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Nonzero Lie Brackets of Third Order Nonlinear Ordinary Differential Equation

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Abstract

Lie symmetry analysis of Ordinary Differential Equation can be used to obtain exact solution of the equation of the form $F(x, y, y' y'', y''') = 0$. In this paper we use Lie Symmetry analysis approach to obtain the nonzero Lie brackets of a nonlinear Ordinary Differential Equation for heat conduction. The Lie Brackets obtained forms Lie solvable algebra that can be used to reduce the equation to lower order.

Keywords: Infinitesimal generators, Prolongation, Lie symmetry, Ordinary differential equation, determining equation.

Introduction

One of the most interesting properties of systems of differential equations is their symmetries. A symmetry group of a system of differential equation is a transformation that maps any solution to another solution of the system. It allows one to reduce the order of differential equation and further finds a general solution in quadrature [5]. Some of the methods used to solve or integrate nonlinear Ordinary Differential Equations (ODE) are; painleve' singularity structure analysis, Darboux method, Jacobi last multiplier method and Lie symmetry analysis.

The Lie group transformation was done in the nineteenth century by the founder of symmetry analysis Norwegian Mathematician Sophus Lie [15] Symmetry analysis plays an important role in this study. Lie discovered the ad hoc method of integration of differential equations that could be derived by his theory of continuous groups. The most exciting in Lie symmetry is the fact that complicated nonlinear conditions under continuous group of action can be reduced to far simpler conditions [6].

Lie group method is based on the invariance of the differential equation under continuous group of a point transformation. The achievement as a result of this theory is due to perfection of tools of analysis and algebras, especially the availability of sufficiently useful hypothesis of the implicit function theorem and existence uniqueness of Ordinary Differential Equations [6]. Suppose the symmetry group of system of equation is identified, it can be used to find new solutions from old ones, discover whether or not differential equation can be linearized and to construct an explicit linearization when one exists, to derive conserved quantities, to classify and simplify differential equations. Since mathematical models of many phenomena of the real world are formulated in the form of differential equations [3], it suffices to say that their use in general theory of differential equation is one of the most essential applications of Lie group theory. In this study we will apply Lie symmetry analysis to find the Lie brackets of the third order nonlinear equation, $F(x, y, y' y'', y''') = 0$

[1]. Lie symmetry analysis can be regarded as the best approach since the solution does not depend on either boundary or initial values and is not an approximation to the exact solution.

Third Order Nonlinear Ordinary Equation

We consider heat equation which is third order nonlinear ordinary differential equation

$$y'y'''\left(y'^2+1\right)^{-1}=\left(y+y'\right). \tag{1}$$

The equation can be solved analytically or numerically which depends on boundary and initial conditions to approximation.
To obtain Lie brackets for the equation using Lie symmetry approach the following applies

(i) We construct the determining equations through prolongation.
(ii) The determining equations are solved to give infinitesimal generators and then one parameter symmetries.
(iii) We then obtain non zero Lie brackets.

Equation (1) can be expressed as $y'''-yy'-y\left(y'\right)^{-1}-y'^2-1=0$ (2)

Writing (2) in the form $y^{(3)}=f(x,y,y',y'')=0$ (3)

Gives $y'''=yy'+y(y')^{-1}+y'^2+1$ (4)

We subject the equation to third extension; the nth extension is of the form

$$G^{[n]}=G+\sum_{i=1}^{n}\left\{\beta^{(i)}-\sum_{j=1}^{i}\binom{i}{j}y^{(i+1-j)}\alpha^{(j)}\right\}\frac{\partial}{\partial y^{(i)}}$$

Therefore

$$\begin{aligned}G^{[3]}&=G^{(2)}+(\beta'''-3y'''\alpha'-3y''\alpha''-y'\alpha''')\frac{\partial}{\partial y'''}\\ G^{[3]}&=\alpha\frac{\partial}{\partial x}+\beta\frac{\partial}{\partial y}+(\beta'-\alpha'y')\frac{\partial}{\partial y'}+(\beta''-2y''\alpha'-y'\alpha'')\frac{\partial}{\partial y''}\\ &+(\beta'''-3y'''\alpha'+3y''\alpha''-y'\alpha''')\frac{\partial}{\partial y''}\end{aligned} \tag{5}$$

