Finite Difference Solution of (1+1) Sine-Gordon Equation: A Mathematical Model for the Rigid Pendula Attached to a Stretched Wire

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Abstract**—**The nonlinear (1+1) Sine-Gordon equation that governs the vibrations of the rigid pendula attached to a stretched wire is solved. The equation is discretized and solved by Finite Difference Method with specific initial and boundary conditions. A Crank Nicolson numerical scheme is developed with concepts of stability of the scheme analysed using matrix method. The resulting systems of linear algebraic equations are solved using Mathematica software. The solutions are presented graphically in three dimensions and interpreted. The numerical results obtained indicate that the amplitudes of the rigid pendula attached to a stretched wire vary inversely as the position of the travelling waves produced on the stretched wire. The efficacy of the proposed approach and the results obtained are acceptable and in good agreement with earlier studies on the rigid pendula attached to a stretched wire.

Keywords**—**Crank Nicolson Numerical Scheme; Finite Difference Method; Pendula Attached to a Stretched Wire; Sine-Gordon Equation.

Abbreviations**—**Crank Nicolson Scheme (CNS); Differential Transform Method (DTM); Discrete Singular Convolution (DSC); Forward Difference Scheme (FDS); Modified Variation Iteration Method (MVIM); Partial Differential Equation (PDE); Sine-Gordon Equation (SG).

I. INTRODUCTION

SINE-GORDON Equation was originally considered in the nineteenth century in the course of study of surfaces of constant negative curvatures [Purring & Skyrme, 4]. This equation attracted a lot of attention in 1970s due to the presence of solitons solution.This equation which arises in the study of differential geometry of surfaces with Gaussian curvature, has wide applications in the propagation of fluxon in Josephson Junctions (a junction between two superconductors) [Sirendaoreji & Jiong, 17],the motion of rigid pendula attached to a stretched wire [Purring & Skyrme, 4], solid state physics, nonlinear optics, stability of fluid motions, dislocations in crystals [Purring & Skyrme, 4] and other scientific fields.The Sine-Gordon equation is one of the most important equations of nonlinear physics and describes many physical applications, one of them being periodic pendulum problem. Most of the research on the periodic pendulum suggests that the period of swing depends on length and the local strength of gravity if amplitude is limited to small swings. In an ideal situation, where friction plays no part, an object would continue to oscillate with constant amplitude indefinitely. Objects in the real world do not experience perpetual oscillation; instead, they are subject to damping, or the dissipation of energy, primarily as a result of friction. If damping effect is small, the amplitude will gradually decrease as the object continues to oscillate, until eventually oscillation ceases.

The Sine-Gordon Equation we consider is of the form;

$$u_{tt} - u_{xx} + \sin u = 0 \tag{1}$$

Equation (1) is solved subject to;

$$\text{Initial conditions } u(x,0) = 0\,,\; u_t(x,0) = 0 \tag{2}$$

$$\text{Boundary conditions } u(0,t) = 1 - e^{-t}\,,\; u(M,t) = 0,\; t > 0 \tag{3}$$

Where M is the largest position value of the wave considered in this work.In the case of mechanical transmission line, $u(x, t)$ describes amplitude of the wave particles at position x and time t. A wave is the result of a disturbance moving through a medium such as water, air, or a crowd of people. As the disturbance is transferred from one part of the medium to another, we are able to observe the location of the disturbance as it moves with speed in a particular direction. Any quantitative measurement or feature of the medium which clearly identifies the location and velocity of the disturbance is called a signal. The signal may distort; however, as long as it remains recognizable, it can be used to identify the motion of the disturbance. A wave is any recognizable signal that is transferred from one part of the medium to another with recognizable velocity of propagation. One-dimension waves are represented mathematically by functions of two variables $u(x,t)$, where u represents the value of some quantitative measurement made at every position x in the medium at time t [Roger Knobel, 16].

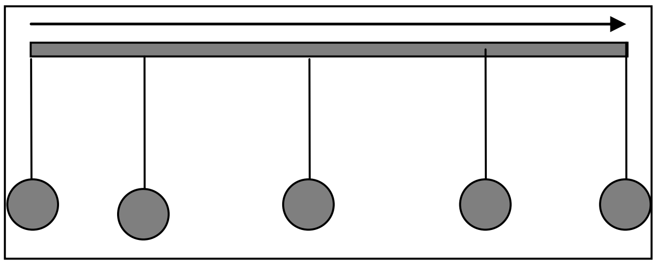

Figure 1: Pendula Attached to a Stretched Wire with Arrow Showing Increasing Wave Position x

The set up in "Figure 1" consists of a series of pendula attached to a stretched horizontal wire and hanging vertically on thin strings. Each pendulum is free to swing in a plane perpendicular to a stretched wire. The interactions between adjacent pendula permits a disturbance in one part of the set up to propagate and mechanically transmit a signal along the line of pendula. If a pendulum at one end of the set up is disturbed slightly, the transmitted disturbance results in a small "wavy" motion along a stretched wire.

