Finite Difference Solution of (2+1)-Dimensional Sine-Gordon Equation: A Model for Investigating the Effects of Varying Surface Damping Parameter on Josephson Current Flowing through the Long Josephson Junction

Tinega Abel Kurura[1], Oduor Okoya[2], Omolo Ongati[3]

[1, 2, 3]School of Mathematics and Actuarial Science, Jaramogi Oginga Odinga University of Science and Technology, Bondo, Kenya

Abstract: ***Modeling of some physical phenomena and technological processes taking into account dissipation leads to the Sine-Gordon equation with the first time derivative. The (2+1) Sine-Gordon equation with the first time derivative is used in explaining a number of physical phenomena including the propagation of fluxons in Josephson junctions. This study uses Finite Difference Method to solve (2+1) dimensional Sine-Gordon equation with the first time derivative that models dissipation of the current flow through the long Josephson junction. An Alternating Direction Implicit numerical scheme for the equation is developed with concepts of stability tested using Matrix Method. The value of surface damping parameter used are β=1.1μΩ, β =3.7 μΩ, β=7.2 μΩ and β=9.3 μΩ for Alluminium (Al), Tin (Sn), Lead (Pb), and Niobium (Nb) respectively. The numerical results obtained are presented in tables and graphs. The computational results obtained indicate that as the length of long Josephson junction increases, the current flowing through the long Josephson junction decreases to zero. The results also indicate that when the surface damping parameter increases, the current flowing through the long Josephson junction also increases.***

Keywords: Alternating Direction implicit Scheme, Finite Difference Method, Sine-Gordon Equation, Matrix Method, and Stability Analysis, Surface damping parameter, and Partial Differential Equation

Abbreviations: Sine-Gordon Equation (SGE),Alternating Direction Explicit Scheme (ADES), Alternating Direction implicit Scheme (ADIS), Differential Transform Method (DTM), Discrete Fourier Transform (DFT), Explicit Scheme (ES),Partial Differential Equation (PDE),Sine-Gordon Equation (SGE),Strong Stability Preserving Runge–Kutta scheme (SSP-RK54),Alluminium(Al),Tin (Sn),Lead(Pb), and Niobium (Nb).

1. Introduction

The Sine–Gordon equation appears in the propagation of fluxons in Josephson junctions, dislocations in crystals, solid state physics, nonlinear optics, stability of fluid motions and the motion of a rigid pendulum attached to a stretched wire, see [6, 21, 26, 28]. Josephson junction model [21] consists of two superconducting layers separated by isolated barriers. According to [7], the (2+1) dimensional sine-Gordon equation with first time derivative that governs current vibration flow through Josephson junction is given by;

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{1}{\overline{C}^2}\frac{\partial^2 u}{\partial t^2}-\frac{\beta}{\overline{C}^2}\frac{\partial u}{\partial t}=\frac{1}{\lambda_J^2}\sin u \qquad (1)$$

where $u = u(x,y)$ represents the current flow at position (x,y) and at time t along Josephson junction $\overline{c}$ is the Swihart velocity (velocity of the electron), λ_J is the Josephson penetration depth. The parameter β is known as surface damping parameter (measure of resistance of superconductors), which is supposed to be a real number with $\beta > 0$. This model of the two-dimensional damped sine-Gordon equation has various applications in physics, electronics etc. Methods of solving the Sine-Gordon equation have been the focus of many recent research works. Drazin [19] discusses the stability of the finite difference schemes for solving the nonlinear Klein-Gordon equation. Olusola and Emmanuel [24] employed the Reduced Differential Transform Method (RDTM) to obtain solutions to the (2+1) SG equation. The approximate solution for the equation converges to solution of Adomian Decomposition when compared. They showed that the results for the two methods had closeness between them. Aero [3] used a method proposed by Lamb for solving the two-dimensional SGE.The method was extended to the generalized (3+1)-dimensional SGE to model some physical phenomena and technological processes taking into account dissipation leading to the sine-Gordon equation with the first time derivative;. The proposed Lamp approach transforms the problem of integration of the classical SGE and its generalizations to searching of unknown functions from the system of algebraic equations. A particular solution was found satisfying the definite initial and boundary conditions. The approach supposes natural generalization in the case of integration of the equation considered in the space of any number of dimensions giving a solution of sinh-Gordon equation. Abdul-Majid [1] examined for (1+1), (2+1) and (3+1) dimension multiple soliton solutions of SGE using the simplified form of the Hirota's analytic method. One and two soliton solutions were obtained for the higher dimension SGE. Paul [25] considered the (1+1), (2+1) and (3+1) dimensional cases of sine-Gordon equation. He solved the equations numerically using the Discrete Fourier Transform (DFT) and leapfrog method to approximate the second derivative in time with a central difference. A numerical

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solution with a stationary moving and colliding breathers were constructed based on Dirichlet's boundary conditions. The given equations were decomposed into a system of equations and the modified cubic B-spline basis functions used for spatial variable and their derivatives, which resulted in a system of ordinary differential equations. The resulting systems of equations were subsequently solved by SSP-RK54 scheme giving 1D, 2D and 3D pulse breather solution.

