Applied Mathematical Sciences, Vol. 7, 2013, no. 15, 717 - 725

Symmetry Group Approach to the Solution of Generalized Burgers Equation: $U_t + UU_x = \lambda U_{xx}$

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Abstract

Symmetry of a system of differential equations is a transformation that maps any solution to another solution of the system. In Lie's framework such transformations are groups that depend on continuous parameters and consist of point transformations (point symmetries), acting on the system's space of independent and dependent variables, or, more generally, contact transformations (contact symmetries), acting on independent and dependent variables as well as on all first derivatives of the dependent variables. Lie groups, and hence their infinitesimal generators, can be naturally prolonged to act on the space of independent variables, dependent variables, and derivatives of the dependent variables. We present a Lie symmetry approach in solving Burgers Equation:$U_t + UU_x = \lambda U_{xx}$ which is a nonlinear partial differential equation, which arises in model studies of turbulence and shock wave theory. In physical application of shock waves in fluids, coefficient λ, has the meaning of viscosity. So far in both analytic and numerical approaches the solutions have only been established for $0 \leq \lambda \leq 1$. In this paper, we give a global solution to the Burgers equation with no restriction on λ i.e. for $\lambda \in (-\infty, \infty)$.

Keywords: Symmetries, Lie Group, Burgers Equation, Invariant, Global, Solutions

1 The Burgers Equation

We consider the Burgers equation:

$$U_t + UU_x = \lambda U_{xx} \tag{1}$$

1.1 Lie Symmetry Analysis

First we consider a K^{th} order partial differential equation of the form

$$F(x, u, u_1, u_2, u_3, ..., u_k) = 0 \tag{2}$$

where $x = (x_1, x_2, x_3, ..., x_n)$ denotes n independent variables, u_j denotes the set of coordinates corresponding to all the $j-th$ order partial derivatives with respect to x.

1.1.1 Definition

The one-parameter Lie group of transformations

$$x^* = X(x, u; \varepsilon) \tag{3}$$

$$u^* = U(x, u; \varepsilon) \tag{4}$$

leaves the partial differential equation (2) invariant if and only if its $K-th$ extension, $x^*, u^*, u_1^*, u_2^*, ..., u_k^*$ leaves the surface $F(x, u, u_1, u_2, u_3, ..., u_k) = 0$ invariant.

1.1.2 Theorem

Let $F_r(x, u^{(k)}) = 0$ be a non degenerate system of partial differential equations and

$$V = \xi_i(x, u)\frac{\partial}{\partial x_i} + \eta(x, u)\frac{\partial}{\partial u} \tag{5}$$

be the infinitesimal generator of the one parameter Lie group of transformations (3, 4)
Let

$$V^{(k)} = \xi(x, u)\frac{\partial}{\partial x_i} + \eta(x, u)\frac{\partial}{\partial u} + \eta_i^1 + ... + \eta^k_{i_1 i_2 ... i_k}(x, u, u_1, u_2, ...u_k)\frac{\partial}{\partial u_{i_1 i_2 i_3 ... i_k}} \tag{6}$$

be the corresponding k^{th} extended infinitesimal generator of (5) where

$$\eta_i^1 = D_i\eta - (D_i\xi_j)u_j, i = 1, 2, 3, ..., n; \tag{7}$$

and $\eta^j_{i_1 i_2 i_3 i_4 i_5 ... i_j}$ is given by

$$\eta^{(k)}_{i_1 i_2 i_3 i_4 i_5 ... i_k} = D_{i_k}\eta^{(k-1)}_{i_1 i_2 i_3 ... i_{k-1}} - (D_{i_k}\xi_j)u_{i_1 i_2 ... i_{k1} j} \tag{8}$$

$i_j = 1,2,3,...,n$ for $j = 1,2,3,...,k$ with $k = 1,2,3,...$ in terms of $(\xi(x,u), \eta(x,u))$. then a connected local group of transformations G of the form; (3,4) is a symmetrical group of the system of the partial differential equations if and only if

$$V^{(k)}[F_r(x, u^{(k)})] = 0, r = 1,2,3,...l, \quad whenever \quad F(x, u^{(k)}) = 0 \tag{9}$$