II. Literature Review

Broad classes of analytical and numerical solution methods have been presented to study the different solutions and physical phenomena related to Sine-Gordon Equation due to its wide applications and important mathematical properties. Wei [21] explored the utility of a discrete singular convolution algorithm for the integration of the Sine-Gordon Equation. The initial values were chosen close to a homoclinic manifold. The form of Sine-Gordon equation solved is;

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} + \sin[u(x,t)] = 0 \tag{4}$$

A number of new initial values were considered, including a case where the initial value is "exactly" on the homoclinic orbit subject to initial condition values are given as;

$$u(x,0) = 0, \; u_t(x,0) = 4\operatorname{sec}h(x) \tag{5}$$

The analytical solution obtained representing a breather-kink and ant-kink transition associated with double point in the nonlinear spectrum of Equation (5) is

$$u\,(x,t) = 4\tan^{-1}\left[\operatorname{sec}h(x)t\right], \; -\infty < x < \infty \tag{6}$$

Minzoni et al., [11] considered evolution lump and ring solutions of Sine-Gordon Equation in two-space dimensions. Approximate equations governing this evolution were derived using a pulse or ring with variable parameters in an averaged Lagrangian for the Sine-Gordon Equation. The Sine-Gordon considered was;

$$\frac{\partial^2 u}{\partial t^2} - \nabla^2 u + \frac{1}{\epsilon^2}\sin(u) = 0, \tag{7}$$

where ϵ is a variable parameter in an averaged Lagrangian. It was shown that the initial conditions for the lump with wavy boundaries for the Sine-Gordon Equation eventually collapsed due to the shedding of angular momentum into dispersive radiation both asymptotically and numerically. Rodolfo R. Rosales [15] derived the nonlinear wave equation (Sine-Gordon Equation) for torsion coupled pendulums using the continuum modelling techniques. Only gravity and torsion forces induced on the axle was considered when the pendulums are not all aligned. The nonlinear wave equation (the "Sine-Gordon Equation"); obtained (for the continuum limit) is

$$\theta_{tt} - c^2\theta_{xx} = -\omega^2 \sin\theta \tag{8}$$

where $\omega = \sqrt{\frac{K}{L}}$ is the pendulum angular frequency, and $c = \sqrt{\frac{K}{\rho L^2}}$ is a wave propagation speed, g the acceration due to gravity, K a constant depending on axle material, L the distance of attached mass from centre of mass and ρ is the mass density along the rod. He solved the equation using "Pseudo-spectral" Numerical Method and showed that the particle-like solutions obtained preserve their identities and original velocities. Bobenko et al., [1] developed a numerical scheme for solution of the Goursat problem for a class of nonlinear hyperbolic systems with an arbitrary number of independent variables. The results were applied to hyperbolic systems of differential–geometric origin, like the Sine–Gordon Equation describing the surfaces of the constant negative Gaussian curvature (K–surfaces). The construction was illustrated by the well–known Sine-Gordon Equation namely;

$$\partial_x \partial_y \phi = \sin\phi \tag{9}$$

A naive Discretisation of the Sine-Gordon Equation was obtained from (9) by replacing partial derivatives with their difference analogs:

$$\partial_x^{\epsilon} \partial_y^{\epsilon} \phi = \sin\phi \tag{10}$$

where ϵ is the lattice coordinate mesh–size for discrete K–surfaces. The converging results were proved for an integrable Discretisation of the Sine–Gordon Equation applied to discrete K–surfaces and their Backlund transformations. Anton Belyakov & Alexander P. Seyranian [2] studied the pendulum with periodically varying lengths treated as a simple model of a child's swing. The asymptotic expressions for boundaries of instability domains near resonance frequencies were derived. The domains for oscillations motions were found analytically and compared with numerical study results. Wazzan & Ismail [20] used Petrov-Galerkin method to derive a scheme for the *K (2.2)* equation, where cubic B-spines are chosen as test functions and linear functions as trial functions. Product approximation technique was applied for the nonlinear terms. A Crank-Nicolson Scheme was used to discretize in time. The system was solved by Newton's method and linearization technique. The K (2.2) equation is given by;

