{-1}T^* = \begin{pmatrix} 2c_2 & 0 & 0 & 0 & 0 \\ 0 & 2c_2 & 0 & 0 & 0 \\ 0 & 0 & 2c_1 & 0 & 0 \\ 0 & 0 & 0 & 2\mu^{-r} & 0 \\ 0 & 0 & 0 & 0 & 2\mu^{-r} \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} 2c_2 & 0 & 0 & -c_1 & 0 \\ 0 & 2\mu^{-r} & 0 & 0 & 0 \\ 0 & 0 & 2c_1 & 0 & 0 \\ -c_1 & 0 & 0 & 2c_1 & 0 \\ 0 & 0 & 0 & 0 & 2\mu^{-r} \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2c_2 & 0 & 0 & 0 & 0 \\ 0 & 2c_2 & 0 & 0 & 0 \\ 0 & \frac{-\mu^{r+2}}{2} & 2c_1 & 0 & 0 \\ 0 & 0 & 0 & 2\mu^{-r} & 0 \\ 0 & 0 & 0 & 0 & 2\mu^{-r} \end{pmatrix}$$

is positive. □

Proposition 4.37. *The linear map $\Psi_{(\mu,c_1,c_2)}$ is decomposable.*

Proof. From Proposition 4.35 $\Psi_{(\mu,c_1,c_2)}$ is 2-positive(2-copositive)and Proposition 4.36, complete positivity is equivalent to complete copositivity. Observe that, $C_{\Psi_{(\mu,c_1,c_2)}}$ is the sum of

$$\psi_{1(\mu,c_1,c_2)} = \left(\begin{array}{c|ccccc||c|ccccc}
\mu^{-r} & 0 & 0 & 0 & 0 & -c_1 & 0 & 0 & 0 & 0 & 0 & -\mu \\
\hline
0 & c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-c_1 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 & 0 & 0 & 0 & -c_2 & 0 \\
\hline\hline
0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -c_2 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 \\
-\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r}
\end{array}\right) \tag{4.7.3}$$

and

$$\psi_{2(\mu,c_1,c_2)} = \left(\begin{array}{c|ccccc||c|ccccc}
\mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & c_2 & 0 & 0 & -c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 \\
0 & -c_1 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline\hline
0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 & 0 & -c_2 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -c_2 & 0 & 0 & c_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r}
\end{array}\right). \tag{4.7.4}$$

Therefore, $\Psi_{(\mu,c_1,c_2)}$ is decomposable with $\psi_{1(\mu,c_1,c_2)}$ 2-positive and $\psi_{2(\mu,c_1,c_2)}$ 2-copositive.

□

4.7.2 Decomposability of $\Psi_{(\mu,c_1,c_2,c_3)}$

Let $X \in \mathcal{M}_3(\mathbb{C})$ and $0 < \mu < 1$, $c_1, c_2 > 0$ and $r \in \mathbb{N}$. Then we define the family of positive maps Ψ as follows:

$$\Psi_{(\mu,c_1,c_2,c_3)} : \mathcal{M}_4(\mathbb{C}) \longrightarrow \mathcal{M}_5(\mathbb{C}).$$

$$X \mapsto \begin{pmatrix} 2P_1 & -c_1(x_1\bar{x}_2 + x_2\bar{x}_1) & -c_2(x_1\bar{x}_3 + x_3\bar{x}_1) & 0 & -\mu(x_1\bar{x}_4 + x_4\bar{x}_1) \\ -c_1(x_2\bar{x}_1 + x_1\bar{x}_2) & 2P_2 & -c_2(x_2\bar{x}_3 + x_3\bar{x}_2) & -c_3(x_2\bar{x}_4 + x_4\bar{x}_2) & 0 \\ -c_2(x_3\bar{x}_1 + x_1\bar{x}_3) & -c_2(x_3\bar{x}_2 + x_2\bar{x}_3) & 2P_3 & -c_3(x_3\bar{x}_4 + x_4\bar{x}_3) & 0 \\ 0 & -c_3(x_4\bar{x}_2 + x_2\bar{x}_4) & -c_3(x_4\bar{x}_3 + x_3\bar{x}_4) & 2P_4 & 0 \\ -\mu(x_4\bar{x}_1 + x_1\bar{x}_4) & 0 & 0 & 0 & 2P_5 \end{pmatrix},$$

where

$$\begin{aligned}
P_1 &= \mu^{-r}(|x_1| + c_1|x_2|\mu^r + c_2|x_3|\mu^r + c_3|x_4|\mu^r) \\
P_2 &= \mu^{-r}(|x_2| + c_1|x_3|\mu^r + |x_4|c_2\mu^r + |x_1|c_3\mu^r) \\
P_3 &= \mu^{-r}(|x_3| + c_1|x_1|\mu^r + |x_2|c_2\mu^r + |x_3|c_3\mu^r) \\
P_4 &= \mu^{-r}(|x_1| + |x_2| + |x_3| + |x_4|) \\
P_5 &= \mu^{r}(|x_4| + c_1|x_1|\mu^r + c_2|x_2|\mu^r + c_3|x_4|\mu^r)
\end{aligned}$$

Proposition 4.38. *The linear map $\Psi_{(\mu,c_1,c_2,c_3)}$ is positive provided the conditions in Lemma. 4.5 holds.*

Proof. Let $b_{ij} = x_i\bar{x}_j + x_j\bar{x}_i$ Then the map described in (4.7.5) reduce to

$$\Psi(X) = \begin{pmatrix} 2P_1 & -c_1b_{12} & -c_2b_{12} & 0 & -\mu b_{14} \\ -c_1b_{21} & 2P_2 & -c_2b_{23} & -c_3b_{24} & 0 \\ -c_2b_{31} & -c_2b_{32} & 2P_3 & -c_3b_{34} & 0 \\ 0 & -c_3b_{42} & -c_3b_{43} & 2P_4 & 0 \\ -\mu b_{41} & 0 & 0 & 0 & 2P_5 \end{pmatrix}.$$

The proof follows by the same method as in Proposition. 4.6. □

Now we find the conditions for 2-positivity and 2-copositivity of the map $\Psi_{(\mu,c_1,c_2,c_3)}$.

Proposition 4.39. *The linear map $\psi_{(\mu,c_1,c_2,c_3)}$ is 2-positive(2-copositive).*

Proof. Let $\Psi_{(\mu,c_1,c_2,c_3)}$ be positive. We have that,

$\mathcal{I}_2 \otimes \Psi_{(\mu,c_1,c_2,c_3)}(X)$

$$= \left(\begin{array}{ccccc|ccccc} 2\mu^{-r} & . & . & . & . & . & -c_1 & -c_2 & . & -\mu \\ . & 2c_3 & . & . & . & -c_1 & . & . & . & . \\ . & . & 2c_2 & . & . & -c_2 & . & . & . & . \\ . & . & . & 2\mu^{-r} & . & . & . & . & . & . \\ . & . & . & . & 2c_1 & -\mu & . & . & . & . \\ \hline . & -c_1 & -c_2 & . & -\mu & 2c_1 & . & . & . & . \\ -c_1 & . & . & . & . & . & 2\mu^{-r} & -2c_2 & -2c_3 & . \\ -c_2 & . & . & . & . & . & -2c_2 & 2\mu^{-r} & -2c_3 & . \\ . & . & . & . & . & . & -2c_3 & -2c_3 & 2\mu^{-r} & . \\ -\mu & . & . & . & . & . & . & . & . & 2\mu^{-r} \end{array}\right).$$

Since $\psi_{(\mu,c_1,c_2,c_3)}$ is 2-positive, the matrix $\mathcal{I}_2 \otimes \Psi_{(\mu,c_1,c_2,c_3)}(X)$ is positive definite. Therefore,

$$4\mu^{-r} > c_1, \quad 4\mu^{-r} > c_2 \quad \mu^{-r} > c_3 \quad 4c_3 \geq c_1, \quad c_1 \geq \mu \quad \textit{and} \quad 4c_1 \geq c_2.$$

□

Proposition 4.40. *The positive map $\Psi_{(\mu,c_1,c_2,c_3)}$ is completely positive(completely copositive).*

Proof. The computation of the Choi matrix of the linear map $\Psi_{(\mu,c_1,c_2,c_3)}$ gives the Choi matrix, $C_{\Psi_{(\mu,c_1,c_2,c_3)}}$ as (4.7.5).

$$aB - C^*C = \begin{pmatrix} 4c_3\mu^{-r} & 0 & 0 & 0 & -2c_1\mu^{-r} & 0 & 0 & 0 & 0 \\ 0 & 4c_2\mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4\mu^{-2r} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4c_1\mu^{-r} & 0 & 0 & 0 & 0 & 0 \\ -2c_1\mu^{-r} & 0 & 0 & 0 & 4c_1\mu^{-r} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4\mu^{-2r} - c_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4c_3\mu^{-r} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4\mu^{-2r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4c_2\mu^{-r} \end{pmatrix}.$$

The matrix is positive when $2\mu^{-r} > c_1$ and $4c_3 \geq c_1$.