Guo *et al*. [20] proposed two different difference schemes, namely, explicit and implicit schemes. Christiansen and Lomdahl [12] used a generalized leapfrog method for the numerical computation of two-dimensional undamped SGE while a finite elements method was used by Argyris *et al*. [20]. Both methods were successfully applied using the appropriate initial conditions with the latter one gave slightly more accurate results. Xin [29] modeled light bullets with two-dimensional sine–Gordon equation. Light bullets contain only a few electromagnetic oscillations under their envelopes and propagate long distances without essentially changing shapes. Author of [29] performed a modulation analysis and observed that the sine-Gordon pulse envelopes undergo focusing–defocusing cycles. The evolution of lump and ring solitions of the two dimensional sine-Gordon equation and the evolution of standing and travelling breather-type waves are studied in Minzoni *et al*. [22, 23]. Morever, lump and ring solitions can be applied to the Baby Skyrme model and to study the vortex models [22].

Sheng *et al* [27] used a split cosine approach for the numerical simulation of two-dimensional sine-Gordon solitons. An explicit numerical scheme and an improved numerical scheme for the numerical solutions of (2+1) dimension SGE is proposed in [7-11]. Dehghan and his colleagues [13-18] proposed number of schemes for the numerical solutions of two-dimensional damped and undamped sine-Gordon equations.

On the basis of the literature review, it appears that no work was reported on solving (2+1) model of sine-Gordon equation with first time derivative that governs the current flow density through the Josephson junction when $\beta > 0$ using Finite Difference Method. It is the objective of this paper to numerically solve the (2+1) SGE with the first time derivative on a rectangular domain subject to some prescribed boundary conditions. The rest of the paper is organized as follows; in Section II the Method of solution of (2+1) SGE is briefly discussed. After a brief discussion of the numerical Method, Section III describes the numerical schemes developed and stability for the schemes analyzed. Section IV is numerical results and discussion while the last Section V is about conclusion and recommendation.

2. Method of Solution

The numerical methods can be categorized as Finite Difference, Finite element, Finite volume and Boundary element. The method of Finite Difference is one of the most valuable methods of approximating numerical solutions of PDEs. In this study, Finite Differences Method is used to solve a (2+1) dimensional Sine-Gordon Equation (1) with first time derivative. Before numerical computations are made, there are three important properties of finite difference equations that must be considered, namely;

a) *Convergence*: A finite difference equation is convergent if the solution of finite difference equation approaches the exact solution of the partial differential equation as the mesh sizes approaches zero.
b) *Consistency*: When a truncation error goes to zero, a finite difference equation is said to be consistent or compatible with a partial differential equation.
c) *Stability:* The difference between a partial differential equation and the equivalent finite difference expression is referred to as truncation error. A numerical process is said to be stable if it limits amplification of all components of the initial conditions.

The use of finite difference techniques for the solution of partial differential equation is a three step process .These steps are;

1) The partial differential equations are approximated by a set of linear equations relating to the values of the functions at each mesh point.
2) The set of the algebraic equations, generated for equation must be solved and
3) An iteration procedure has to be developed which takes into account the non-linear character of the equation. In our endeavor to solve the (2+1) dimensional sine-Gordon equation, the stability of the scheme developed for the equation is analyzed.

3. Numerical Schemes and Stability Analysis

In this section, a numerical scheme is developed for Equation (1) namely; An Alternating Direction Implicit (ADIS) numerical schemes and its stability analyzed using Matrix method.

3.1 Discretization of Partial Derivatives

The finite difference technique basically involves replacing the partial derivatives occurring in the partial differential equation as well as the boundary and initial conditions by their corresponding finite difference approximations and then solving the resulting linear algebraic system of equations by a direct method or a standard iterative procedure. The numerical values of the dependent variable are obtained at the points of intersection of the parallel lines, called mesh points or nodal point.

3.2 Discretization of Equation (1)

Discretization of Equation (1) is obtained by replacing partial derivatives appearing in the equation with their Finite difference approximations as follows

$$\frac{\partial^2 u}{\partial t^2} = \frac{U_{i,j}^{n+1} - 2U_{i,j} + U_{i,j}^{n-1}}{(\Delta t)^2} \tag{2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{U_{i+1,j}^{n} - 2U_{i,j}^{n} + U_{i-1,j}^{n}}{(\Delta x)^2} \tag{3}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U_{i,j+1}^{n} - 2U_{i,j}^{n} + U_{i,j-1}^{n}}{(\Delta y)^2} \tag{4}$$

