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The Norm Attainability Of Some Elementary Operators*

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Abstract

In this note, we present new results on necessary and sufficient conditions for norm-attainability for Hilbert space operators. Moreover, norm-attainability conditions for elementary operators and generalized derivations are also established. The main results shows that if we let $S \in B(H)$, $\beta \in W_0(S)$ and $\alpha > 0$, then there exists an operator $Z \in B(H)$ such that $\|S\| = \|Z\|$, with $\|S - Z\| < \alpha$. Furthermore, there exists a vector $\eta \in H$, $\|\eta\| = 1$ such that $\|Z\eta\| = \|Z\|$ with $\langle Z\eta, \eta\rangle = \beta$.

1 Introduction

Let H be an infinite dimensional complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. Let both S and T belong $B(H)$ and consider $\mathcal{T} : B(H) \to B(H)$. $\mathcal{T}$ is called an elementary operator if it is represented as $\mathcal{T}(X) = \sum_{i=1}^{n} S_i X T_i,\ \forall\ X \in B(H)$, where $S_i,\ T_i$ are fixed in $B(H)$ or $\mathcal{M}(B(H))$ where $\mathcal{M}(B(H))$ is the multiplier algebra of $B(H)$. For $S, T \in B(H)$ we have the following examples of elementary operators: (i) the left multiplication operator $L_S(X) = SX$, (ii) the right multiplication operator $R_T(X) = XT$, (iii) the inner derivation $\delta_S = SX - XS$, (iv) the generalized derivation $\delta_{S,T} = SX - XT$, (v) the basic elementary operator $M_{S,\,T}(X) = SXT$, (vi) the Jordan elementary operator and $\mathcal{U}_{S,\,T}(X) = SXT + TXS,\ \forall\ X \in B(H)$. Stampfli [3] characterized the norm of the generalized derivation by obtaining that $\|\delta_{S,T}\| = \inf_{\beta\in\mathbb{C}}\{\|S-\beta\| + \|T-\beta\|\}$, where $\mathbb{C}$ is the complex plane. Other studies on derivations and elementary operators have also been carried out with nice results obtained, see [1] and [2] and the references there in.

DEFINITION 1.1. An operator $S \in B(H)$ is said to be norm-attainable if there exists a unit vector $x_0 \in H$ such that $\|Sx_0\| = \|S\|$.

DEFINITION 1.2. For an operator $S \in B(H)$ we define a numerical range by $W(S) = \{\langle Sx, x\rangle :\ x \in H,\ \|x\| = 1\}$ and the maximal numerical range by $W_0(S) = \{\beta \in \mathbb{C} :\ \langle Sx_n, x_n\rangle \to \beta, \text{ where } \|x_n\| = 1,\ \|Sx_n\| \to \|S\|\}$.

The main result in the next section is Theorem 2.1 which deals with the necessary and sufficient conditions for a Hilbert space operator to be norm-attainable.

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2 Main Results

The following is the main theorem.

THEOREM 2.1. Let $S \in B(H),\ \beta \in W_0(S)$ and $\alpha > 0$. There exists an operator $Z \in B(H)$ such that $\|S\| = \|Z\|$, with $\|S - Z\| < \alpha$. Furthermore, there exists a vector $\eta \in H,\ \|\eta\| = 1$ such that $\|Z\eta\| = \|Z\|$ with $\langle Z\eta, \eta\rangle = \beta$.

PROOF. Without loss of generality, we may assume that $\|S\| = 1$ and also that $0 < \alpha < 2$. Let $x_n \in H$ $(n = 1, 2, ...)$ be such that $\|x_n\| = 1$, $\|Sx_n\| \to 1$ and also $\lim_{n\to\infty}\langle Sx_n, x_n\rangle = \beta$. Let $S = GL$ be the polar decomposition of S. Here G is a partial isometry and we write $L = \int_0^1 \beta dE_\beta$, the spectral decomposition of $L = (S^*S)^{\frac{1}{2}}$. Since L is a positive operator with norm 1, for any $x \in H$ we have that $\|Lx_n\| \to 1$ as n tends to ∞ and $\lim_{n\to\infty}\langle Sx_n, x_n\rangle = \lim_{n\to\infty}\langle GLx_n, x_n\rangle = \lim_{n\to\infty}\langle Lx_n, G^*x_n\rangle$. Now for $H = \overline{Ran(L)} \oplus KerL$, we can choose x_n such that $x_n \in \overline{Ran(L)}$ for large n. Indeed, let