$$dU = 2c_2 \begin{pmatrix} 2c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\mu^{-r} & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 \\ 0 & 0 & 2\mu^{-r} & 0 & 0 & 0 & -c_3 & 0 & 0 \\ 0 & 0 & 0 & 2c_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2c_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2c_2 & 0 & 0 & 0 \\ 0 & 0 & -c_3 & 0 & 0 & 0 & 2c_1 & 0 & 0 \\ 0 & -c_3 & 0 & 0 & 0 & 0 & 0 & 2\mu^{-r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\mu^{-r} \end{pmatrix}.$$

$dU \geq 0$ is positive when $4\mu^{-r} > c_3$ and $4c_1\mu^{-r} > c_3^2$.

$$aU - Y^*Y \quad = \quad \begin{pmatrix} 4c_1\mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4\mu^{-2r} - c_2^2 & 0 & 0 & 0 & 0 & 0 & -2c_3\mu^{-r} & -c_2\mu \\ 0 & 0 & 4\mu^{-2r} & 0 & 0 & 0 & -2c_3\mu^{-r} & 0 & 0 \\ 0 & 0 & 0 & 4c_3\mu^{-r} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4c_3\mu^{-r} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2c_4\mu^{-r} & 0 & 0 & 0 \\ 0 & 0 & -2c_3\mu^{-r} & 0 & 0 & 0 & 4c_1\mu^{-r} & 0 & 0 \\ 0 & -2c_3\mu^{-r} & 0 & 0 & 0 & 0 & 0 & 4\mu^{-2r} & 0 \\ 0 & -c_2\mu & 0 & 0 & 0 & 0 & 0 & 0 & 4\mu^{-2r} - \mu^2 \end{pmatrix}$$

is positive provided the inequalities, $2\mu^{-r} > c_2, \quad 4\mu^{-2r} > c_2^2 + c_3^2 \quad and \quad 4\mu^{-2r} > c_2^2 + \mu^2$ are satisfied.

$$aU - Z^*Z \quad = \quad \begin{pmatrix} 4c_1\mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4\mu^{-2r} - c_2^2 & 0 & 0 & 0 & 0 & 0 & -2c_3\mu^{-r} & 0 \\ 0 & 0 & 4\mu^{-2r} & 0 & 0 & 0 & -2c_3\mu^{-r} & 0 & 0 \\ 0 & 0 & 0 & 4c_3\mu^{-r} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4c_3\mu^{-r} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2c_4\mu^{-r} & 0 & 0 & 0 \\ 0 & 0 & -2c_3\mu^{-r} & 0 & 0 & 0 & 4c_1\mu^{-r} & 0 & 0 \\ 0 & -2c_3\mu^{-r} & 0 & 0 & 0 & 0 & 0 & 4\mu^{-2r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4\mu^{-2r} - \mu^2 \end{pmatrix}$$

is positive satisfying the inequalities, $2\mu^{-r} > c_2$ *and* $4\mu^{-2r} > c_2^2 + c_3^2$.

$$U - TB^{-1}T^* = \begin{pmatrix} 2c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\mu^{-r} & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 \\ 0 & 0 & 2\mu^{-r} & 0 & 0 & 0 & -c_3 & 0 & 0 \\ 0 & 0 & 0 & 2c_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2c_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2c_2 & 0 & 0 & 0 \\ 0 & 0 & -c_3 & 0 & 0 & 0 & 2c_1 & 0 & 0 \\ 0 & -c_3 & 0 & 0 & 0 & 0 & 0 & 2\mu^{-r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\mu^{-r} \end{pmatrix}$$

$$- \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -c_2 & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 \\ -c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -c_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2c_3 & 0 & 0 & 0 & -c_1 & 0 & 0 & 0 & 0 \\ 0 & 2c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2c_1 & 0 & 0 & 0 & 0 & 0 \\ -c_1 & 0 & 0 & 0 & 2c_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu^{-r} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2c_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\mu^{-r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2c_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -c_2 & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 \\ -c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -c_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T$$

$$= \begin{pmatrix} 2c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\mu^{-r} & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 \\ 0 & 0 & 2\mu^{-r} & 0 & 0 & 0 & -c_3 & 0 & 0 \\ 0 & 0 & 0 & 2c_3 & 0 & 0 & \frac{c_2\mu}{c_1+c_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2c_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2c_2-\frac{1}{2}(c_2+c_3\mu^r) & 0 & 0 & 0 \\ 0 & 0 & -c_3 & 0 & 0 & 0 & 2c_1-\frac{c_2^2}{c_1+c_3} & 0 & 0 \\ 0 & -c_3 & 0 & \frac{1}{2}c_3\mu^{r+1} & 0 & 0 & 0 & 2\mu^{-r}-\frac{1}{2}c_3^2\mu^r & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\mu^{-r} \end{pmatrix}.$$

The minors of $U - TB^{-1}T^*$ are nonnegative when , $2\mu^{-r} > c_3$, $c_1 \geq c_2$ and $4c_1\mu^{-r} > c_3^2$. holds. □

Remark 4.41. It is clear from (4.7.5) that $C_{\Psi_{(\mu,c_1,c_2,c_3)}}$ and $C^{\Gamma}_{\Psi_{(\mu,c_1,c_2,c_3)}}$ are equal.

$$
\left(\begin{array}{c|ccccccccc||c|ccccccccc}
2\mu^{-r} & 0 & 0 & 0 & 0 & 0 & -c_1 & 0 & 0 & 0 & 0 & 0 & -c_2 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu \\
\hline
0 & 2c_3 & 0 & 0 & 0 & -c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2\mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 \\
0 & -c_1 & 0 & 0 & 0 & 2c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-c_1 & 0 & 0 & 0 & 0 & 0 & 2\mu^{-r} & 0 & 0 & 0 & 0 & 0 & -c_2 & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2c_3 & 0 & 0 & 0 & -c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline\hline
0 & 0 & -c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & -c_2 & 0 & 0 & 0 & 2c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-c_2 & 0 & 0 & 0 & 0 & 0 & -c_2 & 0 & 0 & 0 & 0 & 0 & 2\mu^{-r} & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\mu^{-r} & 0 & 0 & 0 & -c_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2c_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2c_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2c_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 & 0 & 0 & 2c_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 & 0 & 0 & 0 & 0 & 2\mu^{-r} & 0 \\
-\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\mu^{-r}
\end{array}\right) \tag{4.7.5}
$$

.

Proposition 4.42. *The linear map* $\Psi_{(\mu,c_1,c_2,c_3)}$ *is decomposable.*

Proof. From Proposition. 4.40 $\Psi_{(\mu,c_1,c_2,c_3)}$ is 2-positive(2-copositive). Since the Choi matrix of complete positivity is equal to the Choi matrix of complete copositivity, we observe that from Proposition 4.17 and Proposition 4.18,

$$C_{\Psi_{(\mu,c_1,c_2,c_3)}} = C_{\psi_{(\mu,c_1,c_2,c_3)}} + C^{\Gamma}_{\psi_{(\mu,c_1,c_2,c_3)}}.$$

Therefore, $\Psi_{(\mu,c_1,c_2,c_3)}$ is decomposable. □

Proposition 4.43. *For every linear map* ψ *in* $\mathbf{B}(\mathcal{M}_n(\mathbb{C}), \mathcal{M}_m(\mathbb{C}))$*, if the matrix transpose of* $[\psi(x_{ij})]$ *is equal to* $[\psi(x_{ji})]$*, then* ψ *decomposable.*

Proof. Assume that $\mathcal{M}_n \subset \mathbf{B}(\mathcal{S})$ for some Hilbert space $\mathcal{S}$. Let

$$S = \{\begin{pmatrix} x & 0 \\ 0 & x^T \end{pmatrix} \in \mathcal{M}_2(\mathbf{B}(\mathcal{S})) : x \in \mathcal{M}_n\}, \tag{4.7.6}$$

where T is the transposition map with respect to some orthonormal basis in $\mathcal{S}$. Then S is a subspace of $\mathcal{M}_2(\mathbf{B}(\mathcal{S}))$ with the identity. One can observe that both $[x_{ij}]$ and $[x_{ji}]$ are both in $\mathcal{M}_k(\mathcal{M}_n)^+$ if and only if

$$\begin{pmatrix} \begin{bmatrix} x_{11} & 0 \\ 0 & x_{kk}^T \end{bmatrix} & \cdots & \begin{bmatrix} x_{1k} & 0 \\ 0 & x_{1k}^T \end{bmatrix} \\ \vdots & & \vdots \\ \begin{bmatrix} x_{k1} & 0 \\ 0 & x_{k1}^T \end{bmatrix} & \cdots & \begin{bmatrix} x_{kk} & 0 \\ 0 & x_{kk}^T \end{bmatrix} \end{pmatrix} \in \mathcal{M}_k(\mathcal{S})^+.$$

Therefore ψ is k-positive. Since $[\psi(x_{ij})] = [\psi(x_{ji})] \in \mathcal{M}_k(\mathcal{A})^+$. By ([92], Theorem 1.) ψ is decomposable. □

Chapter 5

CONCLUSION AND RECOMMENDATIONS

5.1 Introduction

In this chapter we give a summary of main results in this thesis. In chapter one, we gave background information, basic concepts and definitions which were essential in the study. In chapter two, we looked at the related literature in line with the stated objectives in Section 1.4. In the Third chapters we have reviewed the necessary techniques that have been employed to achieve the stated objectives. In chapter four, we gave the results obtained in the discussions. Here we give conclusions and make recommendations based on our objective of study and the results obtained.