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$$\frac{\partial u}{\partial t}=\frac{U_{i,j}^{n+1}-U_{i,j}^{n}}{(\Delta t)} \tag{5}$$

$$\sin u \approx \frac{\partial^2 u}{\partial x \partial y} \approx \left[\frac{U_{i+1,j+1}^{n}-U_{i+1,j-1}^{n}+U_{i-1,j-1}^{n}-U_{i,j-1}^{n}+U_{i-1,j+1}^{n}}{4\Delta x \Delta y}\right] \tag{6}$$

3.3 Alternating Direction Implicit Scheme (ADIS)

In this scheme we advance the solution of the (2+1) sine-Gordon partial differential Equation (1) from n[th] plane to (n+1)[th] plane by replacing u_{xx} by implicit finite difference approximation at the (n+1)[th] plane. Similarly, u_{tt} , u_{yy} and u_t are replaced by an explicit central finite difference approximation at the n[th] plane as in equations (2), (4) and (5) respectively and $\sin u$ by (6). With these approximations substituted into Equation (1), the following scheme is obtained

$$\frac{U_{i+1,j}^{n+1}-2U_{i,j}^{n+1}+U_{i-1,j}^{n+1}}{(\Delta x)^2}+\frac{U_{i,j+1}^{n}-2U_{i,j}^{n}+U_{i,j-1}^{n}}{(\Delta y)^2}-\left(\frac{U_{i,j}^{n+1}-2U_{i,j}^{n}+U_{i,j}^{n-1}}{\bar{c}^2(\Delta t)^2}\right)-\frac{\beta}{\bar{c}^2}\left(\frac{U_{i,j}^{n+1}-U_{i,j}^{n}}{(\Delta t)}\right)=\left[\frac{U_{i+1,j+1}^{n}-U_{i+1,j-1}^{n}+U_{i-1,j-1}^{n}-U_{i-1,j+1}^{n}}{4(\Delta x\Delta y)\lambda_j^2}\right] \tag{7}$$

We let $\Delta x=\Delta y=\Delta t=s$, $\bar{c}^2=1$ and multiplying Equation (7) by $4\bar{c}^2(\Delta x)^2$, we obtain the scheme;

$$4U_{i+1,j}^{n+1}-(-12-4s)U_{i,j}^{n+1}-4U_{i-1,j}^{n+1}=-4sU_{i,j}^{n}-4U_{i,j+1}^{n}-4U_{i,j-1}^{n}+U_{i+1,j-1}^{n}-U_{i+1,j-1}^{n}-U_{i-1,j+1}^{n}+4U_{i,j}^{n-1}+U_{i-1,j-1}^{n} \tag{8}$$

3.4 Stability Analysis of Alternating Direction Implicit Scheme (ADIS)

We use also the matrix method to analyze stability of the scheme (8). This is done by expanding the scheme in equation (8) by taking $i=1,2,3,...............(N-2),(N-1)$ .We get the system of linear algebraic equations as

$$\left.\begin{array}{l}
4U_{2,j}^{n+1}-(-12-4s)U_{1,j}^{n+1}-4U_{0,j}^{n+1}=-4sU_{1,j}^{n}-4U_{1,j+1}^{n}-4U_{1,j-1}^{n}+U_{2,j-1}^{n}-U_{2,j-1}^{n}-U_{0,j+1}^{n}+4U_{1,j}^{n-1}+U_{0,j-1}^{n}\\
4U_{3,j}^{n+1}-(-12-4s)U_{2,j}^{n+1}-4U_{1,j}^{n+1}=-4sU_{2,j}^{n}-4U_{2,j+1}^{n}-4U_{2,j-1}^{n}+U_{3,j-1}^{n}-U_{3,j-1}^{n}-U_{1,j+1}^{n}+4U_{2,j}^{n-1}+U_{1,j-1}^{n}\\
:\\
:\\
4U_{N-1,j}^{n+1}-(-12-4s)U_{N-2,j}^{n+1}-4U_{N-2,j}^{n+1}=-4sU_{N-2,j}^{n}-4U_{N-2,j+1}^{n}-4U_{N-2,j-1}^{n}+U_{N-1,j-1}^{n}-U_{N-1,j-1}^{n}\\
-U_{N-3,j+1}^{n}+4U_{N-1,j}^{n-1}+U_{N-3,j-1}^{n}\\
4U_{N,j}^{n+1}-(-12-4s)U_{N-1,j}^{n+1}-4U_{N-1,j}^{n+1}=-4sU_{N-1,j}^{n}-4U_{N-1,j+1}^{n}-4U_{N-1,j-1}^{n}+U_{N,j-1}^{n}-U_{N,j-1}^{n}\\
-U_{N-2,j+1}^{n}+4U_{N,j}^{n-1}+U_{N-2,j-1}^{n}
\end{array}\right\} \tag{9}$$