$$u_t + \left(u^2\right)_x + \left(u^2\right)_{xxx} = 0 \tag{11}$$

with initial and boundary conditions given by;

$$u(x,0) = f(x),\ u_t(x\ ,t) = g(x) \tag{12}$$

Syed Tausesef Mohyud-Din et al., [18] applied the Modified Variation Iteration Method which was formulated by the elegant coupling of Adomian's polynomials and the correctional functional for solving Sine-Gordon Equations. The standard form of such equations is given by

$$u_{tt}(x,t) - c^2 u_{xx}(x,t) + \alpha \sin u = 0 \tag{13}$$

with initial conditions

$$u(x,t) = f(x),\ u_t = g(x) \tag{14}$$

A series of solutions of Equation (14) were obtained using various initial and boundary conditions. The method applied a direct way without using linearization, transformation, perturbation, discretization or restrictive assumptions. The solutions given consist of implicit functions without physical interpretation. Nimmo & Schief [13] found superposition principles, linear and nonlinear, associated with the Moutard transformation. For an integrable discrete nonlinear and its associated linear system, it was shown that in a particular form, this system was an integrable discretization. The of a (2+1)-dimensional Sine-Gordon system. The Solutions of discrete nonlinear systems were constructed by means of a discrete analogue of the Moutard transformation. Taking $x = mh$ and $t = nk$, for small h and k they got;

$$\sin\left(hku_{xt}\right) = hk \sin u + O(hk); \tag{15}$$

This leads to order u satisfying the Sine-Gordon Equation;

$$u_{xt} = \sin u \tag{16}$$

Where the discrete variables n and m are viewed as true discrete versions of the continuous variables x and t in the Sine-Gordon Equation, and there are solutions of the discrete equation that are closely related to the well-known kink solutions of the continuous one. Houde Han & Zhiwen Zhang [3] studied the numerical solution of the two-dimensional Sine-Gordon Equation. The Split local artificial boundary conditions were obtained by the operator splitting method. Then the original problem was reduced to an initial boundary value problem on a bounded computational domain, which was solved by the FDM. The initial value problem of the two-dimensional Sine-Gordon Equation considered is given by the problem;

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + \sin(u) = 0, \qquad x, y \in \Re^1,\ \text{t.>0} \tag{17}$$

$$u\ \big|_{t=0} = \varphi_o(x,y),\ u_t\ \big|_{t=0} = \varphi_1(x,y),\ x,y \in \Re^1\ \text{t.>0} \tag{18}$$

where $u = u(x,\ y,\ t)$ represents the wave displacement at position $(x,\ y)$ and at time $t, \varphi_o(x,y), \varphi_1(x,y)$ are the initial displacement and velocity respectively, and sin (u) is a nonlinear force term. Soliton solution to the Sine-Gordon Equation (18) was obtained. Lin Jin [10] considered the initial value problem for the Sine-Gordon Equation by using the Homotopy perturbation method. Based upon the Homotopy perturbation method, a small parameter was introduced and Taylor series expansion used to modify the method. The modified method provided a new analytical approach to solve the initial value problem of Equation (17) subject to the initial conditions

$$u(x,t_o) = g_1(x) \qquad u_t(x,t_o) = g_2(x) \tag{19}$$

The Analytical and approximate solutions obtained are of implicit functions without physical interpretations. Jung [5] gave a numerical study on the spectral methods and the high order WENO Finite Difference Scheme for the solution of linear and nonlinear hyperbolic Partial Differential Equations with stationary and non-stationary singular sources. The results were compared with those computed by the second order FDM. They solved a non-linear scalar P.D.E with a stationary singular source namely; the Sine-Gordon Equation with a disordered media possessing a point-like defect. The defect was modeled as a δ-function which is a source term in the equation;

$$u_{tt} - u_{xx} + \sin(u) = p(u), -\infty < \ x < \infty, t > 0 \tag{20}$$

where $p(u)$ is the potential term due to the defect described as $p(u) = \in \delta(x)\sin(u)$. The non-zero constant $\in$ is the measure of the strength of the defect. The Sine-Gordon Equation was found to be integrable with the existence of a soliton solution, known as the kink solution. Jafar Biazar & Wai Sun Don [6] applied Differential Transform Method (DTM) based on Taylor series expansion to the Sine-Gordon Equation;