5.2 Conclusion

The first objective was to construct positive maps. We have employed the use of vectors $x = x_1, \ldots, x_i$ for $i = 2, \ldots, n$ and their conjugate transpose to generate positive semidefinite matrices. The linear map $\psi_{(\mu, c_1, \ldots, c_{n-1})}$ constructed in (4.2.2) maps that take the positive semidefinite matrices $\mathcal{M}_n$ to the space $\mathcal{M}_n(\mathcal{M}_{n+1})$ with entries from the field of complex numbers. Canonical reshuffle of the linear map $\psi_{(\mu,c_1,c_2)}$ allowed us to define a special map in (4.6.1) that takes positive semidefinite matrices $\mathcal{M}_3$ to the space $\mathcal{M}_3(\mathcal{M}_2)$ with entries from the field of 2×2 matrices. The maps in (4.7.1) and (4.7.5) were constructed by merging of two positive maps.

In objective two, the concept of positive semidefinite polynomials was applied to show positivity of linear maps. It has been shown that the family of linear maps $\psi_{(\mu,c_1,\ldots,c_{n-1})}$ and $\Psi_{(\mu,c_1,\ldots,c_{n-1})}$ from $\mathcal{M}_n$ to $\mathcal{M}_{n+1}, n = 2, 3, 4$ is positive preserving for all positive semidefinite matrices X in $\mathcal{M}_n(\mathbb{C})$. The range of values for which the maps are positive have been established for the parameters $\mu, c_1, \ldots, c_{n-1}$. We also observed that from Proposition 4.2, Proposition 4.4, and Proposition 4.6 that the map $\psi_{(\mu,c_1,\ldots,c_{n-1})}$ is positive for all $c_n \in [0, 1]$ whenever $0 < \mu < 1$ for $n = 2, 3, 4$ for all $r \in \mathbb{R}^+$.

The third objective was to characterize the structure of Choi matrix for 2-positive maps that are similar to the Choi matrices of the maps $\psi_{(\mu,c_1,\ldots,c_{n-1})}$. The Choi matrix (4.4.3)canonical shuffling can be visualized i $\mathcal{M}_2(\mathcal{M}_{m+1})$. This shows that $\mathcal{M}_n \otimes \mathcal{M}_{(n+1)}$ is isomorphic to the block matrices $\mathcal{M}_n(\mathcal{M}_{(n+1)})$.

$$\mathcal{M}_n \otimes \mathcal{M}_{(n+1)} \cong \mathcal{M}_n(\mathcal{M}_{(n+1)}) \cong \mathcal{M}_2(\mathcal{M}_{m+1}).$$

The conditions given in the Choi matrix 4.4.3 which also implies that the linear maps $\psi_{(\mu,c_1,\ldots,c_{n-1})}$ are completely positive whenever the conditions (i) through to (v) are necessary. We observe that, if a positive map $\psi_{(\mu,c_1,\ldots,c_{n-1})}$ is $2-$positive, then equivalently it is completely positive for all $r \in \mathbb{R}^+$ for $n \leq 4$.

Indeed, in the final objective of the thesis we considered decomposition of the linear maps $\psi_{(\mu,c_1,\ldots,c_{n-1})}$, $\Phi_{(\mu,c_1,c_2)}$, and the merger maps $\Psi_{(\mu,c_1,c_2)}$ and $\Psi_{(\mu,c_1,c_2,c_3)}$. This was demonstrated in Proposition 4.20, Proposition 4.24, Proposition 4.26, Proposition 4.33, Proposition 4.37 and Proposition 4.42 respectively. This gives an affirmative result in [105], Theorem 3.1.6, that if ψ is a positive linear map from $\mathcal{M}_n(\mathbb{C})$ to $\mathcal{M}_m(\mathbb{C})$, then the map is decomposable provided $nm \leq 6$. It is important to note that the decomposition of the map $\psi_{(\mu,c_1)}$ is not unique. This is one of the reasons decomposition of positive maps even in low dimensions is such complicated to be expressed with a unique algorithm. We have generalized them as the sum of the Choi matrices of the 2-positive and 2-copositive maps. However, we have also used Størmer decomposition criteria to show decomposability. It should be noted that this criteria does not give us the conditions but rather the existence of a decomposable map. A special decomposition technique is shown for the map $\Psi_{(\mu,c_1,c_2)}$ form $\mathcal{M}_3$ to $\mathcal{M}_2(\mathcal{M}_2)$ as in Proposition 4.33 where the partial transposition is operated with respect to the 2×2 matrices as the entry elements as opposed to the conventional way where the elements are complex numbers. It is important to note that the decomposition of these linear maps are not unique.

5.3 Recommendations

From this study, although several properties of linear positive maps have been investigated, it is evident that it remains an interesting and rich area of research in pure mathematics. There has been a vast growth on both the mathematical and physical literature on indecomposable positive maps. However, the questions of decomposability of positive seems to have been ignored. The motivation is that this problem is twofold: on one hand, the mathematical structure of positive maps with respect the block-matrix element transpose will be useful to witness the presence of entanglement, on the other, one may hope that they could shed some light on the process of entanglement generation by the application of such maps.

For two positive operators P and Q acting on $\mathbb{C}^n \otimes \mathbb{C}^m$, W is a decomposable entanglement witness if and only if there exist $a, b \geq 0$ such that $W = aP + bQ^{\Gamma}$, where Γ is Partial Transposition. Otherwise, an entanglement witness is indecomposable if and only if it detects Partial positive entangled states. It is clear that if W is a Choi matrix of a positive map ψ. Then P and Q^Γ are Choi matrices of completely positive, ψ_1 and completely copositive, ψ_2 maps respectively. The Example 4.19 gives another special case where the linear map is 2-positive and its Choi matrix is both completely positive and completely copositive as in Equation. Based on this we ask, if there exist 2-positive maps from $\mathcal{M}_{2n}(\mathbb{C})$ to $\mathcal{M}_{2n}(\mathbb{C})$ that is decomposable if and only if $C_\psi = C_\psi^\Gamma$. It is our belief that generally this should hold even when the dimensions of the underlying spaces are large enough.

We have shown the existence of a 2-decomposable map from $\mathcal{M}_3(\mathbb{C})$ to $\mathcal{M}_4(\mathbb{C})$. In closing we ask; For $n(n+1) \geq 12$, does there exist 2-positive map from $\mathcal{M}_n(\mathbb{C})$ to $\mathcal{M}_{n+1}(\mathbb{C})$ that is indecomposable? We end by the question;

Question 5.1. *If Ψ is a 2-decomposable and there are ψ_1 a 2-positive map and ψ_2 a 2-copositive map, $\psi_1, \psi_2 : \mathcal{A} \longrightarrow \mathbf{B}(\mathcal{H})$ such that $\Psi = \psi_1 + \psi_2$, does there exist a linear map ψ_1 that is 2-decomposable?*

Chapter 6

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Appendix A

Choi matrices positive map $\psi_{(\mu,c_1,c_2,c_3)}$

A.1 Choi matrix for complete positivity map $\psi_{(\mu,c_1,c_2,c_3)}$

```
m={{\[Mu]^(-n),0,0,0,0,0,-\[Alpha],0,0,0,0,0,-\[Kappa],0,0,0,0,0,0,-\[Mu]},
 {0,\[Beta],0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0},{0,0,\[Kappa],0,0,0,0,0,0,
 0,0,0,0,0,0,0,0,0,0,0},{0,0,0,\[Mu]^(-n),0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0},
 {0,0,0,0,\[Alpha],0,0,0,0,0,0,0,0,0,0,0,0,0,0,0},{0,0,0,0,0,\[Alpha],0,0,
 0,0,0,0,0,0,0,0,0,0,0,0},{-\[Alpha],0,0,0,0,0,\[Mu]^(-n),0,0,0,0,0,-\[Kappa],
 0,0,0,0,0,-\[Beta],0},{0,0,0,0,0,0,0,\[Beta],0,0,0,0,0,0,0,0,0,0,0,0},
 {0,0,0,0,0,0,0,0,\[Mu]^(-n),0,0,0,0,0,0,0,0,0,0,0},{0,0,0,0,0,0,0,0,0,
 \[Kappa],0,0,0,0,0,0,0,0,0,0},{0,0,0,0,0,0,0,0,0,0,\[Kappa],0,0,0,0,0,0,
 0,0,0},{0,0,0,0,0,0,0,0,0,0,0,\[Alpha],0,0,0,0,0,0,0,0},
 {-\[Kappa],0,0,0,0,0,-\[Kappa],0,0,0,0,0,\[Mu]^(-n),0,0,0,0,0,-\[Beta],0},
 {0,0,0,0,0,0,0,0,0,0,0,0,0,\[Mu]^(-n),0,0,0,0,0,0},{0,0,0,0,0,0,0,0,0,0,0,
 0,0,0,\[Beta],0,0,0,0,0},{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\[Beta],0,0,0,0},
 {0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\[Kappa],0,0,0},{0,0,0,0,0,0,0,0,0,0,0,0,
 0,0,0,0,0,\[Alpha],0,0},{0,0,0,0,0,0,-\[Beta],0,0,0,0,0,-\[Beta],0,0,0,0,
 0,\[Mu]^(-n),0},{-\[Mu],0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\[Mu]^(-n)}}
```

