JARAMOGI OGINGA ODINGA UNIVERSITY OF SCIENCE AND TECHNOLOGY

SCHOOL OF MATHEMATICS AND ACTUARIAL SCIENCE

UNIVERSITY EXAMINATION FOR DEGREE OF BACHELOR OF EDUCATION AND ACTUARIAL SCIENCE

RESIT 2

MAIN CAMPUS

COURSE CODE: SMA 402

COURSE TITLE: MEASURE THEORY

EXAM VENUE: **STREAM: BED AND ACT SCIENCE**

DATE: **EXAM SESSION:**

TIME: 2.00 HOURS

Instructions:

1. **Answer question one (compulsory) and any other two questions.**
2. **Candidates are advised not to write on the question paper.**
3. **Candidates must hand in their answer booklets to the invigilator while in the examination room.**

QUESTION ONE (30 MARKS)

a) i) Define the Lebesgue outer measure of the set $E \subseteq \mathbb{R}$ (2mks)

ii) Prove that the Lebesgue outer measure of an empty set is zero (5mks)

b) Calculate the outer measure of the following sets (6mks)

i) $\bigcup_{k=1}^{\infty} \left\{x: 0 < x \leq \frac{1}{3^k}\right\}$

ii) $\bigcup_{k=1}^{\infty} \left\{x: \frac{1}{k+1} < x \leq \frac{1}{k}\right\}$

c) i) Prove that if E is a countable set, then $m^*(E) = 0$ (5mks)

ii) Show that every interval is not countable (2mks)

d) i) Describe three forms of measure (3mks)

ii) Define a property of almost everywhere in a set (2mks)

e) Prove that if $m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$ for any set B (5mks)

QUESTION TWO (20 MARKS)

a) Suppose f and g are measurable function and λ is a scalar, prove measurability of the following:

i) λf (4mks)

ii) $f + g$ (5mks)

iii) fg (3mks)

b) i) State Caratheodory's measurability criteria (2mks)

ii) Describe the differences and similarities between the two integrals (6mks)

QUESTION THREE (20 MARKS)

a) Show that if function $f\ (x)$ is measurable on a measurable set E, then $|f(x)|$ is also measurable (5mks)

b) i) Give an example of a set with outer measure zero but not countable. (1mks)

ii) Construct Cantor set (9mks)

c) Prove that the Lebesgue outer measure is translation invariant (5mks)

QUESTION FOUR (20 MARKS)

a) i) State two properties of measurable sets (2mks)

ii) Show that if $m^*(E) = 0$, then E is measurable (5mks)

b) Show that if f is an extended real valued function defined on a measurable set, then the following statements are equivalent

i) f is a measurable function (2mks)

ii) $\forall \alpha \in \mathbb{R}\, ; \{x: f(x) \geq \alpha\}$ is measurable (2mks)

iii) $\forall \alpha \in \mathbb{R}\, ; \{x: f(x) < \alpha\}$ is measurable (2mks)

iv) $\forall \alpha \in \mathbb{R}\, ; \{x: f(x) \leq \alpha\}$ is measurable (2mks)

c) Prove that if $f(x)$ and $g(x)$ are equivalent functions a set E and $f(x)$ is measurable, then $g(x)$ is also measurable (5mks)

QUESTION FIVE (20 MARKS)

a) State and prove Fatous Lemma (10mks)

b) State and prove Monotone convergence theorem (10mks)