Where $\alpha\frac{\partial}{\partial x}+\beta\frac{\partial}{\partial y}$ is symmetry of Differential Equation of $f(x,y,y',y'',\ldots y^{(n)})=0$ (6)

if and only if $G^{(n)}f\big|_{f=0}=0$ [2]

Applying $G^{(3)}$ on (2) yields

$$\begin{aligned}&[\alpha\frac{\partial}{\partial x}+\beta\frac{\partial}{\partial y}+(\beta'-\alpha'y')\frac{\partial}{\partial y'}+(\beta''-2y''\alpha'-y'\alpha'')\frac{\partial}{\partial y''}(\beta'''-3y'''\alpha'-3y''\alpha''\\ &-y'\alpha''')\frac{\partial}{\partial y'''}](y'''-yy'-y(y')^{-1}-y'^2-1)=0\end{aligned} \tag{7}$$

Expanding we obtain

$$\begin{aligned}&\alpha(y^{(4)}-y'^2-yy''-1+yy''(y')^{-2}-2y''y')+\beta(-y'-(y')^{-1})+(\beta'-\alpha'y')(-y+y(y')^{-2}y''\\ &-2y'y'')+(\beta''-2y''\alpha'-y'\alpha'')[0]+(\beta'''-3y'''\alpha'-3y''\alpha''-y'\alpha''')[1]=0\end{aligned} \tag{8}$$

But,

$$y^{(4)} = (y''')'$$
$$= (yy' + y(y')^{-1} + y'^2 + 1)' \tag{9}$$
$$= y'^2 + yy'' + y(y')^{-2}y'' + 2y'y''$$

Substituting (9) in (8) we have

$$\begin{aligned}&\alpha(y'^2 + yy'' + 1 - y(y')^{-2}y'' + 2y'y'' - y'^2 - yy'' - 1 + y(y')^{-2}y'' - 2y'y'') + \beta(-y' - (y')^{-1})\\&+(\beta' - \alpha'y')(-y + y(y')^{-2}y'' - 2y'y'') + \beta''' - 3y'''\alpha' - 3y''\alpha'' - y'\alpha''' = 0\end{aligned} \tag{10}$$

Simplifies to

$$-\beta y' - \beta(y') + \beta' y + \beta' y(y')^{-2}y'' - 2\beta' y'y'' + \alpha' yy' - \alpha' y(y')^{-1}y'' - 2\alpha'(y')^2y'' + \beta''' - 3y'''\alpha' - y'\alpha''' = 0 \tag{11}$$

The first, second and third total derivatives of α can be expressed as

$$\alpha' = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} \tag{12}$$

$$\alpha'' = \frac{\partial^2\alpha}{\partial x^2} + 2y'\frac{\partial^2\alpha}{\partial x\partial y} + y'^2\frac{\partial^2\alpha}{\partial y^2} + y''\frac{\partial\alpha}{\partial y} \tag{13}$$

$$\alpha''' = \frac{\partial^3\alpha}{\partial x^3} + 3y'\frac{\partial^3\alpha}{\partial x^2\partial y} + 3y''\frac{\partial^2\alpha}{\partial x\partial y} + y'''\frac{\partial\alpha}{\partial y} + 3y'^2\frac{\partial^3\alpha}{\partial x\partial y^2} + 3y'y''\frac{\partial^2\alpha}{\partial y^2} + y'^3\frac{\partial^3\alpha}{\partial y^3} \tag{14}$$

