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RESEARCH ARTICLE

APPLICATION OF THREE PARAMETER GENERALIZED BETA I DISTRIBUTION IN BINOMIAL MIXTURE

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ABSTRACT

Two or more individual distributions can be mixed together to form a new distribution. According to Feller, this can be done using weights that sum up to unit. Also by considering a parameter which is a random variable taking another distribution then a new distribution ca be formed. The nature of the mixing distribution has effect on the new distribution formed. If the mixing distribution is continuous random variable then the new mixture formed is also continuous. Skellam pioneered this study when he constructed binomial mixture with varying parameter also taking beta distribution. McDonald generalized beta distribution to a three parameter. This paper focusses on binomial mixture with a three parameter generalized beta distribution considered as the mixing distribution. The three parameters considered are McDonald and Libby-Novick three parameter beta distributions. Two methods of construction of binomial mixture are discussed and proved to obtain identical results. The distribution derived is proved to be a probability density function. Moments of the mixture are obtained. The binomial mixture obtained is useful in addressing the challenges of over-dispersion which is common with probability distributions having binomial outcome.

Key words: Binomial mixtures;three parameter generalized distributions; mixing distributions.

1.0 Introduction

A mixed distribution can have a physical interpretation whereby one distribution can be mixed with another one to form a new distribution. According to Feller[3], let $f(i)$ and $g(i)$ be two probability distribution. If $\propto > 0, \beta > 0$ and $\alpha + \beta = 1$, then $\alpha fi + \beta gi$ is also a probability distribution. This can be generalized to more individual distributions. For instance, let $f_1(x), f_2(x)...f_k(x)$ be k different component probability distribution functions with cumulative distribution functions $F_1(x), F_2(x)...F_k(x)$ having weights $w_1, w_2....w_k$ respectively, where $w_j > 0$ and $\sum_{j=1}^{k} w_j = 1$. Then the joint probability density function of the new mixture becomes $\Pr ob(X = x) = \sum_{j=1}^{k} w_j p_j(x)$ with cumulative density function $F(x) = \sum_{j=1}^{k} w_j f_j(x)$. A mixed distribution can as well be formed when the cumulative distribution function of a random variable depends on a varying parameter p such that the cumulative density function is expressed as $F(x \mid p_1...p_k)$.The new distribution then has the expectation $E[F(x \mid p_1...p_k)]$ which has k varying parameters. If the varying parameter has a discrete distribution then the mixture formed is discrete. However if the varying has a continuous distribution then the resulting mixture is continuous. A binomial mixture in which beta distribution is used as the mixing distribution called beta-binomial distribution. This beta-binomial mixture was first proposed by Skellam[8] in 1948 when he mixed a binomial distribution with its parameter being probability of success taking beta distribution. There are many probability distributions whose outcome data have a binomial distribution. However the binomial distribution often fail to fit the empirical data due to the fact that the actual variance tend to be greater than the theoretical variance leading to over-dispersion. This makes the success probability p be a random variable within the limit(0,1). This random effect needs to be captured in the modeling of the data set to cater for the influence caused by over-dispersion. This challenge can be addressed by use of beta binomial mixture in which the parameter p is treated as a random variable taking beta distribution. This study can benefit researchers especially in the field of health where the data set collected are more often in the form binomial distribution with varying prevalence rates caused by different factors. This paper considers a generalized beta I distribution with three parameters as the mixing distribution. In particular the focus is on McDonald and Libby-Novick three parameter distributions as the mixing distributions. The main objective is to construct a binomial mixture taking a three parameter generalized beta I distributions of McDonald and Libby-Novick. The methods of construction to be applied are explicit and methods of moments. The two methods will be proved to be identical. The resulting mixtures will be proved to meet conditions of a probability density function.

2.0 Method

2.1 Formulation of mathematical problem

2.11 McDonald three parameter mixing distribution.

McDonald[6] parameterized beta classical distribution using $x = p^c$

Let

$x = p^c \quad c > 0,$ ………….[1]

where x is the classical beta variable with parameters a and b.