Writing the system of algebraic Equations (9) in matrix-vector form;

$$\begin{bmatrix}(-12-4s) & -4 & ... & 0 & 0\\ -4 & (-12-4s) & -4 & 0 & 0\\ \vdots & -4 & \ddots & -4 & \vdots\\ 0 & 0 & -4 & (-12-4s) & -4\\ 0 & 0 & \cdots & -4 & (-12-4s)\end{bmatrix}\begin{bmatrix}U_{1,j+1}^{n+1}\\ U_{2,j}^{n+1}\\ U_{2,j}^{n+1}\\ \vdots\\ U_{N-2,j}^{n+1}\\ U_{N-1,j}^{n+1}\end{bmatrix}=\begin{bmatrix}-4s & -4 & ... & 0 & 0\\ -4 & -4s & -4 & 0 & 0\\ \vdots & -4 & \ddots & -4 & \vdots\\ 0 & 0 & -4 & -4s & -4\\ 0 & 0 & \cdots & -4 & -4s\end{bmatrix}\begin{bmatrix}U_{1,j+1}^{n}\\ U_{2,j+1}^{n}\\ \vdots\\ U_{N-2,j+1}^{n+1}\\ U_{N-1,j+1}^{n+1}\end{bmatrix}+\begin{bmatrix}-4sU_{1,j}^{n} & -4U_{1,j-1}^{n} & -U_{2,j-1}^{n} & +U_{0,j-1}^{n} & +4U_{1,j}^{n-1}\\ -4sU_{2,j}^{n} & -4U_{2,j-1}^{n} & -U_{2,j-1}^{n} & +U_{0,j-1}^{n} & +4U_{2,j}^{n-1}\\ : & : & : & : & :\\ -4sU_{N-2,j}^{n} & -4U_{N-2,j-1}^{n} & -U_{N-1,j-1}^{n} & +U_{N-3,j-1}^{n} & +4U_{N-2,j}^{n-1}\\ -4sU_{N-1,j}^{n} & -4U_{N-1,j-1}^{n} & -U_{N,j-1}^{n} & +U_{N-2,j-1}^{n} & +4U_{N-1,j}^{n-1}\end{bmatrix} \tag{10}$$

The system in Equation (10) can be written as

$$((-12-4s)I_{N-1}-4A_{N-1})U_{N-1,j}^{n+1}=[-4I_{N-1}+A_{N-1}]U_{N-1,j}^{n}+\vec{f} \tag{11}$$

where I_{N-1} and are A_{N-1} identity and square matrices respectively of order $(N-1)\times(N-1)$, $\vec{f}$ is a constant vector. The matrices and a vector are given as;

$$B_{N-1}=\begin{bmatrix}0 & 1 & ... & 0 & 0\\ 1 & 0 & 1 & 0 & 0\\ \vdots & 1 & \ddots & 1 & \vdots\\ 0 & 0 & 1 & 0 & 1\\ 0 & 0 & \cdots & 1 & 0\end{bmatrix}, I_{N-1}=\begin{bmatrix}1 & 0 & ... & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ \vdots & 0 & \ddots & 0 & \vdots\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & \cdots & 0 & 1\end{bmatrix} \text{and}$$

$$\vec{f}=\begin{bmatrix}-4sU_{1,j}^{n} & -4U_{1,j-1}^{n} & -U_{2,j-1}^{n} & +U_{0,j-1}^{n} & +4U_{1,j}^{n-1}\\ -4sU_{2,j}^{n} & -4U_{2,j-1}^{n} & -U_{2,j-1}^{n} & +U_{0,j-1}^{n} & +4U_{2,j}^{n-1}\\ : & : & : & : & :\\ -4sU_{N-2,j}^{n} & -4U_{N-2,j-1}^{n} & -U_{N-1,j-1}^{n} & +U_{N-3,j-1}^{n} & +4U_{N-2,j}^{n-1}\\ -4sU_{N-1,j}^{n} & -4U_{N-1,j-1}^{n} & -U_{N,j-1}^{n} & +U_{N-2,j-1}^{n} & +4U_{N-1,j}^{n-1}\end{bmatrix}$$

Equation (11) can again be written as

$$U_{N-1,j}^{n+1}=[-4I_{N-1}+A_{N-1}]((-12-4s)I_{N-1}-4A_{N-1})^{-1}U_{N-1,j+1}^{n}+((-12-4s)I_{N-1}-4A_{N-1})^{-1}\vec{f} \tag{12}$$

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Therefore, Equation (12) is compactly written as

$$U_{N-1,j}^{n+1} = EU_{N-1,j+1}^{n} + \vec{g} \quad (13)$$

Where

$E = [-4I_{N-1} + A_{N-1}]((-12-4s)I_{N-1} - 4A_{N-1})^{-1}$ and

$\vec{g} = ((-12-4s)I_{N-1} - 4A_{N-1})^{-1}\vec{f}$.