A.2 Choi matrix of completely copositive map $\psi^{\Gamma}_{(\mu,c_1,c_2,c_3)}$

```
m={{\[Mu]^(-n),0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0},
```

```
{0,\[Beta],0,0,0,-\[Alpha],0,0,0,0,0,0,0,0,0,0,0,0,0,0},
{0,0,\[Kappa],0,0,0,0,0,0,0,-\[Kappa],0,0,0,0,0,0,0,0,0},
{0,0,0,\[Mu]^(-n),0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0},
{0,0,0,0,\[Alpha],0,0,0,0,0,0,0,0,0,0,-\[Mu],0,0,0,0},
{0,-\[Alpha],0,0,0,\[Alpha],0,0,0,0,0,0,0,0,0,0,0,0,0,0},
{0,0,0,0,0,0,\[Mu]^(-n),0,0,0,0,0,0,0,0,0,0,0,0,0},
{0,0,0,0,0,0,0,\[Beta],0,0,0,-\[Kappa],0,0,0,0,0,0,0,0},
{0,0,0,0,0,0,0,0,\[Mu]^(-n),0,0,0,0,0,0,0,-\[Beta],0,0,0},
{0,0,0,0,0,0,0,0,0,\[Kappa],0,0,0,0,0,0,0,0,0,0},
{0,0,-\[Kappa],0,0,0,0,0,0,0,\[Kappa],0,0,0,0,0,0,0,0,0},
{0,0,0,0,0,0,0,-\[Kappa],0,0,0,\[Alpha],0,0,0,0,0,0,0,0},
{0,0,0,0,0,0,0,0,0,0,0,0,\[Mu]^(-n),0,0,0,0,0,0,0},
{0,0,0,0,0,0,0,0,0,0,0,0,0,\[Mu]^(-n),0,0,0,-\[Beta],0,0},
{0,0,0,0,0,0,0,0,0,0,0,0,0,0,\[Beta],0,0,0,0,0},
{0,0,0,0,-\[Mu],0,0,0,0,0,0,0,0,0,0,\[Beta],0,0,0,0},
{0,0,0,0,0,0,0,0,-\[Beta],0,0,0,0,0,0,0,\[Kappa],0,0,0},
{0,0,0,0,0,0,0,0,0,0,0,0,0,-\[Beta],0,0,0,\[Alpha],0,0},
{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\[Mu]^(-n),0},
{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\[Mu]^(-n)}}
```

A.2.1 B^{-1}-Completely copositive matrix

In[3]:=
```
B = {{β, 0, 0, 0, −α, 0, 0, 0, 0}, {0 , κ , 0 , 0 , 0 , 0, 0, 0, 0}, {0 , 0 , μ^−r , 0 , 0 , 0 , 0 , 0 , 0},
  {0 , 0 , 0 , α, 0, 0 , 0 , 0 , 0}, {−α , 0 , 0 , 0 , α , 0 , 0 , 0 , 0}, {0 , 0 , 0 , 0 , 0 , μ^−r , 0 , 0 , 0},
  {0 , 0 , 0 , 0 , 0 , 0 , β , 0 , 0}, {0 , 0 , 0 , 0 , 0 , 0 , 0 , μ^−r , 0}, {0 , 0, 0 , 0 , 0 , 0 , 0 , 0 , κ}}
```

Out[3]=
$$\{\{\beta, 0, 0, 0, -\alpha, 0, 0, 0, 0\}, \{0, \kappa, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, \mu^{-r}, 0, 0, 0, 0, 0, 0\},$$
$$\{0, 0, 0, \alpha, 0, 0, 0, 0, 0\}, \{-\alpha, 0, 0, 0, \alpha, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, \mu^{-r}, 0, 0, 0\},$$
$$\{0, 0, 0, 0, 0, 0, \beta, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, \mu^{-r}, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, \kappa\}\}$$

In[4]:=
```
Inverse[B]
```

Out[4]=
$$\Big\{\Big\{\frac{\alpha^2\,\beta\,\kappa^2\,\mu^{3-r}}{- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0, 0, 0, -\frac{-\alpha\,\alpha\,\beta\,\kappa^2\,\mu^{3-r}}{- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0, 0, 0, 0\Big\},$$
$$\Big\{0, \frac{- -\alpha^2\,\alpha\,\beta\,\kappa\,\mu^{3-r} + \alpha^2\,\beta^2\,\kappa\,\mu^{3-r}}{- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0, 0, 0, 0, 0, 0, 0\Big\},$$
$$\Big\{0, 0, \frac{- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{2-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{2-r}}{- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0, 0, 0, 0, 0, 0\Big\},$$
$$\Big\{0, 0, 0, \frac{- -\alpha^2\,\beta\,\kappa^2\,\mu^{3-r} + \alpha\,\beta^2\,\kappa^2\,\mu^{3-r}}{- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0, 0, 0, 0, 0\Big\},$$
$$\Big\{-\frac{-\alpha\,\alpha\,\beta\,\kappa^2\,\mu^{3-r}}{- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0, 0, 0, \frac{\alpha\,\beta^2\,\kappa^2\,\mu^{3-r}}{- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0, 0, 0, 0\Big\},$$
$$\Big\{0, 0, 0, 0, 0, \frac{- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{2-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{2-r}}{- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0, 0, 0\Big\},$$
$$\Big\{0, 0, 0, 0, 0, 0, \frac{- -\alpha^2\,\alpha\,\kappa^2\,\mu^{3-r} + \alpha^2\,\beta\,\kappa^2\,\mu^{3-r}}{- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0, 0\Big\},$$
$$\Big\{0, 0, 0, 0, 0, 0, 0, \frac{- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{2-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{2-r}}{- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0\Big\},$$
$$\Big\{0, 0, 0, 0, 0, 0, 0, 0, \frac{- -\alpha^2\,\alpha\,\beta\,\kappa\,\mu^{3-r} + \alpha^2\,\beta^2\,\kappa\,\mu^{3-r}}{- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}\Big\}\Big\}$$

$TB^{-1}T^*$

In[7]:=

$$\begin{aligned}
m = \{&\{0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0\} \{0, 0, 0, 0, 0, 0, 0, 0, 0\},\\
&\{0, 0, 0, 0, -\mu, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0\},\\
&\{-\kappa, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, -\beta, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0\}\}.
\end{aligned}$$

$$\begin{aligned}
\Big\{&\Big\{\frac{\alpha^2\,\beta\,\kappa^2\,\mu^{3-r}}{- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0, 0, 0, -(-\alpha\,\alpha\,\beta\,\kappa^2\,\mu^{3-r}) / (- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}), 0, 0, 0, 0\Big\},\\
&\Big\{0, \frac{- -\alpha^2\,\alpha\,\beta\,\kappa\,\mu^{3-r} + \alpha^2\,\beta^2\,\kappa\,\mu^{3-r}}{- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0, 0, 0, 0, 0, 0, 0\Big\}, \Big\{0, 0, \frac{- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{2-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{2-r}}{- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}},\\
&\quad 0, 0, 0, 0, 0, 0\Big\}, \Big\{0, 0, 0, \frac{- -\alpha^2\,\beta\,\kappa^2\,\mu^{3-r} + \alpha\,\beta^2\,\kappa^2\,\mu^{3-r}}{- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0, 0, 0, 0, 0\Big\},\\
&\Big\{-(-\alpha\,\alpha\,\beta\,\kappa^2\,\mu^{3-r}) / (- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}), 0, 0, 0, \frac{\alpha\,\beta^2\,\kappa^2\,\mu^{3-r}}{- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0, 0, 0, 0\Big\},\\
&\Big\{0, 0, 0, 0, 0, \frac{- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{2-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{2-r}}{- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0, 0, 0\Big\},\\
&\Big\{0, 0, 0, 0, 0, 0, \frac{- -\alpha^2\,\alpha\,\kappa^2\,\mu^{3-r} + \alpha^2\,\beta\,\kappa^2\,\mu^{3-r}}{- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0, 0\Big\}, \Big\{0, 0, 0, 0, 0, 0, 0,\\
&\quad \frac{- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{2-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{2-r}}{- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0\Big\}, \Big\{0, 0, 0, 0, 0, 0, 0, 0, \frac{- -\alpha^2\,\alpha\,\beta\,\kappa\,\mu^{3-r} + \alpha^2\,\beta^2\,\kappa\,\mu^{3-r}}{- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}\Big\}\Big\}.
\end{aligned}$$

$$\begin{aligned}
\{&\{0, 0, 0, 0, 0, 0, -\kappa, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0\},\\
&\{0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, -\mu, 0, 0, 0, -\beta, 0\},\\
&\{0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0\}\}
\end{aligned}$$

Out[7]=

$$\begin{aligned}
\Big\{&\{0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0\},\\
&\Big\{0, 0, 0, 0, 0, 0, -\frac{-\alpha\,\alpha\,\beta\,\kappa^3\,\mu\,\mu^{3-r}}{- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0, 0\Big\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0\},\\
&\{0, 0, 0, 0, 0, 0, 0, 0, 0\}, \Big\{0, 0, 0, 0, 0, 0, \frac{\alpha^2\,\beta\,\kappa^4\,\mu^{3-r}}{- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0, 0\Big\},\\
&\Big\{0, 0, 0, \frac{\beta\,\mu\,(- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{2-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{2-r})}{- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0, 0, 0, \frac{\beta^2\,(- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{2-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{2-r})}{- -\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0\Big\},\\
&\{0, 0, 0, 0, 0, 0, 0, 0, 0\}\Big\}
\end{aligned}$$