And it follows for β. Thus we express (11) as

$$\begin{aligned}&-\beta y' - \beta(y')^{-1} + y\frac{\partial}{\partial x} + yy'\frac{\partial\beta}{\partial y} + y(y')^{-2}y''\frac{\partial\beta}{\partial x} + y(y')^{-1}y''\frac{\partial\beta}{\partial y} - 2y'y''\frac{\partial\beta}{\partial x} - 2y'^2y''\frac{\partial\beta}{\partial y} + yy'\frac{\partial\alpha}{\partial x}\\
&+y(y')^2\frac{\partial\alpha}{\partial y} - y(y')^{-1}y''\frac{\partial\alpha}{\partial x} - yy''\frac{\partial\alpha}{\partial y} - 2y'^2y''\frac{\partial\alpha}{\partial x} - 2(y')^3y''\frac{\partial\alpha}{\partial y} + \frac{\partial^3\beta}{\partial x^3} + 3y'\frac{\partial^3\beta}{\partial x^2\partial y} + 3y''\frac{\partial^2\beta}{\partial x\partial y} + yy'\frac{\partial\beta}{\partial y}\\
&+yy'\frac{\partial\beta}{\partial y} + y'^2\frac{\partial\beta}{\partial y} + \frac{\partial\beta}{\partial y} + 3y'^2\frac{\partial^3\beta}{\partial x\partial^2} + 3y'y''\frac{\partial^2\beta}{\partial y^2} + y'^3\frac{\partial^3\beta}{\partial y^3} - 3yy'\frac{\partial\alpha}{\partial x} - 3y(y')^{-1}\frac{\partial\alpha}{\partial x} - 3y'^2\frac{\partial\alpha}{\partial x} - 3\frac{\partial\alpha}{\partial x}\\
&-3yy'^2\frac{\partial\alpha}{\partial y} - 3y\frac{\partial\alpha}{\partial y} - 3y'^3\frac{\partial\alpha}{\partial y} - 3y'\frac{\partial\alpha}{\partial y} - 3y''\frac{\partial^2\alpha}{\partial x^2} - 9y'y''\frac{\partial^2\alpha}{\partial x\partial y} - 6y'^2y''\frac{\partial^2\alpha}{\partial y^2} - 3y''^2\frac{\partial\alpha}{\partial y} - y'\frac{\partial^3\alpha}{\partial x^3}\\
&-3y'\frac{\partial^3\alpha}{\partial x\partial y^2} - y'^4\frac{\partial^3\alpha}{\partial y^3} = 0\end{aligned} \tag{15}$$

Constructing the determining equations from (15) since α and β are functions of x and y only, we equate the coefficients of the powers of y', y'' and their combinations to zero. [7], [8]. We obtain

$$y'^3y'': \frac{\partial\alpha}{\partial y} = 0 \tag{16}$$

$$y'^2y'': -2\frac{\partial\beta}{\partial y} - 2\frac{\partial\alpha}{\partial x} - 6\frac{\partial^2\alpha}{\partial y^2} = 0 \tag{17}$$

$$y'y'': -2\frac{\partial\beta}{\partial x} + 3\frac{\partial^2\beta}{\partial y^2} - 9\frac{\partial^2\alpha}{\partial x\partial y} = 0 \tag{18}$$

$$(y')^0 y'': -\frac{\partial\alpha}{\partial y} + 3\frac{\partial^2\beta}{\partial x\partial y} + \frac{\partial^2\alpha}{\partial x^2} = 0 \quad (19)$$

$$(y')^{-1} y'': y\frac{\partial\beta}{\partial y} - y\frac{\partial\alpha}{\partial x} = 0 \quad (20)$$

$$(y')^{-2} y'': y\frac{\partial\beta}{\partial x} = 0 \quad (21)$$

Since $\frac{\partial\beta}{\partial y} - \frac{\partial\alpha}{\partial x} = 0$ from (20) equation (17) becomes $-3\frac{\partial^2\alpha}{\partial y^2} = 0$ Integrating this equation we obtain

$$\alpha = c_1 y + c_2 \quad (22)$$

Where c_1 and c_2 are arbitrary functions of x. We substitute equation (22) in (18) and integrate,

Since $\frac{\partial\beta}{\partial x} = 0$ from (21) the equation (18) is

$$\Rightarrow \beta = \frac{3}{2}c'_1 y^2 + c_3 y + c_4 \quad (23)$$

Substituting (22) and (23) in (20)

$$\frac{\partial\beta}{\partial y} - \frac{\partial\alpha}{\partial x} = 0$$

$$2c'_1 + c_3 - c'_2 = 0 \quad (24)$$

We equate the coefficients of powers of y to zero since c_1, c_2 and c_3 depend on x only. This gives

$$y^1 : 2c'_1 = 0 \quad (25)$$

$$y^0 : c_3 - c_{2'} = 0 \quad (26)$$

Now we substitute (23) in (21)

$$\frac{\partial\beta}{\partial x} = 0 \Rightarrow \frac{3}{2}c''_1 y^2 + c'_3 y + c'_4 = 0 \quad (27)$$