$$x_n = x_n^{(1)} \oplus x_n^{(2)},\ n = 1, 2, ...$$

Then we have that

$$Lx_n = Lx_n^{(1)} \oplus Lx_n^{(2)} = Lx_n^{(1)}$$

and that

$$\lim_{n\to\infty}\|x_n^{(1)}\| = 1,\ \lim_{n\to\infty}\|x_n^{(2)}\| = 0$$

since

$$\lim_{n\to\infty}\|Lx_n\| = 1.$$

Replacing x_n with $\frac{x_n^{(1)}}{\|x_n^{(1)}\|}$, we obtain

$$\lim_{n\to\infty}\left\|L\frac{1}{\|x_n^{(1)}\|}x_n^{(1)}\right\| = \lim_{n\to\infty}\left\|S\frac{1}{\|x_n^{(1)}\|}x_n^{(1)}\right\| = 1,$$

$$\lim_{n\to\infty}\left\langle S\frac{1}{\|x_n^{(1)}\|}x_n^{(1)}, \frac{1}{\|x_n^{(1)}\|}x_n^{(1)}\right\rangle = \beta.$$

Now assume that $x_n \in \overline{RanL}$. Since G is a partial isometry from $\overline{RanL}$ onto $\overline{RanS}$, we have that $\|Gx_n\| = 1$ and $\lim_{n\to\infty}\langle Lx_n, G^*x_n\rangle = \beta$. Since L is a positive operator, $\|L\| = 1$ and for any $x \in H$,

$$\langle Lx, x\rangle \leq \langle x, x\rangle = \|x\|^2.$$

Replacing x with $L^{\frac{1}{2}}x$, we get that $\langle L^2x, x\rangle \leq \langle Lx, x\rangle$, where $L^{\frac{1}{2}}$ is the positive square root of L. Therefore we have that $\|Lx\|^2 = \langle Lx, Lx\rangle \leq \langle Lx, x\rangle$. It is obvious that $\lim_{n\to\infty}\|Lx_n\| = 1$ and that

$$\|Lx_n\|^2 \leq \langle Lx_n, x_n\rangle \leq \|Lx_n\|^2 = 1.$$

Hence, $\lim_{n\to\infty}\langle Lx_n, x_n\rangle = 1 = \|L\|$. Moreover, Since $I - L \geq 0$, we have $\lim_{n\to\infty}\langle (I - L)x_n, x_n\rangle = 0$. thus $\lim_{n\to\infty}\|(I-L)^{\frac{1}{2}}x_n\| = 0$. Indeed,

$$\lim_{n\to\infty}\|(I-L)x_n\| \leq \lim_{n\to\infty}\|(I-L)^{\frac{1}{2}}\| \cdot \|(I-L)^{\frac{1}{2}}x_n\| = 0.$$

For $\alpha > 0$, let $\gamma = [0, 1 - \frac{\alpha}{2}]$ and let $\rho = (1 - \frac{\alpha}{2}, 1]$. We have

$$\begin{aligned} L &= \int_\gamma \mu dE_\mu + \int_\rho \mu dE_\mu \\ &= LE(\gamma) \oplus LE(\rho). \end{aligned}$$

Next we show that $\lim_{n\to\infty}\|E(\gamma)x_n\| = 0$. If there exists a subsequence $x_{n_i}, (i = 1, 2, ...,)$ such that $\|E(\gamma)x_{n_i}\| \geq \epsilon > 0,\ (i = 1, 2, ...,)$, then since $\lim_{i\to\infty}\|x_{n_i} - Lx_{n_i}\| = 0$, it follows that

$$\begin{aligned} \lim_{i\to\infty}\|x_{n_i} - Lx_{n_i}\|^2 &= \lim_{i\to\infty}(\|E(\gamma)x_{n_i} - LE(\gamma)x_{n_i}\|^2 + \|E(\rho)x_{n_i} - LE(\rho)x_{n_i}\|^2) \\ &= 0. \end{aligned}$$

Hence we have that $\lim_{i\to\infty}\|E(\gamma)x_{n_i} - LE(\gamma)x_{n_i}\|^2 = 0$. Now it is clear that

$$\begin{aligned} \|E(\gamma)x_{n_i} - LE(\gamma)x_{n_i}\| &\geq \|E(\gamma)x_{n_i}\| - \|LE(\gamma)\|.\|E(\gamma)x_{n_i}\| \\ &\geq (I - \|LE(\gamma)\|)\|E(\gamma)x_{n_i}\| \\ &\geq \frac{\alpha}{2}\epsilon \\ &> 0. \end{aligned}$$