Then

$$\frac{dx}{dp} = cp^{c-1} \quad 0 < x < 1,$$

and

$$g(p) = f(x)\left|\frac{dx}{dp}\right|$$

$$= \frac{x^{a-1}(1-x)^{b-1} cp^{c-1}}{B(a,b)}$$

$$= \frac{cp^{c(c-1)}\left(1-p^c\right)^{b-1} p^{c-1}}{B(a,b)}$$

$$= \frac{cp^{ca} p^{-c}\left(1-p^c\right)^{b-1} p^{c-1}}{B(a,b)}$$

$$g(p) = \frac{cp^{ca-1}\left(1-p^c\right)^{b-1}}{B(a,b)}\ 0 < p < 1, \qquad a,b,c > 0, \qquad \text{..............[2]}$$

which is McDonald three parameter generalized (G3B) distribution.

The j[th] moment of McDonald G3B mixing distribution is

$$E(P^j) = \int_0^1 \frac{c}{B(a,b)} p^{j+ca-1}\left(1-p^c\right)^{b-1} dp$$

Let

$p^c = z; \qquad cp^{c-1}dp = dz$ and $p = z^{\frac{1}{c}}$

$$E(P^j) = \int_0^1 \frac{cz^{\frac{1}{c}(j+ca-1)}(1-z)^{b-1} dz}{B(a,b) \quad cp^{c-1}}$$

$$= \frac{1}{B(a,b)}\int_0^1 \frac{z^{\frac{j}{c}+a-\frac{1}{c}}(1-z)^{b-1} dz}{z^{1-\frac{1}{c}}}$$

$$= \frac{1}{B(a,b)}\int_0^1 z^{\frac{j}{c}+a-1}(1-z)^{b-1} dz$$

Therefore

$$E(P^j) = \frac{B\left(\frac{j}{c}+a,b\right)}{B(a,b)}. \qquad \text{............. [3]}$$

2.12 Libby- Novick G3B mixing distribution.

Libby and Novick [5] introduced this G3B distribution using the following transformation:

Let

$Y_0 = X_0$ and $Y_1 = \dfrac{X_1}{X_0 + X_1}$

where $X_0 \sim \Gamma(b,\beta)$ and $X_1 \sim \Gamma(a,\delta)$.

Furthermore, X_0 and X_1 are independent. Therefore $Y_0 = X_0$ and

$$Y_1(X_0 + X_1) = X_1$$

$$Y_1X_0 + Y_1X_1 = X_1$$

Y $Y_1X_0 = (1-Y_1)X_1$

$$X_1 = \frac{Y_0 Y_1}{1-Y_1}$$

The Jacobian of transformation becomes

$$J = \begin{vmatrix} \frac{dx_0}{dy_0} & \frac{dx_0}{dy_1} \\ \frac{dx_1}{dy_0} & \frac{dx_1}{dy_1} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \frac{y_1}{1-y_1} & \frac{y_0}{(1-y_1)^2} \end{vmatrix} = \frac{y_0}{(1-y_1)^2}$$

The joint pdf of Y_0 and Y_1 is given by

$$g(y_0, y_1) = f(x_0, x_1)|J| = f(x_0)f(x_1)|J|$$

$$g(y_0, y_1) = \frac{\beta^b}{\Gamma b} e^{-\beta x_0} x_0^{\,b-1} \frac{\delta^a}{\Gamma a} e^{-\delta x_1} x_1^{\,a-1} \frac{y_0}{(1-y_1)^2}$$

$$= \frac{\delta^a \beta^b}{\Gamma a \Gamma b} e^{-\beta y_0} y_0^{\,b-1} e^{\frac{-\delta y_0 y_1}{1-y_1}} \left(\frac{y_0 y_1}{1-y_1}\right)^{a-1} \frac{y_0}{(1-y)^2}$$

$$= \frac{\delta^a \beta^b}{\Gamma a \Gamma b} \frac{y_0^{\,a+b-1} y_1^{\,a-1}}{(1-y_1)^{a+1}} \exp\left\{-\left(\beta + \frac{\delta y_1}{1-y_1}\right) y_0\right\} \qquad \text{............. [4]}$$

for $0 < y_0 < \infty$ and $0 < y_1 < \infty$.