E is the amplification matrix . According to Wen-Chyuan [30], the Eigenvalue of a tridiagonal (N-1) by (N-1) matrix

$$M = \begin{bmatrix} b & c & .. & 0 \\ a & b & \ddots & : \\ : & \ddots & \ddots & c \\ 0 & .. & a & b \end{bmatrix} \quad (14)$$

Is given by

$$\lambda_M = b + 2\sqrt{ac}\cos\left(\frac{m\pi}{N+1}\right) \quad (15)$$

Using the formula in (15), the eigenvalues of A_{N-1} and I_{N-1} matrix are

Eigenvalue o Eigenvalue of

$$A_{N-1} = 2\cos\left(\frac{m\pi}{N+1}\right) = 2 - 4\sin^2\left(\frac{m\pi}{2(N+1)}\right)$$

Eigenvalue of A_{N-1} is 1

Hence Eigenvalue of the amplification matrix E is given as

$$|\lambda_E| = \left|\frac{4I + -4A_{N-1}}{((-12-4s)I_{N-1} - 4A_{N-1})}\right| = \left|\frac{4 + 2 - 4\sin^2\left(\frac{m\pi}{2(N+1)}\right)}{-12 - 4s - 8 + 16\sin^2\left(\frac{m\pi}{2(N+1)}\right)}\right|$$

$$|\lambda_E| = \left|\frac{6 - 4\sin^2\left(\frac{m\pi}{2(N+1)}\right)}{-20 - 4s + 16\sin^2\left(\frac{m\pi}{2(N+1)}\right)}\right| \quad (16)$$

For a tridiagonal matrix, the modulus of the eigenvalue of the amplification matrix E should be less than or equal to unity

$$|\lambda_E| = \left|\frac{6 - 4\sin^2\left(\frac{m\pi}{2(N+1)}\right)}{-20 - 4s + 16\sin^2\left(\frac{m\pi}{2(N+1)}\right)}\right| \le 1 \quad (17)$$

a) For $\sin^2\left(\frac{m\pi}{2(N+1)}\right) = 0$, Equation (17) becomes;

$$|\lambda_E| = \left|\frac{6}{-20-4s}\right| \le 1 \quad (18)$$

b) For $\sin^2\left(\frac{m\pi}{2(N+1)}\right) = 1$, Equation (17) becomes;

$$|\lambda_E| = \left|\frac{6-4}{-20-4s+16}\right| \le 1$$

$$|\lambda_E| = \left|\frac{1}{-1-s}\right| \le 1 \quad (19)$$

The Equation (18) and (19) satisfies the stability conditions. The condition on the right is always satisfied as the left inequality requires. All the eigenvalues in Equations (18) and (19) are bounded by 1 since the denominator is larger than the numerator. Thus the ADIS scheme is unconditionally stable.

4. Numerical Solution of Equation (1)

In this section, Equation (1) is solved using finite difference method and results presented graphically.

4.1Alternating Direction Implicit scheme

If we take Δx=Δy=0.25 on a square mesh and Δt = 0.01, $\bar{c} = \lambda_J = \beta = 1$ and $\beta = 1.1\Omega$ and substituting these values into Equation (1), the following scheme is obtained;

$$0.16U_{i+1,j}^{n+1} - 0.43U_{i,j}^{n+1} + 0.16U_{i-1,j}^{n+1} = -0.8U_{i,j}^{n} - 0.16U_{i,j+1}^{n} - 0.16U_{i,j-1}^{n} + 0.01U_{i,j}^{n-1} \quad (20)$$

If i is varied from $i = 1,2,3,..............10$, $j = 1$ and $n = 0$, the systems of linear algebraic equations are obtained as follows matrix equation is obtained as