$U - TB^{-1}T^*$

```
In[1]:= d = {{β , 0, 0, 0, 0, 0, 0, 0, 0}, {0, μ^(-r), 0, 0, 0, 0, 0, 0, 0}, {0, 0, μ^(-r), 0, 0, 0, -β , 0, 0},
    {0, 0, 0, β , 0, 0, 0, 0, 0}, {0, 0, 0, 0, β , 0, 0, 0, 0}, {0, 0, 0, 0, 0, κ, 0, 0, 0},
    {0, 0, -β , 0, 0, 0, α, 0, 0}, {0, 0, 0, 0, 0, 0, 0, μ^(-r), 0}, {0, 0, 0, 0, 0, 0, 0, 0, μ^(-r)}} -
   {{0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, -(—α α β κ^3 μ μ^(3-r)) / (-—α^2 α β κ^2 μ^(3-r) + α^2 β^2 κ^2 μ^(3-r)), 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, (α^2 β κ^4 μ^(3-r)) / (-—α^2 α β κ^2 μ^(3-r) + α^2 β^2 κ^2 μ^(3-r)), 0, 0},
    {0, 0, 0, (β μ (-—α^2 α β κ^2 μ^(2-r) + α^2 β^2 κ^2 μ^(2-r))) / (-—α^2 α β κ^2 μ^(3-r) + α^2 β^2 κ^2 μ^(3-r)), 0, 0, 0,
     (β^2 (-—α^2 α β κ^2 μ^(2-r) + α^2 β^2 κ^2 μ^(2-r))) / (-—α^2 α β κ^2 μ^(3-r) + α^2 β^2 κ^2 μ^(3-r)), 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

```
Out[1]= {{β, 0, 0, 0, 0, 0, 0, 0, 0}, {0, μ^-r, 0, 0, 0, 0, 0, 0, 0}, {0, 0, μ^-r, 0, 0, 0, -β, 0, 0},
   {0, 0, 0, β, 0, 0, (—α α β κ^3 μ μ^(3-r)) / (-—α^2 α β κ^2 μ^(3-r) + α^2 β^2 κ^2 μ^(3-r)), 0, 0}, {0, 0, 0, 0, β, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, κ, 0, 0, 0}, {0, 0, -β, 0, 0, 0, α - (α^2 β κ^4 μ^(3-r)) / (-—α^2 α β κ^2 μ^(3-r) + α^2 β^2 κ^2 μ^(3-r)), 0, 0},
   {0, 0, 0, -(β μ (-—α^2 α β κ^2 μ^(2-r) + α^2 β^2 κ^2 μ^(2-r))) / (-—α^2 α β κ^2 μ^(3-r) + α^2 β^2 κ^2 μ^(3-r)), 0, 0, 0,
    μ^-r - (β^2 (-—α^2 α β κ^2 μ^(2-r) + α^2 β^2 κ^2 μ^(2-r))) / (-—α^2 α β κ^2 μ^(3-r) + α^2 β^2 κ^2 μ^(3-r)), 0}, {0, 0, 0, 0, 0, 0, 0, 0, μ^-r}}
```

```
In[3]:= MatrixForm[d]
```

Out[3]//MatrixForm=

$$\begin{pmatrix} \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu^{-r} & 0 & 0 & 0 & -\beta & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 & 0 & \frac{-\alpha\alpha\beta\kappa^3\mu\mu^{3-r}}{--\alpha^2\alpha\beta\kappa^2\mu^{3-r}+\alpha^2\beta^2\kappa^2\mu^{3-r}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \kappa & 0 & 0 & 0 \\ 0 & 0 & -\beta & 0 & 0 & 0 & \alpha - \frac{\alpha^2\beta\kappa^4\mu^{3-r}}{--\alpha^2\alpha\beta\kappa^2\mu^{3-r}+\alpha^2\beta^2\kappa^2\mu^{3-r}} & 0 & 0 \\ 0 & 0 & 0 & -\frac{\beta\mu\left(--\alpha^2\alpha\beta\kappa^2\mu^{2-r}+\alpha^2\beta^2\kappa^2\mu^{2-r}\right)}{--\alpha^2\alpha\beta\kappa^2\mu^{3-r}+\alpha^2\beta^2\kappa^2\mu^{3-r}} & 0 & 0 & 0 & \mu^{-r} - \frac{\beta^2\left(--\alpha^2\alpha\beta\kappa^2\mu^{2-r}+\alpha^2\beta^2\kappa^2\mu^{2-r}\right)}{--\alpha^2\alpha\beta\kappa^2\mu^{3-r}+\alpha^2\beta^2\kappa^2\mu^{3-r}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} \end{pmatrix}$$

In[4]:= **d47** $= \dfrac{\text{—}\alpha\,\alpha\,\beta\,\kappa^3\,\mu\,\mu^{3\,\text{—}r}}{-\text{—}\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3\,\text{—}r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3\,\text{—}r}}$

Out[4]= $\dfrac{\text{—}\alpha\,\alpha\,\beta\,\kappa^3\,\mu\,\mu^{3\,\text{—}r}}{-\text{—}\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3\,\text{—}r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3\,\text{—}r}}$

In[5]:= **Factor[d47]**

Out[5]= $-\dfrac{\text{—}\alpha\,\kappa\,\mu}{\text{—}\alpha^2 - \alpha\,\beta}$

In[6]:= **d77** $= \dfrac{\alpha^2\,\beta\,\kappa^4\,\mu^{3\,\text{—}r}}{-\text{—}\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3\,\text{—}r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3\,\text{—}r}}$

Out[6]= $\dfrac{\alpha^2\,\beta\,\kappa^4\,\mu^{3\,\text{—}r}}{-\text{—}\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3\,\text{—}r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3\,\text{—}r}}$

In[7]:= **Factor[d77]**

Out[7]= $-\dfrac{\alpha\,\kappa^2}{\text{—}\alpha^2 - \alpha\,\beta}$

In[8]:= **d84** $= \dfrac{\beta\,\mu\,(-\text{—}\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{2\,\text{—}r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{2\,\text{—}r})}{-\text{—}\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3\,\text{—}r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3\,\text{—}r}}$

Out[8]= $\dfrac{\beta\,\mu\,(-\text{—}\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{2\,\text{—}r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{2\,\text{—}r})}{-\text{—}\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3\,\text{—}r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3\,\text{—}r}}$

In[9]:= **Factor[d84]**

Out[9]= $\beta\,\mu\,\mu^{-\text{—}r}$

In[10]:= **d88** $= \dfrac{\beta^2\,(-\text{—}\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{2\,\text{—}r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{2\,\text{—}r})}{-\text{—}\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3\,\text{—}r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3\,\text{—}r}}$

Out[10]= $\dfrac{\beta^2\,(-\text{—}\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{2\,\text{—}r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{2\,\text{—}r})}{-\text{—}\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3\,\text{—}r} + \alpha^2\,\beta^2\,\kappa^2\,\mu^{3\,\text{—}r}}$

In[11]:= **Factor[d88]**

Out[11]= $\beta^2\,\mu^{-\text{—}r}$

Appendix B

Choi matrix positive map $\Phi_{(\mu,c_1,c_2,c_3)}$

B.0.1 Completely copositive matrix

```
d = {{2\[Mu]^(-r),0,0,0,0,0,-\[Alpha],0,0,0,0,0,-\[Kappa],0,0,0,0,0,0,-\[Mu]},
{0,2 \[Beta],0,0,0,-\[Alpha],0,0,0,0,0,0,0,0,0,0,0,0,0,0},{0,0,2 \[Kappa],
0,0,0,0,0,0,0,-\[Kappa],0,0,0,0,0,0,0,0,0},{0,0,0,2\[Mu]^(-r),0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0},{0,0,0,0,2\[Alpha],0,0,0,0,0,0,0,0,0,0,-\[Mu],0,0,0,0},
{0,-\[Alpha],0,0,0,2\[Alpha],0,0,0,0,0,0,0,0,0,0,0,0,0,0},{-\[Alpha],0,0,
0,0,0,2\[Mu]^(-r),0,0,0,0,0,-\[Kappa],0,0,0,0,0,-\[Beta],0},{0,0,0,0,0,0,0,
2 \[Beta],0,0,0,-\[Kappa],0,0,0,0,0,0,0,0},{0,0,0,0,0,0,0,0,2\[Mu]^(-r),0,0,
0,0,0,0,0,-\[Beta],0,0,0},{0,0,0,0,0,0,0,0,0,2 \[Kappa],0,0,0,0,0,0,0,0,0,0},
{0,0,-\[Kappa],0,0,0,0,0,0,0,2\[Kappa],0,0,0,0,0,0,0,0,0},{0,0,0,0,0,0,0,
-\[Kappa],0,0,0,2\[Alpha],0,0,0,0,0,0,0,0},{-\[Kappa],0,0,0,0,0,-\[Kappa],0,
0,0,0,0,2\[Mu]^(-r),0,0,0,0,0,-\[Beta],0},{0,0,0,0,0,0,0,0,0,0,0,0,0,
2\[Mu]^(-r),0,0,0,-\[Beta],0,0},{0,0,0,0,0,0,0,0,0,0,0,0,0,0,2 \[Beta],
0,0,0,0,0},{0,0,0,0,-\[Mu],0,0,0,0,0,0,0,0,0,0,2 \[Beta],0,0,0,0},{0,0,0,
0,0,0,0,0,-\[Beta],0,0,0,0,0,0,0,2\[Kappa],0,0,0},{0,0,0,0,0,0,0,0,0,0,0,
0,0,-\[Beta],0,0,0,2\[Alpha],0,0},{0,0,0,0,0,0,-\[Beta],0,0,0,0,0,-\[Beta],0,
0,0,0,0,2\[Mu]^(-r),0},{-\[Mu],0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,2\[Mu]^(-r)}}
```