Therefore the marginal density of y_1 becomes

$$g(y_1) = \int_0^\infty g(y_0, y_1) dy_0$$

$$= \frac{\delta^a \beta^b y_1^{a-1}}{\Gamma a \Gamma b (1-y_1)^{a+1}} \int_0^\infty y_0^{a+b-1} \exp\left\{-\left(\beta + \frac{\delta y_1}{(1-y_1)}\right) y_0\right\} dy_0$$

$$= \frac{\Gamma(a+b)\delta^a \beta^b y_1^{a-1} (1-y_1)^{b-1}}{\Gamma a \Gamma b (\beta + \beta y_1 + \delta y_1)^{a+b}}$$

$$= \frac{\delta^a \beta^b y_1^{a-1} (1-y_1)^{b-1}}{B(a,b)(\beta + \beta y_1 + \delta y_1)^{a+b}}$$

$$= \frac{\delta^a \beta^b y_1^{a-1} (1-y_1)^{b-1}}{B(a,b)\beta^{a+b}\left[1-\left(1-\frac{\delta}{\beta}\right)y_1\right]^{a+b}}$$

$$= \frac{\left(\frac{\delta}{\beta}\right)^a y_1^{a-1} (1-y_1)^{b-1}}{B(a,b)\left[1-\left(1-\frac{\delta}{\beta}\right)y_1\right]^{a+b}}$$

Therefore

$$g(y_1) = \frac{c^a y_1^{a-1} (1-y_1)^{b-1}}{B(a,b)[1-(1-c)y_1]^{a+b}} \qquad \text{............. [5]}$$

where $\frac{\delta}{\beta} = c$

Letting $y_1 = p$ we get

$$g(p)=\frac{c^a y_1^{a-1}(1-y_1)^{b-1}}{B(a,b)\left[1-(1-c)y_1\right]^{a+b}} \quad \text{............. [6]}$$

$$0<p<1 \quad a,b,c>0$$

The j[th] moment is becomes

$$E(P^j)=\frac{c^a}{B(a,b)}\int_0^1 \frac{p^{j+a-1}(1-p)^{b-1}}{\left[1-(1-c)p\right]^{a+b}}dp$$

$$=\frac{c^a B(j+a,b)}{B(a,b)}\,{}_2F_1(a+b;\, j+a;\, j+a+b;1-c). \quad \text{............. [7]}$$

2.2 Construction of the Binomial mixtures.

2.21 Binomial-McDonald G3B distribution

By integration, Binomial-McDonald G3B distribution becomes

$$f(x)=\int_0^1 \binom{n}{x} p^x(1-p)^{n-x}\frac{c}{B(a,b)}p^{ca-1}\left(1-p^c\right)^{b-1}dp$$

$$=\frac{c}{B(a,b)}\binom{n}{x}\int_0^1 p^{x+ca-1}(1-p)^{n-x}\left(1-p^c\right)^{b-1}dp$$

$$=\frac{c}{B(a,b)}\binom{n}{x}\int_0^1 p^{x+ca-1}(1-p)^{n-x}\sum_{k=0}^{\infty}\binom{b-1}{k}\left(-p^c\right)^k dp$$

$$=\frac{c}{B(a,b)}\binom{n}{x}\sum_{k=0}^{\infty}(-1)^k\binom{b-1}{k}\int_0^1 p^{x+ca+ck-1}(1-p)^{n-x}dp$$

$$f(x)=\frac{c}{B(a,b)}\sum_{k=0}^{\infty}(-1)^k\binom{b-1}{k}B(x+ca+ck,n-x+1) \quad \text{.............[8]}$$

By method of moment, Binomial-McDonald G3B distribution is obtained as

$$f(x)=\binom{n}{x}\sum_{j=x}^{n}(-1)^{j-x}\binom{n-x}{j-x}E\left(P^j\right)$$

$$=\binom{n}{x}\sum_{j=x}^{n}(-1)^{j-x}\binom{n-x}{j-x}\frac{B\left(\frac{j}{c}+a,b\right)}{B(a,b)}. \quad \text{.............[9]}$$