This is a contradiction. Therefore,

$$\lim_{n\to\infty}\|E(\gamma)x_n\| = 0.$$

Since

$$\lim_{n\to\infty}\langle Lx_n, x_n\rangle = 1,$$

we have that

$$\lim_{n\to\infty}\langle LE(\rho)x_n, E(\rho)x_n\rangle = 1$$

and

$$\lim_{n\to\infty}\langle E(\rho)x_n, G^*E(\rho)x_n\rangle = \beta.$$

It is easy to see that

$$\lim_{n\to\infty}\|E(\rho)x_n\| = 1,\ \lim_{n\to\infty}\left\langle L\frac{E(\rho)x_n}{\|E(\rho)x_n\|}, \frac{E(\rho)x_n}{\|E(\rho)x_n\|}\right\rangle = 1$$

and

$$\lim_{n\to\infty}\left\langle L\frac{E(\rho)x_n}{\|E(\rho)x_n\|}, G^*\frac{E(\rho)x_n}{\|E(\rho)x_n\|}\right\rangle = \beta$$

Replacing x with $\frac{E(\rho)x_n}{\|E(\rho)x_n\|}$, we can assume that $x_n \in E(\rho)H$ for each n and $\|x_n\| = 1$. Let

$$\begin{aligned} J &= \int_\gamma \mu dE_\mu + \int_\rho \mu dE_\mu \\ &= J_1 \oplus E(\rho). \end{aligned}$$

Then it is evident that

$$\|J\| = \|S\| = \|L\| = 1, Jx_n = x_n,$$

and $\|J - L\| \leq \frac{\alpha}{2}$. If we can find a contraction V such that $V - G \leq \frac{\alpha}{2}$ and $\|Vx_n\| = 1$ and $\langle Vx_n, x_n\rangle = \beta$, for a large n then letting $Z = VJ$, we have that $\|Zx_n\| = \|VJx_n\| = 1$, and that

$$\langle Zx_n, x_n\rangle = \langle VJx_n, x_n\rangle = \langle Vx_n, x_n\rangle = \beta,$$

$$\begin{aligned} \|S - Z\| &= \|GL - VJ\| \\ &\leq \|GL - GJ\| + \|GJ - VJ\| \\ &\leq \|G\| \cdot \|L - J\| + \|G - V\| \cdot \|J\| \\ &\leq \frac{\alpha}{2} + \frac{\alpha}{2} \\ &= \alpha. \end{aligned}$$

To finish the proof, we now construct the desired contraction V. Clearly,

$$\lim_{n\to\infty} \langle x_n, G^* x_n\rangle = \beta,$$

because $\lim_{n\to\infty}\langle Lx_n, G^*x_n\rangle = \beta$ and

$$\lim_{n\to\infty} \|x_n - Lx_n\| = 0.$$

Let $Gx_n = \phi_n x_n + \varphi_n y_n,\ (y_n \perp x_n,\ \|y_n\| = 1)$ then $\lim_{n\to\infty} \phi_n = \beta$, because

$$\lim_{n\to\infty} \langle Gx_n, x_n\rangle = \lim_{n\to\infty} \langle x_n, G^* x_n\rangle = \beta$$

but $\|Gx_n\|^2 = |\phi_n|^2 + |\varphi_n|^2 = 1$, so we have that $\lim_{n\to\infty} |\varphi_n| = \sqrt{1 - |\beta|^2}$. Now for without loss of generality, there exists an integer M such that $|\phi_M - \beta| < \frac{\alpha}{8}$. Choose φ_M^0 such that $|\varphi_M^0| = \sqrt{1 - |\beta|^2},\ |\varphi_M - \varphi_M^0| < \frac{\alpha}{8}$. We have that

$$\begin{aligned} Gx_M &= \phi_M x_M + \varphi_M y_M - \beta x_M + \beta x_M - \varphi_M^0 y_M + \varphi_M^0 y_M \\ &= (\phi - \beta)x_M + (\varphi_M - \varphi_M^0)y_M + \beta x_M + \varphi_M^0 y_M. \end{aligned}$$

Let $q_M = \beta x_M + \varphi_M^0 y_M$,

$$Gx_M = (\phi - \beta)x_M + (\varphi_M - \varphi_M^0)y_M + q_M.$$

Suppose that $y \perp x_M$, then

$$\begin{aligned}\langle Gx_M, Gy\rangle &= (\phi-\beta)\langle x_M, Gy\rangle + (\varphi_M - \varphi_M^0)\langle y_M, Gy\rangle + \langle q_M, Gy\rangle \\ &= 0,\end{aligned}$$

because G^*G is a projection from H to $RanL$. It follows that

$$|\langle q_M, Gy\rangle| \leq |\phi_M - \beta| \cdot \|y\| + |\varphi_M - \varphi_M^0| \cdot \|y\| \leq \frac{\alpha}{4}\|y\|.$$