$$\left.\begin{array}{l}
\text{i=1} \;:\; 0.16U_{2,1}^{1} - 0.43U_{1,1}^{1} + 0.16U_{0,1}^{1} = -0.8U_{1,1}^{0} - 0.16U_{1,2}^{0} - 0.16U_{1,0}^{0} + 0.01U_{1,0}^{-1} \\
\text{i=2} \;:\; 0.16U_{3,1}^{1} - 0.43U_{2,1}^{1} + 0.16U_{1,1}^{1} = -0.8U_{2,1}^{0} - 0.16U_{2,2}^{0} - 0.16U_{2,0}^{0} + 0.01U_{2,0}^{-1} \\
\text{i=3} \;:\; 0.16U_{4,1}^{1} - 0.43U_{3,1}^{1} + 0.16U_{2,1}^{1} = -0.8U_{3,1}^{0} - 0.16U_{3,2}^{0} - 0.16U_{3,0}^{0} + 0.01U_{3,0}^{-1} \\
\text{i=4} \;:\; 0.16U_{5,1}^{1} - 0.43U_{4,1}^{1} + 0.16U_{3,1}^{1} = -0.8U_{4,1}^{0} - 0.16U_{4,2}^{0} - 0.16U_{4,0}^{0} + 0.01U_{4,0}^{-1} \\
\text{i=5} \;:\; 0.16U_{6,1}^{1} - 0.43U_{5,1}^{1} + 0.16U_{4,1}^{1} = -0.8U_{5,1}^{0} - 0.16U_{5,2}^{0} - 0.16U_{5,0}^{0} + 0.01U_{5,0}^{-1} \\
\text{i=6} \;:\; 0.16U_{7,1}^{1} - 0.43U_{6,1}^{1} + 0.16U_{5,1}^{1} = -0.8U_{6,1}^{0} - 0.16U_{6,2}^{0} - 0.16U_{6,0}^{0} + 0.01U_{6,0}^{-1} \\
\text{i=7} \;:\; 0.16U_{8,1}^{1} - 0.43U_{7,1}^{1} + 0.166U_{6,1}^{1} = -0.8U_{7,1}^{0} - 0.16U_{7,2}^{0} - 0.16U_{7,0}^{0} + 0.01U_{7,0}^{-1} \\
\text{i=8} \;:\; 0.16U_{9,1}^{1} - 0.43U_{8,1}^{1} + 0.16U_{7,1}^{1} = -0.8U_{8,1}^{0} - 0.16U_{8,2}^{0} - 0.16U_{8,0}^{0} + 0.01U_{8,0}^{-1} \\
\text{i=9} \;:\; 0.16U_{10,1}^{1} - 0.43U_{9,1}^{1} + 0.16U_{8,1}^{1} = -0.8U_{9,1}^{0} - 0.16U_{9,2}^{0} - 0.16U_{9,0}^{0} + 0.01U_{9,0}^{-1} \\
\text{i=10} \;:\; 0.16U_{11,1}^{1} - 0.43U_{10,1}^{1} + 0.16U_{9,1}^{1} = -0.8U_{10,1}^{0} - 0.16U_{10,2}^{0} - 0.16U_{10,0}^{0} + 0.01U_{10,0}^{-1}
\end{array}\right\} \quad (21)$$

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With the initial and boundary conditions $u(x,y,0)=e^{-(x+y)}$ and $u(0,y,t)=0$, $u(10,y,t)=0$, for $0\le x\le 10$, $u(x,0,t)=0$, $u(x,2,t)=0$, for $0\le y\le 2$ respectively the above systems of equations (21) can be written in matrix-vector form as The above systems Equation (21) can be written in matrix –vector form as

$$\begin{bmatrix} -0.43 & 0.16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.16 & -0.43 & 0.16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.16 & -0.43 & 0.16 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.16 & -0.43 & 0.16 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.16 & -0.43 & 0.16 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.16 & -0.43 & 0.16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.16 & -0.43 & 0.16 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.16 & -0.43 & 0.16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.16 & -0.43 & 0.16 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.16 & -0.43 \end{bmatrix} \begin{bmatrix} U_{11}^1 \\ U_{21}^1 \\ U_{31}^1 \\ U_{41}^1 \\ U_{51}^1 \\ U_{61}^1 \\ U_{71}^1 \\ U_{81}^1 \\ U_{91}^1 \\ U_{101}^1 \end{bmatrix} = \begin{bmatrix} -0.10400 \\ -0.0400 \\ -0.0160 \\ -0.00536 \\ -0.00200 \\ -0.000728 \\ -0.000272 \\ -0.000096 \\ -0.000036 \\ -0.0000136 \end{bmatrix} \quad (22)$$

When the parameter β is changed to 3.7µΩ, 7.2 µΩ and 9.3 µΩ and with use of MATLAP, we get the solutions of matrix –vector Equation (22) and tabulate the results in table 1 below

Table 1: (2+1) –dimension SGE Numerical Solution u(x,y) with varying surface damping parameter β

L	β =1.1 µΩ	β =3.7µΩ	β =7.2 µΩ	β =9.3 µΩ
1	5.40116×10^{-3}	8.558000×10^{-3}	1.231601×10^{-2}	1.423109×10^{-2}
2	2.514496×10^{-2}	4.636755×10^{-2}	5.018436×10^{-2}	8.208707×10^{-2}
3	1.030042×10^{-2}	2.101222×10^{-2}	1.950365×10^{-2}	3.161732×10^{-2}
4	3.61024×10^{-3}	3.802803×10^{-3}	6.605925×10^{-3}	6.930631×10^{-3}
5	1.327587×10^{-3}	1.502139×10^{-3}	2.450177×10^{-3}	2.588760×10^{-3}
6	4.845999×10^{-4}	5.060644×10^{-4}	8.930032×10^{-4}	6.055359×10^{-4}
7	1.7991210^{-4}	2.423222×10^{-4}	3.330352×10^{-4}	4.619052×10^{-4}
8	6.430021×10^{-5}	6.89082×10^{-4}	1.179672×10^{-4}	1.305606×10^{-4}
9	2.38406×10^{-5}	2.52046×10^{-5}	4.379454×10^{-5}	4.533537×10^{-5}
10	2.207248×10^{-5}	1.605775×10^{-5}	1.705775×10^{-5}	1.862447×10^{-5}