To prove that the above methods of construction are identical we show that

$$\sum_{j=x}^{n}(-1)^{j-x}\binom{n-x}{j-x}B\left(\tfrac{j}{c}+a,b\right)=c\sum_{k=0}^{\infty}(-1)^k\binom{b-1}{k}B(x+ck+ca,n-x+1). \quad \text{.............[10]}$$

Starting from the RHS we have

$$c\sum_{k=0}^{\infty}(-1)^k\binom{b-1}{k}B(x+ck+ca,n-x+1)$$

$$=c\sum_{k=0}^{\infty}(-1)^k\binom{b-1}{k}\int_0^1 p^{x+ck+ca-1}(1-p)^{n-x}dp$$

$$=c\int_0^1 p^{x+ca-1}(1-p)^{n-x}\sum_{k=0}^{\infty}(-1)^k\binom{b-1}{k}p^{ck}dp$$

$$=c\int_0^1 p^{x+ca-1}(1-p)^{n-x}\left(1-p^c\right)^{b-1}dp$$

Let

$p^c = y, \qquad p = y^{\frac{1}{c}}$ and $dp = \frac{1}{c} y^{\frac{1}{c}-1} dy$

Therefore

$$c\int_0^1 y^{\frac{1}{c}(x+ca-1)}\left(1-y^{\frac{1}{c}}\right)^{n-x}(1-y)^{b-1}\frac{1}{c}y^{\frac{1}{c}-1}dy$$

$$=c\int_0^1 y^{\left(\frac{x}{c}+a-\frac{1}{c}\right)}\left(1-y^{\frac{1}{c}}\right)^{n-x}(1-y)^{b-1}\frac{1}{c}y^{\frac{1}{c}-1}dy$$

$$=\int_0^1 y^{\frac{x}{c}+a-1}\left(1-y^{\frac{1}{c}}\right)^{n-x}(1-y)^{b-1}dy$$

$$=\int_0^1 y^{\frac{x}{c}+a-1}(1-y)^{b-1}\sum_{k=0}^{n-x}\binom{n-x}{k}\left(-y^{\frac{1}{c}}\right)^k dy$$

$$=\sum_{k=0}^{n-x}\binom{n-x}{k}(-1)^k\int_0^1 y^{\frac{x}{c}+a-1+\frac{k}{c}}(1-y)^{b-1}dy$$

$$=\sum_{k=0}^{n-x}\binom{n-x}{k}(-1)^k B\left(\frac{x}{c}+\frac{k}{c}+a,b\right)$$

$$=\sum_{j=x}^{n}(-1)^{j-x}\binom{n-x}{j-x}B\left(\frac{j}{c}+a,b\right),$$

where $k+x=j$ and $k=j-x$ which is identical to LHS of equation [10] above.

2.22 Binomial-Libby-Novick G3B distribution.

Binomial-Libby-Novick G3B distribution is obtained by direct integration as

$$f(x)=\int_0^1 \frac{\binom{n}{x}p^x(1-p)^{n-x}c^a p^{a-1}(1-p)^{b-1}}{B(a,b)[1-(1-c)p]^{a+b}}dp$$

$$=\frac{c^a\binom{n}{x}}{B(a,b)}\int_0^1 \frac{p^{x+a-1}(1-p)^{n-x+b-1}}{[1-(1-c)p]^{a+b}}dp$$

$$=\frac{c^a\binom{n}{x}}{B(a,b)}B(x+a,n-x+b)\,{}_2F_1(a+b;x+a;n+a+b;1-c) \qquad \text{………….[11]}$$

for $x = 0,1,2 \ldots n$ and $a,b > 0$.