If we suppose that $Gy = \phi q_M + y^0,\ (y^0 \perp q_M,)$ then y^0 is uniquely determined by y. Hence we can define V as follows

$$V : x_M \to q_M,\ y \to y^0,\ \phi x_M + \varphi_M y \to \phi q_M + \varphi_M y^0,$$

with both ϕ, φ being complex numbers. V is a linear operator. We prove that V is a contraction. Now,

$$\|Vx_M\|^2 = \|q_M\|^2 = |\beta|^2 = |\varphi_M^0|^2 = 1,$$
$$\|Vy\|^2 = \|Gy\|^2 - |\phi y|^2 \leq \|Gy\|^2 \leq \|y\|^2.$$

It follows that

$$\|V\phi\|^2 = \|\phi\|^2\|Vx_M\|^2 + |\varphi|^2\|Vy\|^2 \leq |\phi|^2 + |\varphi|^2 = 1,$$

for each $x \in H$ satisfying that $x = \phi x_M + \varphi_M y,\ \|x\| = 1,\ x_M \perp y$, which is equivalent to that V is a contraction. From the definition of V, we can show that

$$\|Gx_M - Vx_M\|^2 = |\phi - \beta|^2 + |\varphi_M - \varphi_M^0|^2 \leq \frac{2\alpha^2}{16} = \frac{1}{8}\alpha^2.$$

If $y \perp x_M,\ \|y\| \leq 1$ then obtain

$$\|Gy - Vy\| = |\phi|\|Vx_M\| = |\langle Gy, Vx_M\rangle| = |\langle q_M, Gy\rangle| < \frac{\alpha}{4}.$$

Hence for any $x \in H,\ x = \phi x_M + \varphi_M y,\ \|x\| = 1,$

$$\begin{aligned}\|Gx - Vx\|^2 &= \|\phi(G-V)x_M + \varphi(G-V)y\|^2 \\ &= |\phi|^2\|(G-V)x_M\|^2 + |\varphi|^2\|(G-V)y\|^2 \\ &< |\phi|^2\frac{\alpha^2}{16} + |\varphi|^2\frac{\alpha^2}{16} \\ &< \frac{\alpha^2}{8},\end{aligned}$$

which implies that

$$\|(G-V)x\| < \frac{\alpha}{2},\ \ \|x\| = 1,$$

and hence $\|(G-V)\| < \frac{\alpha}{2}$. Let $Z = VJ$. Then Z is what we want and this completes the proof.

3 Norm Attainability for Elementary Operators

In this section we consider norm-attainability for inner derivation, generalized derivations and general elementary operators. We utilize the technique and the conditions in Theorem 2.1 in our work in this section. We start with the inner derivation in the lemma below.

LEMMA 3.1. Let $S \in B(H)$. δ_S is norm-attainable if there exists a vector $\zeta \in H$ such that $\|\zeta\| = 1$, $\|S\zeta\| = \|S\|$, $\langle S\zeta, \zeta\rangle = 0$.

PROOF. For any x satisfying that $x \perp \{\zeta, S\zeta\}$, define X as follows

$$X : \zeta \to \zeta, \ \ S\zeta \to -S\zeta, \ \ x \to 0.$$

Since X is a bounded operator on H and $\|X\zeta\| = \|X\| = 1$,

$$\|SX\zeta - XS\zeta\| = \|S\zeta - (-S\zeta)\| = 2\|S\zeta\| = 2\|S\|.$$

It follows that $\|\delta_S\| = 2\|S\|$ via the result in [3, Theorem 1], because $\langle S\zeta, \zeta\rangle = 0 \in W_0(S)$. Hence we have that $\|SX - XS\| = 2\|S\| = \|\delta_S\|$. Therefore, δ_S is norm-attainable.

THEOREM 3.2. Let $S, T \in B(H)$ If there exists vectors $\zeta, \eta \in H$ such that $\|\zeta\| = \|\eta\| = 1$, $\|S\zeta\| = \|S\|$, $\|T\eta\| = \|T\|$ and $\frac{1}{\|S\|}\langle S\zeta, \zeta\rangle = -\frac{1}{\|T\|}\langle T\eta, \eta\rangle$, then $\delta_{S,T}$ is norm-attainable.