[6, 3]

1.2 Lie Group Solution of the Burgers Equation

From (1) the symmetry groups of transformations are of the form
$t^* = T(t,x,u;\varepsilon),\ x^* = X(t,x,u;\varepsilon),\ u^* = U(t,x,u;\varepsilon)$ with corresponding infinitesimals
$\xi(t,x,u) = \frac{\partial X(t,x,u;\varepsilon)}{\partial \varepsilon}|_{\varepsilon=0}, \tau(t,x,u) = \frac{\partial T(t,x,u;\varepsilon)}{\partial \varepsilon}|_{\varepsilon=0}, \phi(t,x,u) = \frac{\partial U(t,x,u;\varepsilon)}{\partial \varepsilon}|_{\varepsilon=0}$
We let the generator V, of (1), be of the form $V = \xi(t,x,u)\frac{\partial}{\partial x} + \tau(t,x,u)\frac{\partial}{\partial t} + \phi(t,x,u)\frac{\partial}{\partial u}$
From equation (1), $F = u_t + uu_x - \lambda u_{xx} = 0$. By theorem (1.1.2) it follows that

$$V^{(2)}F = V^{(2)(u_t + uu_x - \lambda u_{xx}) = 0} \tag{10}$$

when $F = 0$, and so we obtain

$$[\xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u} + \eta^t\frac{\partial}{\partial u_t} + \eta^x\frac{\partial}{\partial u_x} + \eta^{tt}\frac{\partial}{\partial u_{tt}} + \eta^{tx}\frac{\partial}{\partial u_{tx}}$$
$$+\eta^{xx}\frac{\partial}{\partial u_{xx}}][u_t + uu_x - \lambda u_{xx}] = 0$$

The infinitesimal condition above reduces to

$$\phi u_x + \eta^t + \eta^x u - \lambda\eta^{xx} = 0 \tag{11}$$

with $\eta^t,\ \eta^x,\ \eta^{xx}$ explicitly defined [3]
Substituting $\eta^t,\ \eta^x,\ \eta^{xx}$ into equation (11), we obtain the equation

$$\begin{aligned}
[\phi u_x + \{\phi_t - \xi_t - \xi_t u_x + (\phi_u - \tau_t)u_t - \xi_u u_x u_t &- \tau_u u_t^2\} \\
+u\{\phi_x - \tau_x u_t + (\phi_u - \xi_x)u_x - \xi_u u_x^2 - \tau_u u_t u_x\} &- \lambda[\phi_{xx} \\
+(2\phi_{xu} - \xi_{xx})u_x - \tau_{xx}u_t + (\phi_{uu} - 2\xi_{xu})u_x^2 - 2\tau_{xu}u_x u_t &- \xi_{uu}u_x^3 \\
-\tau_{uu}u_x^2 u_t + (\phi_u - 2\xi_x)u_{xx} - 2\tau_x u_{xt} - 3\xi_u u_x u_{xx} - \tau_u u_t u_{xx} &- 2\tau_u u_x u_{xx}] = 0
\end{aligned} \tag{12}$$

Equate to zero the coefficients of monomials in the first and second partial derivatives of u and on substituting $u_t + uu_x = \lambda u_{xx}; \quad u_t = \lambda uu_{xx} - u_{xx}; \quad uu_x = \lambda u_{xx} - u_t$; wherever it occurs in equation(11) we arrive at the solutions of the infinitesimal functions τ, ξ, ϕ as

$$\xi = c_2 + c_4 x + c_5 xt \tag{13}$$
$$\tau = c_1 + (c_3 + 2c_4)t + c_5 t^2 \tag{14}$$
$$\phi = c_3 - c_4 u + c_5 x - c_5 tu + \alpha(x,t) : \quad \alpha_t = \lambda \alpha_{xx} \tag{15}$$