And by method of moments

$$f(x)=\binom{n}{x}\sum_{j=x}^{n}(-1)^{j-x}\binom{n-x}{j-x}\frac{c^a}{B(a,b)}B(j+a,b)\,{}_2F_1(a+b;\,j+a;\,j+a+b;1-c). \qquad \text{…………[12]}$$

Equating the two equations we have

$$\frac{c^a\binom{n}{x}}{B(a,b)}B(x+a,n-x+b)\,{}_2F_1(a+b;x+a;n+a+b;1-c)$$
$$=\binom{n}{x}\sum_{j=x}^{n}(-1)^{j-x}\binom{n-x}{j-x}\frac{c^a}{B(a,b)}B(j+a,b)\,{}_2F_1(a+b;j+a;j+a+b;1-c) \qquad \text{.............[13]}$$

To show the identity of the two equations in equation [13] above we have LHS

$$B(x+a,n-x+b)\,{}_2F_1(a+b;x+a;n+a+b;1-c)$$
$$=\int_0^1 p^{x+a-1}(1-p)^{n-x+b-1}[1-(1-c)p]^{-(a+b)}dp$$
$$=\int_0^1 p^{x+a-1}(1-p)^{b-1}(1-p)^{n-x}[1-(1-c)p]^{-(a+b)}dp$$
$$=\int_0^1 p^{x+a-1}(1-p)^{b-1}\sum_{k=0}^{n-x}(-1)^k\binom{n-x}{k}p^k[1-(1-c)p]^{-(a+b)}dp$$
$$=\sum_{k=0}^{n-x}(-1)^k\binom{n-x}{k}\int_0^1 p^{k+x+a-1}(1-p)^{b-1}[1-(1-c)p]^{-(a+b)}dp$$
$$=\sum_{k=0}^{n-x}(-1)^k\binom{n-x}{k}B(k+x+a,b)\,{}_2F_1(a+b;k+x+a;k+x+a+b;1-c)$$
$$=\sum_{j=x}^{n}(-1)^{j-x}\binom{nx}{j-x}B(j+a,b)\,{}_2F_1(a+b;j+a;j+a+b;1-c)$$

where $j=k+x$. Which is the RHS of the equation [13].

3.0 Results

For a continuous random variable x to be a pdf, $\int_{-\infty}^{\infty} f(x)dx=1$.

3.1 Binomial-McDonald G3B distribution.

For Binomial-McDonald G3B distribution we have

$$f(x)=\frac{c}{B(a,b)}\binom{n}{x}\sum_{k=0}^{\infty}(-1)^k\binom{b-1}{k}B(x+ca+ck,n-x+1)$$
$$=\frac{c}{B(a,b)}\binom{n}{x}\sum_{k=0}^{\infty}(-1)^k\binom{b-1}{k}\int_0^1 p^{x+ca+ck-1}(1-p)^{n-x}dp$$
$$=\frac{c}{B(a,b)}\binom{n}{x}\int_0^1(1-p^c)^{b-1}p^{x+ca-1}(1-p)^{n-x}dp$$
$$=\frac{c}{B(a,b)}\binom{n}{x}p^x(1-p)^{n-x}\int_0^1(1-p^c)^{b-1}p^{ca-1}dp$$

But

$$\sum_{x=0}^{n}\binom{n}{x}p^x(1-p)^{n-x}=1.$$

And putting $c=1$ we have

$$f(x)=\frac{1}{B(a,b)}\int_0^1 p^{a-1}(1-p)^{b-1}dp$$

$$=\frac{B(a,b)}{B(a,b)}$$

=1.

Therefore Binomial-McDonald G3B distribution is a pdf.

The expected value of Binomial-McDonald G3B distribution can be obtained using the rthfactorial moment given as

$E(X^r)=E[X(X-1)(X-2)...(X-r+1)]=n^{(r)}E(P^r)$, where $n^{(r)}=n(n-1)(n-2)...(n-r+1)$.