Also equating coefficients of powers of y to zero since c_4 is also a constant.

$$y^2 : c''_1 = 0 \quad (28)$$

$$y^1 : c'_3 = 0 \quad (29)$$

$$y^0 : c'_4 = 0 \quad (30)$$

From equation (28) we have

$$c''_1 = 0 \Rightarrow c'_1 = H_1$$

$$c_1 = H_1 x + H_2 \quad (31)$$

Now (29) gives

$$c_3 = H_3 \quad (32)$$

And (30) gives

$$c_4 = H_4 \quad (33)$$

Considering (26)

$$c_3 - c'_2 = 0 \Rightarrow c'_2 = c_3 \text{ but } c_3 = H_3$$

Therefore $c_2 = H_3 x + H_5$ (34)

Where H_1, H_2, H_3, H_4 and H_5 are arbitrary constants.
Substituting (31), (32), (33) and (34) to find

$$\alpha(x,y) = H_1xy + H_2y + H_3x + H_5 \quad (35)$$

And $$\beta(x,y) = \frac{3}{2}H_2y^2 + H_3y + H_4 \quad (36)$$

As a result the generator G of the infinitesimal transformation is

$$G = (H_1xy + H_2y + H_3x + H_5)\frac{\partial}{\partial x} + (\frac{3}{2}H_1y^2 + H_3y + H_4)\frac{\partial}{\partial y} \quad (37)$$

Equivalent to

$$G = H_1(xy\frac{\partial}{\partial x} + \frac{3}{2}y^2\frac{\partial}{\partial y}) + H_2(y\frac{\partial}{\partial x}) + H_3(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}) + H_4(\frac{\partial}{\partial y}) + H_5(\frac{\partial}{\partial x}) \quad (38)$$

This given Ordinary Differential Equation thatadmits a set of five one parameter symmetries [10], which are given as

$$G_1 = \frac{\partial}{\partial x},\ G_2 = \frac{\partial}{\partial y},\ G_3 = y\frac{\partial}{\partial x},\ G_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},\ G_5 = xy\frac{\partial}{\partial x} + \frac{3}{2}y^2\frac{\partial}{\partial y} \quad (39)$$

A key feature of symmetry generators is that they form algebra under the operation of Lie brackets. Symmetries are differential operators and one can calculate their Lie Brackets. ([5], [6], [9])
For any two generator n G_i and G_j their Lie Bracket is defined by

$$[G_i, G_j] = G_iG_j - G_jG_i$$

$$3\frac{\partial^2\beta}{\partial y^2} - 9\frac{\partial^2\alpha}{\partial x\partial y} = 0 \Rightarrow \frac{\partial^2\beta}{\partial y^2} = 3\frac{\partial c_1}{\partial x}$$

$$\Rightarrow \frac{\partial^2\beta}{\partial y^2} = 3c'_1 \Rightarrow \frac{\partial\beta}{\partial y} = 3c'_1\,y + c_3$$

$$[G_2, G_3] = G_1, \quad [G_2, G_4] = G_2, \quad [G_1, G_5] = G_3 \quad (40)$$

Thus equation (40) admits solvable Lie algebra and can be solved as outlined by ([10], [14], [12], [2] and [11]) to reduce the third order differential equation (1) to lower order.

Conclusion

In this paper we have subjected the nonlinear ordinary heat equation to prolongation (extended generators), constructed determining equations (system of partial differential equations). We have also obtained set of n-parameter symmetry of the Ordinary Differential Equation which we have used to develop n-one parameter symmetry by letting particular parameters take specific values. This enabled us to get nonzero Lie Brackets. The nonzero Lie Brackets obtained are aimed at reducing the equation from Third Order to a Lower Order [5], [6].

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