```
In[16]:= b = {{2β, 0, 0, 0, −α, 0, 0, 0, 0}, {0 , 2κ , 0 , 0 , 0 , 0, 0, 0, 0},
           {0 , 0 , 2μ^−r , 0 , 0 , 0 , 0 , 0 , 0}, {0 , 0 , 0 , 2α, 0, 0 , 0 , 0 , 0},
          {−α , 0 , 0 , 0 , 2α , 0 , 0 , 0 , 0}, {0 , 0 , 0 , 0 , 0 , 2μ^−r , 0 , 0 , 0},
           {0 , 0 , 0 , 0 , 0 , 0 , 2β , 0 , 0}, {0 , 0 , 0 , 0 , 0 , 0 , 0 , 2μ^−r , 0},
           {0 , 0, 0 , 0 , 0 , 0 , 0 , 0 , 2κ}}
```

Out[16]=
$$\{\{2\beta, 0, 0, 0, -\alpha, 0, 0, 0, 0\}, \{0, 2\kappa, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 2\mu^{-r}, 0, 0, 0, 0, 0, 0\},$$
$$\{0, 0, 0, 2\alpha, 0, 0, 0, 0, 0\}, \{-\alpha, 0, 0, 0, 2\alpha, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 2\mu^{-r}, 0, 0, 0\},$$
$$\{0, 0, 0, 0, 0, 0, 2\beta, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 2\mu^{-r}, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 2\kappa\}\}$$

```
In[15]:=
```

```
In[18]:= Inverse[b]
```

Out[18]=
$$\Big\{\Big\{\frac{256\,\alpha^2\,\beta\,\kappa^2\,\mu^{3-r}}{-128-\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r}+512\,\alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0, 0, 0, -\frac{128-\alpha\,\alpha\,\beta\,\kappa^2\,\mu^{3-r}}{-128-\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r}+512\,\alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0, 0, 0, 0\Big\},$$
$$\Big\{0, \frac{-64-\alpha^2\,\alpha\,\beta\,\kappa\,\mu^{3-r}+256\,\alpha^2\,\beta^2\,\kappa\,\mu^{3-r}}{-128-\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r}+512\,\alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0, 0, 0, 0, 0, 0, 0\Big\},$$
$$\Big\{0, 0, \frac{-64-\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{2-r}+256\,\alpha^2\,\beta^2\,\kappa^2\,\mu^{2-r}}{-128-\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r}+512\,\alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0, 0, 0, 0, 0, 0\Big\},$$
$$\Big\{0, 0, 0, \frac{-64-\alpha^2\,\beta\,\kappa^2\,\mu^{3-r}+256\,\alpha\,\beta^2\,\kappa^2\,\mu^{3-r}}{-128-\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r}+512\,\alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0, 0, 0, 0, 0\Big\},$$
$$\Big\{-\frac{128-\alpha\,\alpha\,\beta\,\kappa^2\,\mu^{3-r}}{-128-\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r}+512\,\alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0, 0, 0, \frac{256\,\alpha\,\beta^2\,\kappa^2\,\mu^{3-r}}{-128-\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r}+512\,\alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0, 0, 0, 0\Big\},$$
$$\Big\{0, 0, 0, 0, 0, \frac{-64-\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{2-r}+256\,\alpha^2\,\beta^2\,\kappa^2\,\mu^{2-r}}{-128-\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r}+512\,\alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0, 0, 0\Big\},$$
$$\Big\{0, 0, 0, 0, 0, 0, \frac{-64-\alpha^2\,\alpha\,\kappa^2\,\mu^{3-r}+256\,\alpha^2\,\beta\,\kappa^2\,\mu^{3-r}}{-128-\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r}+512\,\alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0, 0\Big\},$$
$$\Big\{0, 0, 0, 0, 0, 0, 0, \frac{-64-\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{2-r}+256\,\alpha^2\,\beta^2\,\kappa^2\,\mu^{2-r}}{-128-\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r}+512\,\alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0\Big\},$$
$$\Big\{0, 0, 0, 0, 0, 0, 0, 0, \frac{-64-\alpha^2\,\alpha\,\beta\,\kappa\,\mu^{3-r}+256\,\alpha^2\,\beta^2\,\kappa\,\mu^{3-r}}{-128-\alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r}+512\,\alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}\Big\}\Big\}$$