Thus

$$E(X)=nE(P)$$

But from equation [3] $E(P^j)=\dfrac{B\left(\frac{j}{c}+a,b\right)}{B(a,b)}$

Therefore

$$E(X)=\frac{nB\left(\frac{1}{c}+a,b\right)}{B(a,b)}$$

And the variance becomes

$$Var(X)=n^{(2)}E(P^2)+nE(P)-[nE(P)]^2$$

$$=n^2\left[\frac{B\left(\frac{2}{c}+a,b\right)}{B(a,b)}\right]-n\left[\frac{B\left(\frac{2}{c}+a,b\right)}{B(a,b)}\right]+n\left[\frac{B\left(\frac{1}{c}+a,b\right)}{B(a,b)}\right]-n^2\left[\frac{B\left(\frac{1}{c}+a,b\right)}{B(a,b)}\right]^2$$

$$=n^2\left\{\left[\frac{B\left(\frac{2}{c}+a,b\right)}{B(a,b)}\right]-\left[\frac{B\left(\frac{1}{c}+a,b\right)}{B(a,b)}\right]^2\right\}+n\left\{\left[\frac{B\left(\frac{1}{c}+a,b\right)}{B(a,b)}\right]-\left[\frac{B\left(\frac{2}{c}+a,b\right)}{B(a,b)}\right]\right\}$$

3.2 Binomial-Libby-Novick G3B distribution

To show that Binomial-Libby-Novick G3B distribution is a pdf;

$$f(x)=\frac{c^a\binom{n}{x}}{B(a,b)}\int_0^1\frac{p^{x+a-1}(1-p)^{n-x+b-1}}{[1-(1-c)p]^{a+b}}dp$$

$$=\frac{c^a}{B(a,b)}\int_0^1\binom{n}{x}p^x(1-p)^{n-x}\frac{p^{a-1}(1-p)^{b-1}}{[1-(1-c)p]^{a+b}}dp$$

$$=\sum_{x=0}^{n}\binom{n}{x}p^x(1-p)^{n-x}\frac{c^a}{B(a,b)}\int_0^1\frac{p^{a-1}(1-p)^{b-1}}{[1-(1-c)p]^{a+b}}dp$$

But

$$\sum_{x=0}^{n}\binom{n}{x}p^x(1-p)^{n-x}=1$$

Therefore

$$f(x)=\frac{c^aB(a,b)}{B(a,b)}{}_2F_1(a;a+b;1-c)$$

And putting c=1 then

$_2F_1(a;a+b;0)=1$

Therefore

$$=\frac{B(a,b)}{B(a,b)}$$

=1.

Thus Binomial –Libby-Novick G3B distribution is apdf.

Using r^{th}factorial moment the expected value of Binomial-Libby-Novick G3B distribution becomes

$$E(X)=nE(P)$$

$$=n\frac{c^a B(1+a,b)}{B(a,b)}{}_2F_1(a+b;1+a;1+a+b;1-c)$$

And its variance becomes

$$Var(X)=n^{(2)}E(P^2)+nE(P)-[nE(P)]^2$$

$$=n^2\left\{\left[\frac{c^a B(2+a,b)}{B(a,b)}{}_2F_1(a+b;2+a;2+a+b;1-c)\right]-\left[\frac{c^a B(1+a,b)}{B(a,b)}{}_2F_1(a+b;1+a;1+a+b;1-c)\right]\right\}$$

+

$$n\left\{\left[\frac{c^a B(1+a,b)}{B(a,b)}{}_2F_1(a+b;1+a;1+a+b;1-c)\right]-\left[\frac{c^a B(2+a,b)}{B(a,b)}{}_2F_1(a+b;2+a;2+a+b;1-c)\right]\right\}$$

4.0 Discussion

From the results obtained above, the mixed distribution between Binomial and beta is continuous. The binomial distribution has its parameters n and p while beta has its parameters a,b and c. And it considers p as a random variable within the limits $(0,1)$. The generalized three parameter beta distributions of Libby-Novick and McDonald have provided a good prior distribution for the binomial mixture. Both methods of construction have resulted into mixed distributions which are identical as proved above.

5.0 Conclusion

Binomial mixture discussed in this article is composed ofbinomial and three parameter distributions. The aim was to find mixing distribution beyond those that are naturally available within the limits $(0,1)$. Both McDonald and Libby-Novick three parameter distributions have proved to be suitable mixing distribution. It is therefore recommended that further research be done on how to construct mixtures with generalized mixing distribution that have more parameters.

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