α, is an arbitrary solution t the generalized heat equation
Representation of the infinitesimal functions ξ, τ, ϕ in the standard basis

$$\begin{array}{rcl} & & \quad \underline{v_1} \quad\;\; \underline{v_2} \quad\;\; \underline{v_3} \quad\;\; \underline{v_4} \qquad\;\; \underline{v_5} \qquad\;\; \underline{v_\alpha} \\ & & \quad\; \downarrow \quad\;\;\; \downarrow \quad\;\;\; \downarrow \quad\;\;\; \downarrow \qquad\;\;\; \downarrow \qquad\;\;\; \downarrow \\ \xi &=& 0.c_1 + 1.c_2 + 0.c_3 + \quad c_4 x + \quad c_5 tx \qquad + 0.c_\alpha \\ \tau &=& 1.c_1 + 0.c_2 + 1.c_3 t + 2.c_4 t + \quad 1.c_5 t_2 \qquad + 0.c_\alpha \\ \phi &=& 0.c_1 + 0.c_2 + 1.c_3 - 1.c_4 u + 1.c_5(x - tu) + 1.c_\alpha.\alpha \end{array} \tag{16}$$

enables one to determine infinitesimal generators.
We derive the corresponding basis infinitesimal generators $v_i's$ of the form $v_i = \widehat{\xi}_i \frac{\partial}{\partial x} + \widehat{\tau}_i \frac{\partial}{\partial t} + \widehat{\phi}_i \frac{\partial}{\partial u}$: where $\widehat{\xi}_i$, $\widehat{\tau}_i$, $\widehat{\phi}_i$ are the coefficients c_i in the standard solutions of ξ, τ, ϕ. Hence v_4, v_5 are listed below as the infinitesimal generators of Burgers equation.

$$\begin{aligned} v_4 &= x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}, \\ v_5 &= tx\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - [ut - x]\frac{\partial}{\partial u} \end{aligned} \tag{17}$$

The heat solutions ω referred to here, are the invariant solutions of the generalized heat equation [3]

1.3 Lie Groups Admitted by Equation (1)

The one parameter groups G_i admitted by the Burgers equation, are determined by solving the corresponding Lie equations The one-parameter groups G, admitted by the Burgers equation, are determined by solving the corresponding Lie equations
$v_4 : \frac{d\overline{t}}{d\varepsilon} = 2\overline{t}, \frac{d\overline{x}}{d\varepsilon} = \overline{x}, \frac{d\overline{u}}{d\varepsilon} = -\overline{u}$; $v_5 : \frac{d\overline{x}}{d\varepsilon} = \overline{x}\overline{t}, \frac{d\overline{t}}{d\varepsilon} = \overline{t}^2, \frac{d\overline{u}}{d\varepsilon} = \overline{x} - \overline{t}\overline{u}$ with initial conditions: $\overline{t}_{\varepsilon=0} = t,\ \overline{x}_{\varepsilon=0} = x,\ \overline{u}_{\varepsilon=0} = u$ which lead to;

$$v_4;\ G_4 :\ X(x,t,u;\ \varepsilon) \longrightarrow \qquad X_4(e^{\varepsilon}x, e^{2\varepsilon}t, e^{-\varepsilon}u) \tag{18}$$

$$v_5;\ G_5 :\ X(x,t,u;\ \varepsilon) \longrightarrow X_5[\frac{x}{1-\varepsilon t}, \frac{t}{1-\varepsilon t}, u(1-\varepsilon t) + \varepsilon x] \tag{19}$$

1.4 Group Transformations of Solutions

The method is based on the fact that a symmetry group transforms any solutions of the equation in question into solutions of the same equation.
If $\overline{u}$, $\overline{x}$, $\overline{t}$ are group transformations of equation (1) with $\overline{u}$, of the form $\overline{u} = \Psi(u, x, t, \varepsilon)$, for some explicit function Ψ, then applying the inverse mapping, the new symmetry solution $\widehat{u}$ is defined by

$$\widehat{u} = \Psi(\Phi(g_\varepsilon^{-1}(\overline{x}), g_\varepsilon^{-1}(\overline{t})), g_\varepsilon^{-1}(\overline{x}), g_\varepsilon^{-1}(\overline{t}), \varepsilon^{-1})$$

where $u = \Phi(x, t)$ is any known solution of equation (2). Finally we obtain a one parameter family (with a parameter ε) of new solutions to equation (2) as

$$\widehat{u} = \Psi(\Phi(g_\varepsilon^{-1}(\overline{x}), g_\varepsilon^{-1}(\overline{t})), g_\varepsilon^{-1}(\overline{x}), g_\varepsilon^{-1}(\overline{t}), \varepsilon^{-1}) \tag{20}$$