$$u_{tt}(x,t) - c^2 u_{xx}(x,t) + \alpha \sin u(x,t) = 0 \tag{21}$$

Subject to initial conditions in Equation (21). He obtained a series of solution with use of various conditions. Macías-Díaz & Jerez-Galiano [8] presented two numerical methods to approximate solutions of systems of dissipative Sine-Gordon Equations that arise in the study of one dimensional, semi-infinite arrays of Josephson junctions coupled through superconducting wires. The schemes for the total energy of such systems in association with the Finite Difference Schemes were used to approximate the solutions. The methods were employed in the estimation of the threshold at which nonlinear supratransmission takes place, in the presence of parameters such as internal and external damping, generalized mass and generalized Josephson current. The Finite Difference Method which is use, was first developed as "the method of squares" by Thom Apelt [19] in the 1920s, and was used to solve nonlinear hydrodynamic equations. The technique is based upon the approximations that permit replacement of differential equations by Finite Difference Equation. These Finite Difference approximations are algebraic in form, and the solutions are related to grid points. Much of the work of Finite Difference Schemes has been presented in Jain [7], Morton & Mayers [12] and Rahman [14]. The methods that have been discussed by these authors include The Schmidt, Leap Frog, Du Fort and Frankel, Lax Friedrich's, Crank Nicholson and Douglas. The Schmidt, Leap Frog, Du Fort and Frankel methods are explicit in nature and The Crank Nicholson and Douglas methods are implicit. We develop Crank Nicholson Scheme in our work. In Lin Jin [10], more advanced Finite Difference Method due to Lax Friedrich was discussed. The Von Neumann and Matrix methods of ascertaining stability in Finite Difference Schemes are also discussed.Matrix method is used to analyse stability of our scheme.

III. Numerical Scheme and Stability Analysis

The equation (1) is discretized to come up with Crank-Nicholson Scheme (CNS)

3.1. Crank-Nicholson Scheme (CNS)

In this scheme, we replace u_{tt} by the central difference approximation, u_{xx} by the average of the jth level and the (j+1)th level central differences and u_{xt} by the Forward-central difference as follows to obtain

$$\frac{U_{i,j+1}-2U_{i,j}+U_{i,j-1}}{k^2}-\left[\frac{U_{i+1,j+1}-2U_{i,j+1}+U_{i-1,j+1}}{2h^2}+\frac{U_{i+1,j}-2U_{i,j}+U_{i-1,j}}{2h^2}\right]+\frac{U_{i+1,j+1}-U_{i+1,j-1}-U_{i-1,j+1}+U_{i-1,j-1}}{4hk}=0 \tag{22}$$

Let $\mu=\dfrac{k^2}{h^2}$ and $\phi=\dfrac{k}{h}$, multiplying (21) by $4k^2$, the scheme becomes

$$(4+4\mu)U_{i,j+1}+(\phi-2\mu)U_{i+1,j+1}+(-\phi-2\mu)U_{i-1,j+1}=2\mu U_{i+1,j}-(4\mu-8)U_{i,j}-4U_{i,j-1}+\phi U_{i+1j-1}-\phi U_{i-1,j-1}+2\mu U_{i-1,j} \tag{23}$$

This is for $i=1,2,3,\ldots\ldots\ldots\ \ldots\ldots(N-2),(N-1).$ where N is number of divisions along the x-axis

3.2. Stability Analysis of Crank Nicolson Scheme

We use also the matrix method to analyze stability of the scheme (23)

Expanding this scheme by taking $i=1,2,3,\ldots\ldots\ldots\ \ldots\ldots(N-2),(N-1).$, we get the system of algebraic equations

$$(4+4\mu)U_{1,j+1}+(\phi-2\mu)U_{2,j+1}+(-\phi-2\mu)U_{0,j+1}=2\mu U_{2,j}-(4\mu-8)U_{1,j}-4U_{1,j-1}+\phi U_{2j-1}-\phi U_{0,j-1}+2\mu U_{0,j}$$

$$(4+4\mu)U_{2j+1}+(\phi-2\mu)U_{3j+1}+(-\phi-2\mu)U_{1j+1}=2\mu U_{3,j}-(4\mu-8)U_{2,j}-4U_{2,j-1}+\phi U_{3j-1}-\phi U_{1,j-1}+2\mu U_{1,j}$$

$$(4+4\mu)U_{3j+1}+(\phi-2\mu)U_{4,j+1}+(-\phi-2\mu)U_{2,j+1}=2\mu U_{4,j}-(4\mu-8)U_{3,j}-4U_{3,j-1}+\phi U_{4,+1j-1}-\phi U_{2,j-1}+2\mu U_{2,j}$$

:

:

$$(4+4\mu)U_{N-2,j+1}+(\phi-2\mu)U_{N-1,j+1}+(-\phi-2\mu)U_{N-3,j+1}=2\mu U_{N-1,j}-(4\mu-8)U_{N-2,j}-4U_{N-2,j-1}+\phi U_{N-1j-1}-\phi U_{N-3,j-1}+2\mu U_{N-3,j}$$

$$(4+4\mu)U_{N-1,j+1}+(\phi-2\mu)U_{N,j+1}+(-\phi-2\mu)U_{N-2,j+1}=2\mu U_{N,j}-(4\mu-8)U_{N-1,j}-4U_{N-1,j-1}+\phi U_{N,j-1}-\phi U_{N-2,j-1}+2\mu U_{N-2,j}$$

Writing the algebraic equations in matrix form

$$\begin{bmatrix}(4+8\mu) & (2\mu-\phi) & \ldots & 0 & 0\\ (-2\mu-\phi) & (4+8\mu) & (2\mu-\phi) & 0 & 0\\ \vdots & (-2\mu-\phi) & \ddots & (2\mu-\phi) & \vdots\\ 0 & 0 & (-2\mu-\phi) & (4+8\mu) & (2\mu-\phi)\\ 0 & 0 & \cdots & (-2\mu-\phi) & (4+8\mu)\end{bmatrix}\begin{bmatrix}U_{1,j_{+1}}\\ U_{2,j+1}\\ \vdots\\ U_{N-2,j+1}\\ U_{N-1,j_{+1}}\end{bmatrix}+\begin{bmatrix}(2\mu-\phi)\quad U_{0,j+1}\quad (-2\mu-\phi)U_{0,j-1}\\ 0\\ \vdots\\ 0\\ (2\mu-\phi)\quad U_{N-2,j+1}\quad (-2\mu-\phi)U_{N-2,j+1}\end{bmatrix}$$

$$=\begin{bmatrix}8 & 0 & \ldots & 0 & 0\\ 0 & 8 & 0 & 0 & 0\\ \vdots & 0 & \ddots & 0 & \vdots\\ 0 & 0 & 0 & 8 & 0\\ 0 & 0 & \cdots & 0 & 8\end{bmatrix}\begin{bmatrix}U_{1,j}\\ U_{2,j}\\ \vdots\\ U_{N-2,j}\\ U_{N-1,j}\end{bmatrix}$$

The system can be written compactly as $\left[(2\mu+\phi)G+(4\mu+4)I\right]U_{j_{+1}}=8U_j+\vec{e}$ **,** where I is an identity matrix of order $(N-I)\times(N-I)$, $\vec{e}$ is a constant vector with

$$G=\begin{bmatrix}0 & -1 & \cdots & 0 & 0\\ -1 & 0 & -1 & 0 & 0\\ \vdots & -1 & \ddots & -1 & \vdots\\ 0 & 0 & -1 & 0 & -1\\ 0 & 0 & \cdots & -1 & 0\end{bmatrix},\quad \vec{e}=\begin{bmatrix}(2\mu-\phi)\quad U_{0,j+1}\quad (-2\mu-\phi)U_{0,j-1}\\ 0\\ \vdots\\ 0\\ (2\mu-\phi)\quad U_{N-2,j+1}\quad (-2\mu-\phi)U_{N-2,j+1}\end{bmatrix}$$

Therefore $U_{j+1}=\left[(2\mu+\phi)G+(4\mu+4)I\right]^{-1}[8I]U_j+\vec{f}$ **,** $U_{j+1}=HU_j+\vec{f}$, where $\vec{f}=\left[(2\mu-\phi)G+((4\mu+4)I)\right]^{-1}$, $H=\left[(2\mu-\phi)G+((4\mu+4)I)\right]^{-1}[8I]$

Eigen value of $G=0+2.\cos\left(\frac{m\pi}{m+1}\right)$=2 $\cos\left(\frac{m\pi}{m+1}\right)$=2-4 $\sin^2\cos\left(\frac{m\pi}{m+1}\right)$

Eigen value of H= $\left|\dfrac{8}{8\mu+2\phi+4-(4\mu+2\phi)\sin^2\left(\dfrac{m\pi}{14}\right)}\right|\le 1$

For $\mu=\phi>0$ which is the stability criterion condition.

IV. Numerical Solutions

The numerical solutions for equation (1) are found by developing system of linear algebraic equations and solving their corresponding matrix equations.