The results in table 1 above are represented graphically as shown in figures 1 and 2 below

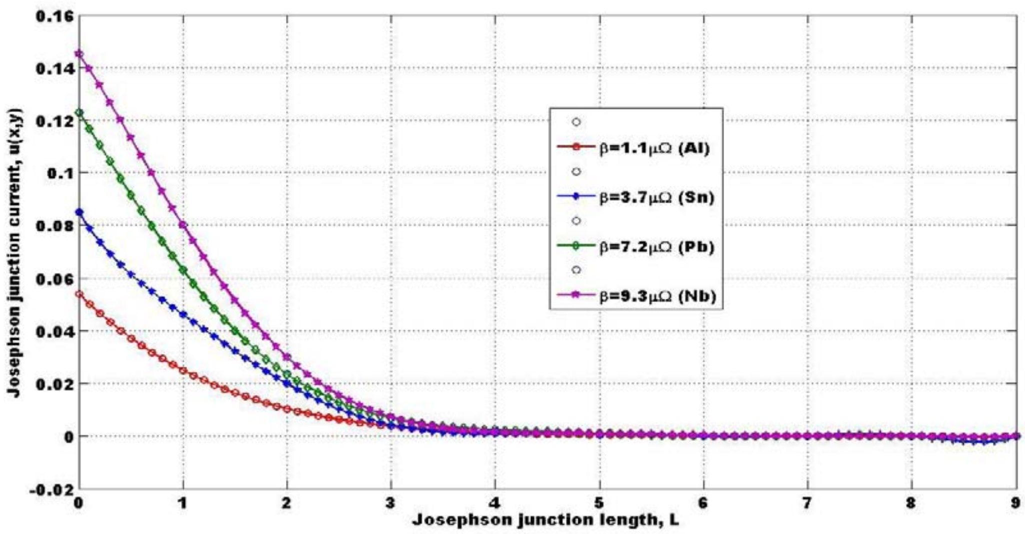

Figure 1: Graph of Josephson junction current against Josephson junction length with varying β=1.1µΩ, β =3.7 µΩ, β=7.2 µΩ and β=9.3 µΩ

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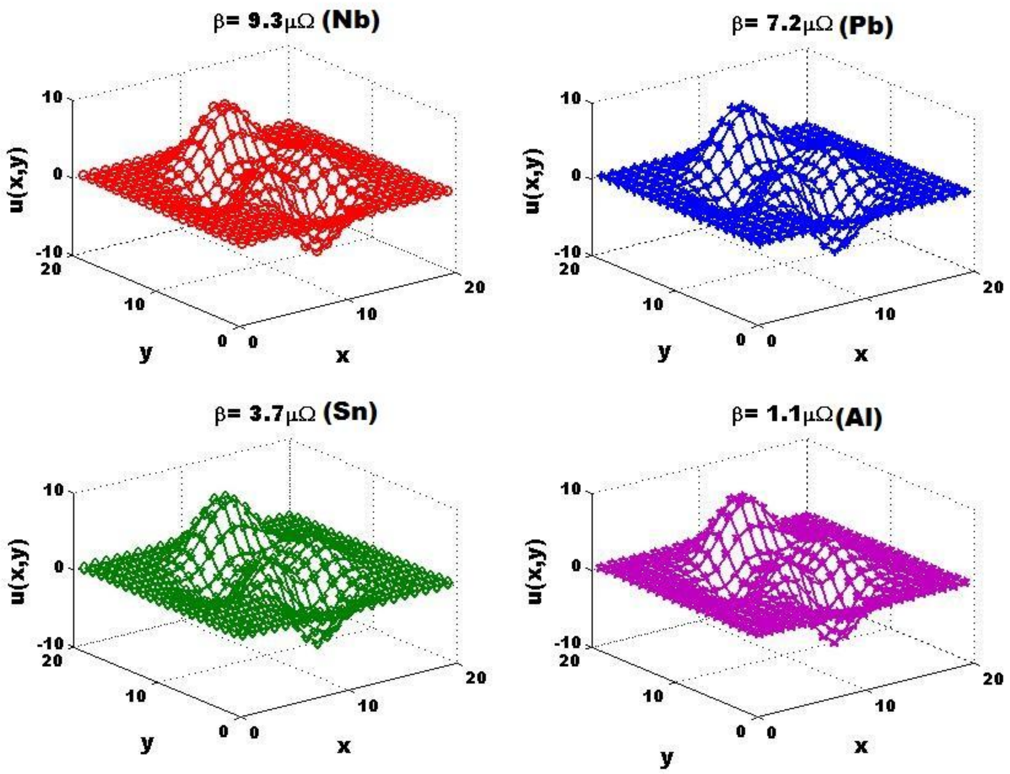