1.5 Symmetry Solutions of the Burgers Equation

By symmetry group inversion theory above, if each G, is a symmetry group and $u = \Phi(x, t)$ is a solution of the Burgers equation (1), then transformation groups of the Burgers equation (1), solve the equation (1).
The above solution can also be written in the new variables: $\overline{u} = \Phi(\overline{x}, \overline{t})$.
If $\overline{u}$, $\overline{x}$, $\overline{t}$ are group transformations of the Burgers equation (1) with $\overline{u}$, of the form $\overline{u} = \Psi(u, x, t, \varepsilon)$, for some explicit function Ψ, then applying the inverse mapping, the new symmetry solution $\widehat{u}$ satisfies relation
$\widehat{u} = \Psi(\Phi(g_\varepsilon^{-1}(\overline{x}), g_\varepsilon^{-1}(\overline{t})), g_\varepsilon^{-1}(\overline{x}), g_\varepsilon^{-1}(\overline{t}), \varepsilon^{-1})$ where $u = \Phi(x, t)$ is any known solution of equation (1). $v_4;\ G_4:\ X(x, t, u;\ \varepsilon) \longrightarrow X_4(e^\varepsilon x, e^{2\varepsilon} t, e^{-\varepsilon} u)$
Then the new symmetry solution $\widehat{u}_4$, $\widehat{u}_5$ are defined by

$$\widehat{u}_4 = e^{-\varepsilon}\Phi(e^{-\varepsilon} x, e^{-2\varepsilon} t) \tag{21}$$

$$\widehat{u}_5 = \frac{\varepsilon x}{1 + \varepsilon t} + \frac{1}{1 + \varepsilon t}\Phi(\frac{x}{1 + \varepsilon t}, \frac{t}{1 + \varepsilon t}) \tag{22}$$

2 Invariant Solutions of the Burgers Equation

If a group of transformations maps a solutions into itself, we arrive at what is called a **self-similar** or **group invariant** solution [1, 4, 5]. Given infinitesimal symmetry of equation (1) the invariant solution under the one-parameter group generated by a generator V are obtained as follows: We calculate two independent invariants $J_1 = k(x, t)$ and $J_2 = \mu(x, t, u)$ by solving the equation

$$V(J) = \tau(t, x, u)\frac{\partial J}{\partial t} + \varsigma(t, x, u)\frac{\partial J}{\partial x} + \eta(t, x, u)\frac{\partial J}{\partial u} = 0 \tag{23}$$

or its system of characteristics

$$\frac{dt}{\tau(t,x,u)} = \frac{dx}{\varsigma(t,x,u)} = \frac{du}{\eta(t,x,u)} \tag{24}$$

Then we designate one of the invariants as a function of the other e.g.

$$\mu = \phi(k) \tag{25}$$

Finally we substitute expression for μ ,in equation (25) and obtain ordinary differential equation for the unknown function $\phi(k)$ of one variable. This procedure reduces the number of independent variables by one [3, 4]. The invariant solution of (1) under the transformation group generated by equation (7) are obtained as u_i, $u_{\omega j}$ in the table below. That is, for each of the infinitesimal generator V_i above we form the corresponding system of characteristics $\frac{dt}{\tau(t,x,u_i)} = \frac{dx}{\varsigma(t,x,u_i)} = \frac{du}{\eta(t,x,u_i)}$
Hence we solve for u_i ,the invariant solution of the Burgers equation corresponding to each of the infinitesimal generator [2, 4, 7].