4.1. Crank Nicolson Scheme (Case 1, t=0.25)

From the CNS (Equation 23), we fix $j=1(t=0.25)$ and $i=1,2\ldots\ldots\ldots\ldots\ldots\ldots.13$, we get the system of algebraic equations

i=1: $8U_{1,2}-U_{2,2}-3U_{0,2}=2U_{0,1}+4U_{1,1}+2U_{2,1}-4U_{1,0}+U_{2,0}-U_{0,0}$

i=2: $8U_{2,2}-U_{3,2}-3U_{1,2}=2U_{1,1}+4U_{3,1}+2U_{1,1}-4U_{2,0}+U_{3,0}-U_{1,0}$

i=3: $8U_{3,2} - U_{4,2} - 3U_{2,2} = 2U_{2,1} + 4U_{4,1} + 2U_{1,1} - 4U_{3,0} + U_{4,0} - U_{2,0}$
i=4: $8U_{4,2} - U_{5,2} - 3U_{3,2} = 2U_{3,1} + 4U_{5,1} + 2U_{1,1} - U_{4,0} + U_{5,0} - U_{2,0}$
i=5: $8U_{5,2} - U_{6,2} - 3U_{4,2} = 2U_{4,1} + 4U_{6,1} + 2U_{1,1} - 4U_{5,0} + U_{6,0} - U_{4,0}$
i=6: $8U_{6,2} - U_{7,2} - 3U_{5,2} = 2U_{5,1} + 4U_{7,1} + 2U_{1,1} - 4U_{6,0} + U_{7,0} - U_{5,0}$
i=7: $8U_{7,2} - U_{8,2} - 3U_{6,2} = 2U_{6,1} + 4U_{8,1} + 2U_{1,1} - 4U_{7,0} + U_{8,0} - U_{6,0}$
i=8: $8U_{8,2} - U_{9,2} - 3U_{7,2} = 2U_{7,1} + 4U_{9,1} + 2U_{1,1} - 4U_{8,0} + U_{9,0} - U_{7,0}$
i=9: $8U_{9,2} - U_{10,2} - 3U_{8,2} = 2U_{8,1} + 4U_{10,1} + 2U_{1,1} - 4U_{9,0} + U_{10,0} - U_{8,0}$
i=10: $8U_{10,2} - U_{11,2} - 3U_{9,2} = 2U_{9,1} + 4U_{11,1} + 2U_{1,1} - 4U_{10,0} + U_{11,0} - U_{9,0}$
i=11: $8U_{11,2} - U_{12,2} - 3U_{10,2} = 2U_{10,1} + 4U_{12,1} + 2U_{1,1} - 4U_{11,0} + U_{12,0} - U_{10,0}$
i=12: $8U_{12,2} - U_{13,2} - 3U_{11,2} = 2U_{11,1} + 4U_{13,1} + 2U_{1,1} - 4U_{12,0} + U_{13,0} - U_{11,0}$
i=13: $8U_{13,2} - U_{14,2} - 3U_{12,2} = 2U_{12,1} + 4U_{14,1} + 2U_{1,1} - 4U_{13,0} + U_{14,0} - U_{12,0}$

This can be written in matrix form as

$$\begin{bmatrix} 8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & 8 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 & 8 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 8 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 8 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 8 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 8 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 8 \end{bmatrix} \begin{bmatrix} U_{1,2} \\ U_{2,2} \\ U_{3,2} \\ U_{4,2} \\ U_{5,2} \\ U_{6,2} \\ U_{7,2} \\ U_{8,2} \\ U_{9,2} \\ U_{10,2} \\ U_{11,2} \\ U_{12,2} \\ U_{13,2} \end{bmatrix} = \begin{bmatrix} 1.93308453 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Using Mathematica the solutions are; $U_{1,2}$=0.2541676, $U_{2,2}$=0.1002561, $U_{3,2}$=0.03954588, $U_{4,2}$=0.01559882, $U_{5,2}$=0.006152936, $U_{6,2}$=0.002427018; $U_{7,2}$=0.000957334, $U_{8,2}$=0.0003776192, $U_{9,2}$=0.0001489514, $U_{10,2}$=0.00005875327, $U_{11,2}$=0.00002317209, $U_{12,2}$=0.0000091168866, $U_{13,2}$=0.000003418832.