PROOF. By linear dependence of vectors, if η and $T\eta$ are linearly dependent, i.e.,$T\eta = \phi\|T\|\eta$, then it is true that $|\phi| = 1$ and $|\langle T\eta, \eta\rangle| = \|T\|$. It follows that $|\langle S\zeta, \zeta\rangle| = \|S\|$ which implies that $S\zeta = \varphi\|S\|\zeta$ and $|\varphi| = 1$. Hence $\left\langle \frac{S\zeta}{\|S\|}, \zeta \right\rangle = \varphi = -\left\langle \frac{T\eta}{\|T\|}, \eta \right\rangle = -\phi$. Defining X as $X : \eta \to \zeta, \ \ \{\eta\}^\perp \to 0$, we have $\|X\| = 1$ and $(SX - XT)\eta = \varphi(\|S\| + \|T\|)\zeta$, which implies that $\|SX - XT\| = \|(SX - XT)\eta\| = \|S\| + \|T\|$. By [3], it follows that

$$\|SX - XT\| = \|S\| + \|T\| = \|\delta_{S,T}\|.$$

That is $\delta_{S,T}$ is norm-attainable. If η and $T\eta$ are linearly independent, then $\left|\left\langle \frac{T\eta}{\|T\|}, \eta \right\rangle\right| < 1$, which implies that $\left|\left\langle \frac{S\zeta}{\|S\|}, \zeta \right\rangle\right| < 1$. Hence ζ and $S\zeta$ are also linearly independent. Let us redefine X as follows: $X : \eta \to \zeta$, $\frac{T\eta}{\|T\|} \to -\frac{S\zeta}{\|S\|}$, $x \to 0$, where $x \in \{\eta, T\eta\}^\perp$. We show that X is a partial isometry. Let

$$\frac{T\eta}{\|T\|} = \left\langle \frac{T\eta}{\|T\|}, \eta \right\rangle \eta + \tau h, \ \ \|h\| = 1, \ h \perp \eta.$$

Since η and $T\eta$ are linearly independent, $\tau \neq 0$. So we have that

$$X\frac{T\eta}{\|T\|} = \left\langle \frac{T\eta}{\|T\|}, \eta \right\rangle X\eta + \tau Xh = -\left\langle \frac{S\zeta}{\|S\|}, \zeta \right\rangle \zeta + \tau Xh,$$

which implies that $\left\langle X\frac{T\eta}{\|T\|}, \zeta\right\rangle = -\left\langle \frac{S\zeta}{\|S\|}, \zeta\right\rangle + \tau\langle Xh, \zeta\rangle = -\left\langle \frac{S\zeta}{\|S\|}, \zeta\right\rangle$. It follows then that $\langle Xh, \zeta\rangle = 0$ i.e., $Xh \perp \zeta (\zeta = X\eta)$. Hence we have that

$$\left\|\left\langle \frac{S\zeta}{\|S\|}, \zeta\right\rangle \zeta\right\|^2 + \|\tau Xh\|^2 = \left\|X\frac{T\eta}{\|T\|}\right\|^2 = \left|\left\langle \frac{T\eta}{\|T\|}, \eta\right\rangle\right|^2 + |\tau|^2 = 1,$$

which implies that $\|Xh\| = 1$. Now it is evident that X a partial isometry and $\|(SX - XT)\zeta\| = \|SX - XT\| = \|S\| + \|T\|$, which is equivalent to $\|\delta_{S,T}(X)\| = \|S\| + \|T\|$. By Lemma 3.1 and [3], $\|\delta_{S,T}\| = \|S\| + \|T\|$. Hence $\delta_{S,T}$ is norm-attainable.

THEOREM 3.3. Let $S, T \in B(H)$ If both S and T are norm-attainable then the basic elementary operator $M_{S,\,T}$ is also norm-attainable.

PROOF. For any pair (S, T) it is known that $\|M_{S,T}\| = \|S\|\|T\|$. We can assume that $\|S\| = \|T\| = 1$. If both S and T are norm-attainable, then there exists unit vectors ζ and η with $\|S\zeta\| = \|T\eta\| = 1$. We can therefore define an operator X by $X = \langle \cdot, T\eta\rangle\zeta$. Clearly, $\|X\| = 1$. Therefore, we have $\|SXT\| \geq \|SXT\eta\| = \|\|T\eta\|^2 S\zeta\| = 1$. Hence, $\|M_{S,T}(X)\| = \|SXT\| = 1$, that is $M_{S,T}$ is also norm-attainable.

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References

[1] N. B. Okelo, J. O. Agure and D. O. Ambogo, Norms of elementary operators and characterization of norm-attainable operators, Int. J. Math. Anal., 24(2010), 687–693.

[2] A. Seddik, On the numerical range and norm of elementary operators, Linear Multilinear Algebra, 52(2004), 293–302.

[3] J. G. Stampfli, The norm of a derivation, Pacific J. Math., 33(3)(1970), 737–747.