```
In[1]:= c = {{2 α, 0, 0, 0, 0, 0, 0, 0, 0}, {0 , 2 μ^(-r) , 0 , 0 , 0 , 0, 0, - β, 0}, {0 , 0 , 2 μ^(-r) , 0 , 0 , 0 , -β , 0 , 0},
    {0 , 0 , 0 , 2 β , 0, 0 , 0 , 0 , 0}, {0 , 0 , 0 , 0 , 2 β , 0 , 0 , 0 , 0}, {0 , 0 , 0 , 0 , 0 , 2 κ, 0 , 0, 0},
    {0 , 0 , - β , 0 , 0 , 0 , 2 α , 0 , 0}, {0 , - β , 0 , 0 , 0 , 0 , 0 , 2 μ^(-r) , 0}, {0 , 0, 0 , 0 , 0 , 0 , 0 , 0 , 2 μ^(-r)}} -
    {{0, 0, 0, 0, 0, 0, 0, 0, 0}, {0 , 0, 0 , 0 , 0, 0, 0, 0, 0},
     {0 , 0 , 0, 0 , 0 , 0 , 0 , 0 , 0}, {0 , 0 , 0 , 0, -μ, 0 , 0 , 0 , 0},
    {0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0}, {0 , -κ , 0 , 0 , 0 , 0, 0 , - β, 0}, {-κ , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0},
     {0 , 0 , 0 , 0 , 0 , - β , 0 , 0, 0}, {0 , 0, 0 , 0 , 0 , 0 , 0 , 0 , 0}}.
    {{(256 α^2 β κ^2 μ^(3 -r)) / (-128 -α^2 α β κ^2 μ^(3 -r) + 512 α^2 β^2 κ^2 μ^(3 -r)), 0, 0, 0,
      -(128 -α α β κ^2 μ^(3 -r)) / (-128 -α^2 α β κ^2 μ^(3 -r) + 512 α^2 β^2 κ^2 μ^(3 -r)), 0, 0, 0, 0},
     {0, (-64 -α^2 α β κ μ^(3 -r) + 256 α^2 β^2 κ μ^(3 -r)) / (-128 -α^2 α β κ^2 μ^(3 -r) + 512 α^2 β^2 κ^2 μ^(3 -r)), 0, 0, 0, 0, 0, 0, 0},
     {0, 0, (-64 -α^2 α β κ^2 μ^(2 -r) + 256 α^2 β^2 κ^2 μ^(2 -r)) / (-128 -α^2 α β κ^2 μ^(3 -r) + 512 α^2 β^2 κ^2 μ^(3 -r)), 0, 0, 0, 0, 0, 0},
     {0, 0, 0, (-64 -α^2 β κ^2 μ^(3 -r) + 256 α β^2 κ^2 μ^(3 -r)) / (-128 -α^2 α β κ^2 μ^(3 -r) + 512 α^2 β^2 κ^2 μ^(3 -r)), 0, 0, 0, 0, 0},
     {-(128 -α α β κ^2 μ^(3 -r)) / (-128 -α^2 α β κ^2 μ^(3 -r) + 512 α^2 β^2 κ^2 μ^(3 -r)), 0, 0,
      0, (256 α β^2 κ^2 μ^(3 -r)) / (-128 -α^2 α β κ^2 μ^(3 -r) + 512 α^2 β^2 κ^2 μ^(3 -r)), 0, 0, 0, 0},
     {0, 0, 0, 0, 0, (-64 -α^2 α β κ^2 μ^(2 -r) + 256 α^2 β^2 κ^2 μ^(2 -r)) / (-128 -α^2 α β κ^2 μ^(3 -r) + 512 α^2 β^2 κ^2 μ^(3 -r)), 0, 0, 0},
     {0, 0, 0, 0, 0, 0, (-64 -α^2 α κ^2 μ^(3 -r) + 256 α^2 β κ^2 μ^(3 -r)) / (-128 -α^2 α β κ^2 μ^(3 -r) + 512 α^2 β^2 κ^2 μ^(3 -r)), 0, 0},
     {0, 0, 0, 0, 0, 0, 0, (-64 -α^2 α β κ^2 μ^(2 -r) + 256 α^2 β^2 κ^2 μ^(2 -r)) / (-128 -α^2 α β κ^2 μ^(3 -r) + 512 α^2 β^2 κ^2 μ^(3 -r)), 0},
     {0, 0, 0, 0, 0, 0, 0, 0, (-64 -α^2 α β κ μ^(3 -r) + 256 α^2 β^2 κ μ^(3 -r)) / (-128 -α^2 α β κ^2 μ^(3 -r) + 512 α^2 β^2 κ^2 μ^(3 -r))}}.
    {{0, 0, 0, 0, 0, 0, -κ, 0, 0}, {0 , 0 , 0 , 0 , 0 , -κ, 0, 0, 0}, {0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0},
     {0 , 0 , 0 , 0, 0, 0 , 0 , 0 , 0}, {0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0}, {0 , 0 , 0 , -μ , 0 , 0 , 0 , -β, 0},
     {0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0}, {0 , 0 , 0 , 0 , 0 , -β , 0 , 0 , 0}, {0 , 0, 0 , 0 , 0 , 0 , 0 , 0 , 0}}
```

Out[1]=

$$\Big\{\{2\alpha, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 2\mu^{-r}, 0, 0, 0, 0, 0, -\beta, 0\}, \{0, 0, 2\mu^{-r}, 0, 0, 0, -\beta, 0, 0\},$$

$$\Big\{0, 0, 0, 2\beta, 0, 0, \frac{128 - \alpha\,\alpha\,\beta\,\kappa^3\,\mu\,\mu^{3-r}}{-128 - \alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r} + 512\,\alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0, 0\Big\}, \{0, 0, 0, 0, 2\beta, 0, 0, 0, 0\},$$

$$\Big\{0, 0, 0, 0, 0, 2\kappa - \frac{\beta^2\,(-64 - \alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{2-r} + 256\,\alpha^2\,\beta^2\,\kappa^2\,\mu^{2-r})}{-128 - \alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r} + 512\,\alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}} - \frac{\kappa^2\,(-64 - \alpha^2\,\alpha\,\beta\,\kappa\,\mu^{3-r} + 256\,\alpha^2\,\beta^2\,\kappa\,\mu^{3-r})}{-128 - \alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r} + 512\,\alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0, 0, 0\Big\},$$

$$\Big\{0, 0, -\beta, 0, 0, 0, 2\alpha - \frac{256\,\alpha^2\,\beta\,\kappa^4\,\mu^{3-r}}{-128 - \alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r} + 512\,\alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0, 0\Big\},$$

$$\Big\{0, -\beta, 0, -\frac{\beta\,\mu\,(-64 - \alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{2-r} + 256\,\alpha^2\,\beta^2\,\kappa^2\,\mu^{2-r})}{-128 - \alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r} + 512\,\alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0, 0, 0, 2\mu^{-r} - \frac{\beta^2\,(-64 - \alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{2-r} + 256\,\alpha^2\,\beta^2\,\kappa^2\,\mu^{2-r})}{-128 - \alpha^2\,\alpha\,\beta\,\kappa^2\,\mu^{3-r} + 512\,\alpha^2\,\beta^2\,\kappa^2\,\mu^{3-r}}, 0\Big\},$$

$$\{0, 0, 0, 0, 0, 0, 0, 0, 2\mu^{-r}\}\Big\}$$

In[1]:= $c66 = 2\kappa - \frac{\beta^2\ (-64 - \alpha^2\ \alpha\ \beta\ \kappa^2\ \mu^{2-r} + 256\ \alpha^2\ \beta^2\ \kappa^2\ \mu^{2-r})}{-128 - \alpha^2\ \alpha\ \beta\ \kappa^2\ \mu^{3-r} + 512\ \alpha^2\ \beta^2\ \kappa^2\ \mu^{3-r}} - \frac{\kappa^2\ (-64 - \alpha^2\ \alpha\ \beta\ \kappa\ \mu^{3-r} + 256\ \alpha^2\ \beta^2\ \kappa\ \mu^{3-r})}{-128 - \alpha^2\ \alpha\ \beta\ \kappa^2\ \mu^{3-r} + 512\ \alpha^2\ \beta^2\ \kappa^2\ \mu^{3-r}}$

Out[1]= $2\kappa - \frac{\beta^2\ (-64 - \alpha^2\ \alpha\ \beta\ \kappa^2\ \mu^{2-r} + 256\ \alpha^2\ \beta^2\ \kappa^2\ \mu^{2-r})}{-128 - \alpha^2\ \alpha\ \beta\ \kappa^2\ \mu^{3-r} + 512\ \alpha^2\ \beta^2\ \kappa^2\ \mu^{3-r}} - \frac{\kappa^2\ (-64 - \alpha^2\ \alpha\ \beta\ \kappa\ \mu^{3-r} + 256\ \alpha^2\ \beta^2\ \kappa\ \mu^{3-r})}{-128 - \alpha^2\ \alpha\ \beta\ \kappa^2\ \mu^{3-r} + 512\ \alpha^2\ \beta^2\ \kappa^2\ \mu^{3-r}}$

In[2]:= **Factor[c66]**

Out[2]= $\frac{1}{2}\ \mu^{--r}\ (-\beta^2 + 3\ \kappa\ \mu^{-r})$

In[3]:= $c47 = \frac{128 - \alpha\ \alpha\ \beta\ \kappa^3\ \mu\ \mu^{3-r}}{-128 - \alpha^2\ \alpha\ \beta\ \kappa^2\ \mu^{3-r} + 512\ \alpha^2\ \beta^2\ \kappa^2\ \mu^{3-r}}$

Out[3]= $\frac{128 - \alpha\ \alpha\ \beta\ \kappa^3\ \mu\ \mu^{3-r}}{-128 - \alpha^2\ \alpha\ \beta\ \kappa^2\ \mu^{3-r} + 512\ \alpha^2\ \beta^2\ \kappa^2\ \mu^{3-r}}$

In[4]:= **Factor[c47]**

Out[4]= $-\frac{-\alpha\ \kappa\ \mu}{-\alpha^2 - 4\ \alpha\ \beta}$

In[5]:= $c77 = 2\ \alpha - \frac{256\ \alpha^2\ \beta\ \kappa^4\ \mu^{3-r}}{-128 - \alpha^2\ \alpha\ \beta\ \kappa^2\ \mu^{3-r} + 512\ \alpha^2\ \beta^2\ \kappa^2\ \mu^{3-r}}$

Out[5]= $2\ \alpha - \frac{256\ \alpha^2\ \beta\ \kappa^4\ \mu^{3-r}}{-128 - \alpha^2\ \alpha\ \beta\ \kappa^2\ \mu^{3-r} + 512\ \alpha^2\ \beta^2\ \kappa^2\ \mu^{3-r}}$

In[6]:= **Factor[c77]**

Out[6]= $-\frac{2\ \alpha\ (- - \alpha^2 + 4\ \alpha\ \beta - \kappa^2)}{-\alpha^2 - 4\ \alpha\ \beta}$

In[7]:= $c84 = -\frac{\beta\ \mu\ (-64 - \alpha^2\ \alpha\ \beta\ \kappa^2\ \mu^{2-r} + 256\ \alpha^2\ \beta^2\ \kappa^2\ \mu^{2-r})}{-128 - \alpha^2\ \alpha\ \beta\ \kappa^2\ \mu^{3-r} + 512\ \alpha^2\ \beta^2\ \kappa^2\ \mu^{3-r}}$

Out[7]= $-\frac{\beta\ \mu\ (-64 - \alpha^2\ \alpha\ \beta\ \kappa^2\ \mu^{2-r} + 256\ \alpha^2\ \beta^2\ \kappa^2\ \mu^{2-r})}{-128 - \alpha^2\ \alpha\ \beta\ \kappa^2\ \mu^{3-r} + 512\ \alpha^2\ \beta^2\ \kappa^2\ \mu^{3-r}}$

In[8]:= **Factor[c84]**

Out[8]= $-\frac{1}{2}\ \beta\ \mu\ \mu^{--r}$

In[9]:= $c88 = 2\ \mu^{-r} - \frac{\beta^2\ (-64 - \alpha^2\ \alpha\ \beta\ \kappa^2\ \mu^{2-r} + 256\ \alpha^2\ \beta^2\ \kappa^2\ \mu^{2-r})}{-128 - \alpha^2\ \alpha\ \beta\ \kappa^2\ \mu^{3-r} + 512\ \alpha^2\ \beta^2\ \kappa^2\ \mu^{3-r}}$

Out[9]= $2\ \mu^{-r} - \frac{\beta^2\ (-64 - \alpha^2\ \alpha\ \beta\ \kappa^2\ \mu^{2-r} + 256\ \alpha^2\ \beta^2\ \kappa^2\ \mu^{2-r})}{-128 - \alpha^2\ \alpha\ \beta\ \kappa^2\ \mu^{3-r} + 512\ \alpha^2\ \beta^2\ \kappa^2\ \mu^{3-r}}$

In[10]:= **Factor[c88]**

Out[10]= $-\frac{1}{2}\ \mu^{-r}\ \mu^{--r}\ (\beta^2\ \mu^{r} - 4\ \mu^{-r})$

B.1 Choi matrices of the linear map $\psi_{(\frac{1}{5},\frac{1}{2},\frac{1}{2})}$

B.1.1 The Choi matrix, $C_{\psi_{(\frac{1}{5},\frac{1}{2},\frac{1}{2})}}$