2.1 Invariant Solutions of the Burgers Equation

Below is a list of Generator (v_i) and their corresponding Invariant Solutions (u_m):

v_4

$$u = [\sqrt{t}C_3 - (2\sqrt{2\lambda}^{-1}\sqrt{t\pi}e^{\frac{\alpha^2}{2}}erf(\alpha)]^{-1} : C = 0 \tag{26}$$

v_5

$$u = \frac{x}{t} - \frac{2\lambda}{x + tc_1} \tag{27}$$

v_5

$$u = \frac{x}{t} - \frac{k}{t}\sqrt{2}\tanh[\frac{\sqrt{2}kx}{2\lambda t} + C_2] : C = k^2,\ C > 0 \tag{28}$$

v_5

$$u = \frac{x}{t} - \frac{r}{t}\sqrt{2}\tan[C_3 - \frac{\sqrt{2}rx}{2\lambda t}] : C = -r^2,\ C < 0 \tag{29}$$

v_{w4}

$$u = \frac{K\sqrt{\frac{\pi}{\lambda}}erf(x\sqrt{\frac{\lambda}{4t}}) + L}{C_3 + F_1(x,t)} \tag{30}$$

v_{w5}

$$u = \frac{1}{\sqrt{t}} e^{\frac{x^2}{4\lambda t}} [e^{-[\frac{4\lambda e^{\frac{x^2}{4\lambda t}}}{(4\lambda t + x^2)\sqrt{t}}]} + K] \tag{31}$$

$$F_1(x,t) = \int [\sqrt{\frac{\pi}{\lambda}} erf[x\frac{\lambda}{4t}]]_x dt,$$

3 Global Symmetry Solutions of the Burgers Equation

Consider the Lie groups G_1, G_2, G_3, G_4, G_5, G_α admitted by the burgers equation in section (1.2). By symmetry group inversion theory of section (1.3) and (1.4), if each G_i is a symmetry group and $u = \Phi(x,t)$ is a known solution of the Burgers equation (1), then the functions $\widehat{u_j}$ expressed new solutions of the Burgers equation (1) [6]
The most general one-parameter group of symmetries is obtained by considering a general linear combination $v_g = c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 + c_5v_5 + c_\alpha v_\alpha$ of the given field vectors. We may represent an arbitrary group transformation g as the composition of transformations in the various one-parameter subgroups G_1, G_2, G_3, G_4, G_5, G_α. In particular if g is near the identity, it can be represented uniquely in the exponential form
$g = exp(\varepsilon_\alpha v_\alpha)^* exp(\varepsilon_5 v_5)^* exp(\varepsilon_4 v_4)^* exp(\varepsilon_3 v_3)^* exp(\varepsilon_2 v_2)^* exp(\varepsilon_1 v_1)^*$
Thus the most general solution u_g, i.e. **global** solution of equation (1) is obtained by

$$u_g = \frac{\varepsilon_5 x - \varepsilon_3\varepsilon_5 - \varepsilon_3}{1+\varepsilon_5 t} + [\frac{e^{-\varepsilon_4}}{1+\varepsilon_5 t}] \times \Phi[\frac{xe^{-\varepsilon_4} - (\varepsilon_2\varepsilon_5 + \varepsilon_3)t - \varepsilon_3\varepsilon_5 t^2 - \varepsilon_2}{1+\varepsilon_5 t}, \frac{te^{-2\varepsilon_4} - \varepsilon_1\varepsilon_5 t - \varepsilon_1}{1+\varepsilon_5 t}] \tag{32}$$

where $u = \Phi[x,t]$ is a known solution of the Burgers equation (1).
Substituting $u_i(x,t) = \Phi(x,t)$, the invariant solution of (1) corresponding to the infinitestimal generator V_i into equation (26), we obtain u_{gi} the global symmetry solutions of the burgers equation as shown below.[3]