Figure 2: 3D Surface plots for Numerical Solution u(x, y) corresponding to β=1.1μΩ, β =3.7 μΩ, β=7.2 μΩ and β=9.3 μΩ

4.2 Discussion

From the tables 1 and the figures 1 and 2, the computational results obtained indicate that as the length L, of the Josephson junction increases, the Josephson junction current decrease till a point where it starts dropping to zero. The results also indicate that when the Josephson junction current increases, the surface damping parameter β (representing resistance of superconductor electrodes of Josephson junction) increases also. The increase in the Josephson junction current makes the temperatures on the junction superconductor electrodes to be very high. Superconductivity is destroyed by high currents (critical current J_c). Superconductivity is also destroyed by magnetic fields created by these high currents. This Critical magnetic field depends typically on temperature. Then the fluxon will move towards the region of the junction with smaller self energy, i.e., from the cold to the hot end of the junction. Certain energy dissipation is associated with such motion. Moreover, the energy is transmitted by fluxons from the cold to the hot end of the junction.

5. Conclusion

It can be concluded that an increase in Josephson junction length leads to a decrease in the magnitude of Josephson junction current flowing through the long Josephson junctions. It is also observed that an increase surface damping parameter of Josephson junction superconductor electrodes leads to an increase in the Josephson junction current.

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Author Profile

Mr. Abel Kurura Tinega was born at Tombe Market, Nyamira County, Kenya. He holds a Bachelor of Education Science degree with specialization in Mathematics and Physics, from Kenyatta University (KU), Nairobi, Kenya, and a Master of Science degree in Applied Mathematics from Masinde Muliro University of Science and Technology (MMUST), Kakamega, Kenya and is currently pursuing a PhD in applied Mathematics at Jaramogi Oginga Odinga University of Science and Technology (JOOUST), Nairobi, Kenya. ***Affiliation:*** Jaramogi Oginga Odinga University of Science and Technology (MMUST), BONDO, KENYA. He is currently a full time Mathematics lecturer at Kisii National Polytechnic, Kisii, Kenya and part-time Mathematics lecture at Kisii University and Mt Kenya University. He has participated in many conferences including Kenya Association of Mathematics Education (KAME) and has been a trainer in Strengthening of Mathematics and science For Secondary Education in Kenya (SMASSE). He has much interest in the study of numerical methods and their respective applications in modeling physical phenomenon in Mathematics, sciences and engineering.

Prof. Oduor Okoya Michael Edmund holds a Bachelor of Science degree from University of Nairobi (1976), Kenya, Master of Science degree in Applied Mathematics from Reading University, U.K (1986), and a PhD in applied mathematics from Maseno University (2005), Kenya. He also holds a Diploma from Leicester Polytechnic, U.K (1985), and Certificate from London University (1990), UK in Distance Education Teaching Course. ***Affiliation****:* Jaramogi Oginga Odinga University of Science and Technology, (JOOUST), Bondo, Kenya. He is currently the chairman Department of Pure and Applied mathematics – JOOUST. He holds the rank of Associate Professor in Applied Mathematics in Pure and Applied Mathematics department (September 2014 to date). He has published 14 papers in respected journals and supervised 18 postgraduate students. Teaching experience is as follows: Assistant Lecturer (University of Nairobi-Kenya), Lecturer (Maseno University -Kenya), Senior Lecturer, Maseno University, Kenya, Senior lecturer (September 2011 todate) at Jaramogi Oginga Odinga University of Science and Technology ,Kenya. He has academic research interests in Numerical Analysis, Lie Groups and Differential Equations, and Mathematical modeling.

Prof. Naftali Omolo Ongati holds a Bachelor of Education (Science) from University of Nairobi, Kenya (1973-1976), Diploma in Mathematics curriculum (Leeds University, UK), Master of Science degree in Applied Mathematics from Odessa state University, Ukraine (1977-1979), and a PhD in applied Mathematics from University of Pretoria (1993-1997), South Africa. ***Affiliation****:* Jaramogi Oginga Odinga University of Science and Technology, (JOOUST), Bondo, Kenya. He is currently the Dean School of Mathematics and Actuarial Sciences– JOOUST. He holds the rank of Full Professor of mathematics department. He has published over 42 papers in respected journals,published 6 books and supervised over 40 postgraduate students and 18 Phds. Teaching experience is as follows:University of Pretoria South Africa, Maseno University-Kenya, Kenyatta University-Kenya, Egerton UniversityKenya, Kenya Science Teachers college-Kenya and Jaramogi Oginga Odinga University of Science and Technology-Kenya. He has academic research interests in Numerical Analysis, Lie Groups and Differential Equations, and Mathematical modeling. E-Mail: nomoloongati@gmail.com

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