4.2. *Crank Nicolson Scheme (Case 2, t=0.5)*

Similarly for $j = 2$ and $i = 1, 2.......13$.This can be written in matrix form as

$$\begin{bmatrix} 8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & 8 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 & 8 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 8 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 8 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 8 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 8 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 8 \end{bmatrix} \begin{bmatrix} U_{1,3} \\ U_{2,3} \\ U_{3,3} \\ U_{4,3} \\ U_{5,3} \\ U_{6,3} \\ U_{7,3} \\ U_{8,3} \\ U_{9,3} \\ U_{10,3} \\ U_{11,3} \\ U_{12,3} \\ U_{13,3} \end{bmatrix} = \begin{bmatrix} 4.3072732 \\ 0.98845136 \\ 0.38989336 \\ 0.15379291 \\ 0.09186106 \\ 0.02392861 \\ 0.00943861 \\ 0.00372305 \\ 0.00146855 \\ 0.00057926 \\ 0.00022843 \\ 0.00008965 \\ 0.00003191 \end{bmatrix}$$

Using Mathematica the solutions are; $U_{1,3}$=0.5843568, $U_{2,3}$=0.3675817, $U_{3,3}$=0.1991315, $U_{4,3}$=0.1004137, $U_{5,3}$=0.052122539, $U_{6,3}$=0.02387797, $U_{7,3}$=0.01072753, $U_{8,3}$=0.004747756, $U_{9,3}$=0.002076396, $U_{10,3}$=0.0008993491, $U_{11,3}$=0.0003863453, $U_{12,3}$=0.0001642852, $U_{13,3}$=0.0000655957.

4.3. *Crank Nicolson Scheme (Case 3, t=0.75)*

Similarly for $j = 3$ and $i = 1, 2.......13$

This can be written in matrix form as

$$\begin{bmatrix} 8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & 8 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 & 8 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 8 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 8 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 8 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 8 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 8 \end{bmatrix} \begin{bmatrix} U_{1,4} \\ U_{2,4} \\ U_{3,4} \\ U_{4,4} \\ U_{5,4} \\ U_{6,4} \\ U_{7,4} \\ U_{8,4} \\ U_{9,4} \\ U_{10,4} \\ U_{11,4} \\ U_{12,4} \\ U_{13,4} \end{bmatrix} = \begin{bmatrix} 5.56444123 \\ 2.42165728 \\ 1.48967600 \\ 0.80837464 \\ 0.41928992 \\ 0.19140465 \\ 0.09428284 \\ 0.04228002 \\ 0.01868512 \\ 0.00816209 \\ 0.00353033 \\ 0.00144858 \\ 0.00056816 \end{bmatrix}$$

Using Mathematica the solutions are; $U_{1,4}$=0.7771392, $U_{2,4}$=0.6526721, $U_{3,4}$=0.4683018, $U_{4,4}$=0.2987221, $U_{5,4}$=0.1764967, $U_{6,4}$=0.0965173, $U_{7,4}$=0.05124369, $U_{8,4}$=0.0261148, $U_{9,4}$=0.0129073, $U_{10,4}$=0.006228868, $U_{11,4}$=0.002946961, $U_{12,4}$=0.001358752, $U_{13,4}$=0.0005805519.

Table 1: Summary for Values of $U_{i,j}$ for Crank Nicolson Difference Scheme cases 1, 2, 3

	t_0=0	t_1=0.25(case1)	t_2=0.50(case2)	t_3=0.75(case3)
X_0=0.00	0	0.24740395	0.479425538	0.841470984
X_1=0.25	0	0.2541676	0.5843568	0.7771392
X_2=0.50	0	0.1002561	0.3675817	0.6526721
X_3=0.75	0	0.03954588	0.1991315	0.4683018
X_4=1.00	0	0.01559882	0.1004137	0.2987221
X_5=1.25	0	0.006152936	0.05212262	0.1764967
X_6=1.50	0	0.002427018	0.02387797	0.0965173
X_7=1.75	0	0.000957334	0.010727553	0.05124369
X_8=2.00	0	0.0003776192	0.004747756	0.0261148
X_9=2.25	0	0.0001489514	0.002076396	0.0129073
X_{10}=2.50	0	0.00005875327	0.0008993491	0.006228868
X_{11}=2.75	0	0.00002317209	0.0003863453	0.002946961
X_{12}=3.00	0	0.000009116886	0.0001642852	0.001358752
X_{13}=3.25	0	0.000003418832	0.0000655957	0.0005805519