3.1 Global Symmetry Solutions of the Burgers Equation

Below is a list of Generator (v_i) and their corresponding Global Symmetry Solutions (u_m) v_1 of the Burgers Equation

v_4

$$\begin{aligned} u_{gw4}(x,t) &= \frac{\varepsilon_5 x - \varepsilon_3\varepsilon_5 t - \varepsilon_3}{1+\varepsilon_5 t} + [\frac{e^{-\varepsilon_4}}{1+\varepsilon_5 t}] * [\sqrt{\frac{te^{-2\varepsilon_4} - \varepsilon_1\varepsilon_5 t - \varepsilon_1}{1+\varepsilon_5 t}} C_3 \\ &- (2\sqrt{2\lambda})^{-1} \sqrt{\frac{(te^{-2\varepsilon_4} - \varepsilon_1\varepsilon_5 t - \varepsilon_1)\pi}{1+\varepsilon t}} e^{\frac{\bar{\alpha}^2}{2}} erf(\bar{\alpha})] \end{aligned} \tag{33}$$

v_5

$$\begin{aligned} u_{g5}(x,t) &= \frac{\varepsilon_5 x - \varepsilon_3\varepsilon_5 t - \varepsilon_3}{1+\varepsilon_5 t} + [\frac{e^{-\varepsilon_4}}{1+\varepsilon_5 t}] * [\frac{xe^{-\varepsilon_4} - (\varepsilon_3 + \varepsilon_2\varepsilon_5)t - \varepsilon_3\varepsilon_5 t^2 - \varepsilon_2}{te^{-2\varepsilon_4} - \varepsilon_1\varepsilon_5 t - \varepsilon_1} \\ &* \sqrt{2}\tanh[\frac{\sqrt{2}k(xe^{-\varepsilon_4} - (\varepsilon_3 + \varepsilon_2\varepsilon_5)t - \varepsilon_3\varepsilon_5 t^2 - \varepsilon_2)}{-2\lambda(te^{-2\varepsilon_4} - \varepsilon_1\varepsilon_5 t - \varepsilon_1)} + C_2] \end{aligned} \tag{34}$$

v_5

$$\begin{aligned} u_{g5}(x,t) &= \frac{\varepsilon_5 x - \varepsilon_3\varepsilon_5 t - \varepsilon_3}{1+\varepsilon_5 t} + [\frac{e^{-\varepsilon_4}}{1+\varepsilon_5 t}] * \frac{xe^{-\varepsilon_4} - (\varepsilon_3 + \varepsilon_2\varepsilon_5)t - \varepsilon_3\varepsilon_5 t^2 - \varepsilon_2 - r(1+\varepsilon_5 t)}{te^{-2\varepsilon_4} - \varepsilon_1\varepsilon_5 t - \varepsilon_1}] \\ &* \sqrt{2}\tan[C_3 - \frac{\sqrt{2}k(xe^{-\varepsilon_4} - (\varepsilon_3 + \varepsilon_2\varepsilon_5)t - \varepsilon_3\varepsilon_5 t^2 - \varepsilon_2)}{-2\lambda(te^{-2\varepsilon_4} - \varepsilon_1\varepsilon_5 t - \varepsilon_1)}] \end{aligned} \tag{35}$$

v_5

$$u_{g5}(x,t) = \frac{\varepsilon_5 x - \varepsilon_3\varepsilon_5 t - \varepsilon_3}{1+\varepsilon_5 t} + [\frac{e^{-\varepsilon_4}}{1+\varepsilon_5 t}] * [\frac{\hat{x}}{\hat{t}} - \frac{2\lambda(1+\varepsilon_5 t)}{\hat{x} + \hat{t}c_1}] \tag{36}$$

$\bar{\alpha}(x,t) = \alpha(\hat{x},\hat{t}), \quad \hat{x} = xe^{-\varepsilon_4} - (\varepsilon_3 + \varepsilon_2\varepsilon_5)t - \varepsilon_3\varepsilon_5 t^2 - \varepsilon_2, \quad \hat{t} = te^{-2\varepsilon_4 - \varepsilon_1\varepsilon_5} t - \varepsilon_1$

4 Conclusions

In this paper, we have managed to find global solutions of the Burgers Equation using the Lie symmetry approach. The solutions (33 - 36) are appearing here in literature for the first time. Other previous attempts to solve this very important equation only managed to find solutions when the coefficient λ is restricted to $\lambda \in [0,1]$.

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Received: October, 2012