```
p ={{25,0,0,0,0,-3/4,0,0,0,0,0,-1/5},{0,3/4,0,0,0,0,0,0,0,0,0,0},
  {0,0,25,0,0,0,0,0,0,0,0,0},{0,0,0,3/4,0,0,0,0,0,0,0,0},
  {0,0,0,0,3/4,0,0,0,0,0,0,0},{-1/2,0,0,0,0,25,0,0,0,0,-3/4,0},
  {0,0,0,0,0,0,25,0,0,0,0,0},{0,0,0,0,0,0,0,3/4,0,0,0,0},
  {0,0,0,0,0,0,0,0,3/4,0,0,0},{0,0,0,0,0,0,0,0,0,3/4,0,0},
  {0,0,0,0,0,-3/4,0,0,0,0,25,0},{-1/5,0,0,0,0,0,0,0,0,0,0,25}}
```

```
In[2]:= p = {{25, 0, 0, 0, 0, -3 / 4, 0, 0, 0, 0, 0, -1 / 5},
          {0, 3 / 4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
          {0, 0, 25, 0, 0, 0, 0, 0, 0, 0, 0, 0},
          {0, 0, 0, 3 / 4, 0, 0, 0, 0, 0, 0, 0, 0},
          {0, 0, 0, 0, 3 / 4, 0, 0, 0, 0, 0, 0, 0},
          {-1 / 2, 0, 0, 0, 0, 25, 0, 0, 0, 0, -3 / 4, 0},
          {0, 0, 0, 0, 0, 0, 25, 0, 0, 0, 0, 0},
          {0, 0, 0, 0, 0, 0, 0, 3 / 4, 0, 0, 0, 0},
          {0, 0, 0, 0, 0, 0, 0, 0, 3 / 4, 0, 0, 0},
          {0, 0, 0, 0, 0, 0, 0, 0, 0, 3 / 4, 0, 0},
          {0, 0, 0, 0, 0,  -3 / 4, 0, 0, 0, 0, 25, 0},
          {-1 / 5, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 25} }
Out[2]= {{25, 0, 0, 0, 0, -3/4, 0, 0, 0, 0, 0, -1/5}, {0, 3/4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
         {0, 0, 25, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 3/4, 0, 0, 0, 0, 0, 0, 0, 0},
         {0, 0, 0, 0, 3/4, 0, 0, 0, 0, 0, 0, 0}, {-1/2, 0, 0, 0, 0, 25, 0, 0, 0, 0, -3/4, 0},
         {0, 0, 0, 0, 0, 0, 25, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 3/4, 0, 0, 0, 0},
         {0, 0, 0, 0, 0, 0, 0, 0, 3/4, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 3/4, 0, 0},
         {0, 0, 0, 0, 0, -3/4, 0, 0, 0, 0, 25, 0}, {-1/5, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 25}}
In[3]:= N[Eigenvalues[p]]
Out[3]= {25.9767, 25.1536, 25., 25., 24.8464, 24.0233, 0.75, 0.75, 0.75, 0.75, 0.75, 0.75}
```

B.1.2 The Choi matrix, $C_{\psi_{1(\frac{1}{3},\frac{1}{4},\frac{1}{4})}}$

```
p1 ={{9,0,0,0,0,-3/4,0,0,0,0,0,-1/5},{0,1/4,0,0,1/4,0,0,0,0,0,0,0},
  {0,0,9,0,0,0,0,0,0,0,0,0},{0,0,0,1/4,0,0,0,0,0,0,0,0},
  {0,1/4,0,0,1/4,0,0,0,0,0,0,0},{-3/4,0,0,0,0,9,0,0,0,0,-3/4,0},
  {0,0,0,0,0,0,9,0,0,1/4,0,0},{0,0,0,0,0,0,0,1/4,0,0,0,0},
  {0,0,0,0,0,0,0,0,1/4,0,0,0},{0,0,0,0,0,0,1/4,0,0,1/4,0,0},
  {0,0,0,0,0,-3/4,0,0,0,0,9,0},{-1/5,0,0,0,0,0,0,0,0,0,0,9}}
```

```
In[4]:= p1 = {{9, 0, 0, 0, 0, -3 / 4, 0, 0, 0, 0, 0, -1 / 5},
    {0, 1 / 4, 0, 0, 1 / 4, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 9, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 1 / 4, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 1 / 4, 0, 0, 1 / 4, 0, 0, 0, 0, 0, 0, 0},
    {-3 / 4, 0, 0, 0, 0, 9, 0, 0, 0, 0, -3 / 4, 0},
    {0, 0, 0, 0, 0, 0, 9, 0, 0, 1 / 4, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 1 / 4, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 1 / 4, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 1 / 4, 0, 0, 1 / 4, 0, 0},
    {0, 0, 0, 0, 0, -3 / 4, 0, 0, 0, 0, 9, 0},
    {-1 / 5, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 9} }
```

Out[4]=
$$\left\{\left\{9, 0, 0, 0, 0, -\frac{3}{4}, 0, 0, 0, 0, 0, -\frac{1}{5}\right\}, \left\{0, \frac{1}{4}, 0, 0, \frac{1}{4}, 0, 0, 0, 0, 0, 0, 0\right\}, \{0, 0, 9, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \left\{0, 0, 0, \frac{1}{4}, 0, 0, 0, 0, 0, 0, 0, 0\right\}, \left\{0, \frac{1}{4}, 0, 0, \frac{1}{4}, 0, 0, 0, 0, 0, 0, 0\right\}, \left\{-\frac{3}{4}, 0, 0, 0, 0, 9, 0, 0, 0, 0, -\frac{3}{4}, 0\right\}, \left\{0, 0, 0, 0, 0, 0, 9, 0, 0, \frac{1}{4}, 0, 0\right\}, \left\{0, 0, 0, 0, 0, 0, 0, \frac{1}{4}, 0, 0, 0, 0\right\}, \left\{0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{4}, 0, 0, 0\right\}, \left\{0, 0, 0, 0, 0, 0, \frac{1}{4}, 0, 0, \frac{1}{4}, 0, 0\right\}, \left\{0, 0, 0, 0, 0, -\frac{3}{4}, 0, 0, 0, 0, 9, 0\right\}, \left\{-\frac{1}{5}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 9\right\}\right\}$$

```
In[5]:= N[Eigenvalues[p1]]

Out[5]= {10.0702, 9.14016, 9.00714, 9., 8.85984, 7.92979, 0.5, 0.25, 0.25, 0.25, 0.242863, 0.}
```

B.1.3 The Choi matrix, $C_{\psi_{2(\frac{1}{3},\frac{1}{4},\frac{1}{4})}}$

```
p2 ={{16,0,0,0,0,0,0,0,0,0,0,0},{0,1/2,0,0,-1/4,0,0,0,0,0,0,0},
  {0,0,16,0,0,0,0,0,0,0,0,0},{0,0,0,1/2,0,0,0,0,0,0,0,0},
  {0,-1/4,0,0,1/2,0,0,0,0,0,0,0},{0,0,0,0,0,16,0,0,0,0,0,0},
  {0,0,0,0,0,0,16,0,0,-1/4,0,0},{0,0,0,0,0,0,0,1/2,0,0,0,0},
  {0,0,0,0,0,0,0,0,1/2,0,0,0},{0,0,0,0,0,0,-1/4,0,0,1/2,0,0},
  {0,0,0,0,0,0,0,0,0,0,16,0},{0,0,0,0,0,0,0,0,0,0,0,16}}
```

```
In[3]:= p2 = {{16, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
          {0, 1 / 2, 0, 0, -1 / 4, 0, 0, 0, 0, 0, 0, 0},
          {0, 0, 16, 0, 0, 0, 0, 0, 0, 0, 0, 0},
          {0, 0, 0, 1 / 2, 0, 0, 0, 0, 0, 0, 0, 0},
          {0, -1 / 4, 0, 0, 1 / 2, 0, 0, 0, 0, 0, 0, 0},
          {0, 0, 0, 0, 0, 16, 0, 0, 0, 0, 0, 0},
          {0, 0, 0, 0, 0, 0, 16, 0, 0, -1 / 4, 0, 0},
          {0, 0, 0, 0, 0, 0, 0, 1 / 2, 0, 0, 0, 0},
          {0, 0, 0, 0, 0, 0, 0, 0, 1 / 2, 0, 0, 0},
          {0, 0, 0, 0, 0, 0, -1 / 4, 0, 0, 1 / 2, 0, 0},
          {0, 0, 0, 0, 0,  0, 0, 0, 0, 0, 16, 0},
          {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 16} }

Out[3]= {{16, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 1/2, 0, 0, -1/4, 0, 0, 0, 0, 0, 0, 0},
         {0, 0, 16, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 1/2, 0, 0, 0, 0, 0, 0, 0, 0},
         {0, -1/4, 0, 0, 1/2, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 16, 0, 0, 0, 0, 0, 0},
         {0, 0, 0, 0, 0, 0, 16, 0, 0, -1/4, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 1/2, 0, 0, 0, 0},
         {0, 0, 0, 0, 0, 0, 0, 0, 1/2, 0, 0, 0}, {0, 0, 0, 0, 0, 0, -1/4, 0, 0, 1/2, 0, 0},
         {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 16, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 16}}

In[4]:= N[Eigenvalues[p2]]

Out[4]= {16.004, 16., 16., 16., 16., 16., 0.75, 0.5, 0.5, 0.5, 0.495969, 0.25}
```