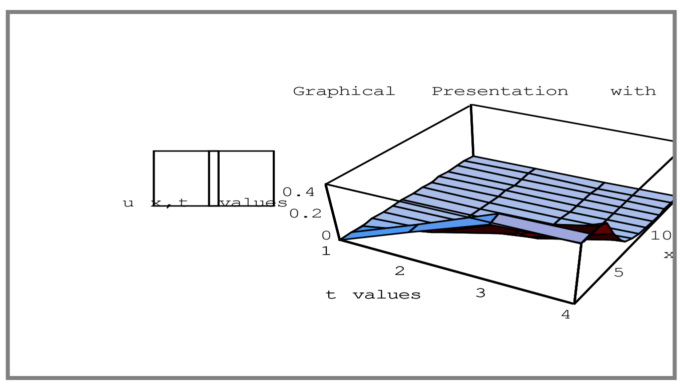

Figure 2: Graphical Presentation for CNS results with $h = k = \frac{1}{4}$

4.4. *Crank Nicolson Scheme (Case 4, for t=0.2, 0.4 and 0.6)*

From the CNS (23).We fix $j = 1, 2.....3 \left(t = 0.2, 0.4, 0.6 \right)$ and $i = 1, 2........13$. Using Mathematica (as in cases 1,2 and 3 above), the results are as shown in table 2.

Table 2: Summary for Values of $U_{i,j}$ for Crank Nicolson Scheme [Case 4, (t=0.2, 0.4, 0.6)]

	t_0=0	t_1=0.2(case4)	t_2=0.4(case4)	t_3=0.6(case4)
X_0=0.0	0	0.19866933	0.389418342	0.564642473
X_1=0.2	0	0.2058488	0.43681	0.6236607
X_2=0.4	0	0.0811968	0.2345961	0.6231907
X_3=0.6	0	0.03202797	01363259	0.3822238
X_4=0.8	0	0.01263339	0.07104636	0.2283048
X_5=1.0	0	0.004983226	0.03483738	0.1308336
X_6=1.2	0	0.001965627	0.01642903	0.07805445
X_7=1.4	0	0.0007753391	0.00754048	0.0398877
X_8=1.6	0	0.0003058315	0.00339247	0.01989448
X_9=1.8	0	0.0001206348	0.001503049	0.0097138
X_{10}=2.0	0	0.00004758392	0.0006576145	0.00465539
X_{11}=2.2	0	0.00001876693	0.000282648	0.002192067
X_{12}=2.4	0	0.0000007383712	0.0001166303	0.0010132
X_{13}=2.6	0	0.0000007268892	0.00004405888	0.0004309062

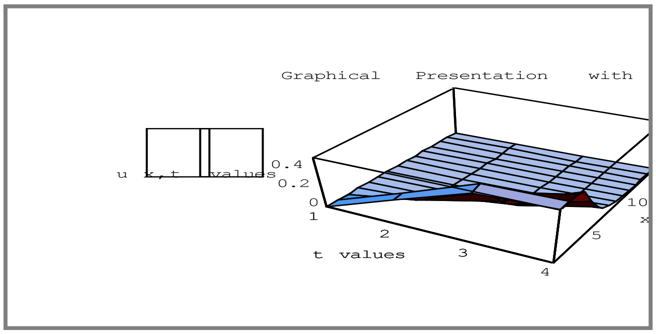

Figure 3: Graphical Presentation for CNS results with $h = k = \frac{1}{5}$

V. Conclusion and Recommendations

5.1. Discussion

From the graphical presentations of the solutions, it can be observed that,

- the surface of the plot is not smooth because the differential equation is satisfied only at a selected number of discrete nodes within the region of integration
- for a given value of t, $U_{i,j}$ decreases to nearly zero as x tends to infinity
- the smaller the mesh sizes, the more finely the result values for CNS

Simple harmonic motion occurs when a particle or object moves back and forth within a stable equilibrium position under the influence of a restoring force proportional to its displacement. In an ideal situation, where friction plays no part, an object would continue to oscillate indefinitely. Objects in the real world do not experience perpetual oscillation; instead, they are subject to damping, or the dissipation of energy, primarily as a result of friction. If damping effect is small, the amplitude will gradually decrease as the object continues to oscillate, until eventually oscillation ceases. Our results from the numerical scheme (CNS) for the two cases are confirming this since the displacement of the wave particles given by $u(x,t)$ is decreasing with an increase in position x from the source (point of wave disturbance) and increases with increase in time t of the wave.

5.2. Conclusion

This study focused on nonlinear Sine-Gordon Equation. A numerical scheme namely Crank Nicolson Schemes was developed and used in this study. It was found out that the scheme is conditionally stable. This conditional stability was restricted to the mesh size $h = k$ with the limits on the values of x as $0 \le x \le 3.25$ for case 1 and $0 \le x \le 2.6$ for case 2.

We have managed to come up with the numerical solutions to the Equation (1) under study and the results interpreted with the physical application to the rigid pendula attached to a stretched wire.

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