Determining Equations of Fourth Order Nonlinear Ordinary Differential Equation

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***Abstract*– Determining Equations are linear partial differential equations. The equation to be solved is subjected to extension generator. The coefficient of unconstrained partial derivatives is equated to zero and since the equations are homogeneous their solutions form vector space [1]. The determining equations obtained leads to n-parameter symmetries.**

***Keywords*– Infinitesimal Generators, Prolongation, Lie Symmetry, Ordinary Differential Equation and Determining Equation**

I. WAVE EQUATION

The solution of fourth order nonlinear wave equation:

$$F(x, y, y', y'', y''', y^{(4)}) = 0 \tag{1}$$

It can be classified either as analytic or as numerical solutions using finite difference approach where the convergence of the numerical schemes depend entirely on the initial and boundary values given. We have attempted in this study to solve a special case of equation:

$$\left(yy'(y(y')^{-1})''\right)' = 0 \tag{2}$$

analytically using Lie Symmetry approach.

Equation (2) can alternatively be expressed as:

$$\left[yy'\left(\frac{y}{y'}\right)''\right]' = 0 \tag{3}$$

The equation (3) can be decomposed in the form:

$y^{(4)} = (x, y, y', y'', y''')$ as follows:

$$4y'y^{-1}y''^{2} - 4y^{2}y'^{-3}y''^{3} + 5y^{2}y'^{-2}y''y''' - y'y'' - 3yy''' - y^{2}y'^{-1}y^{(4)} = 0 \tag{4}$$

We also notice that:

$$y^{(4)} = 4y^{-1}(y'')^{2} - 4(y')^{-2}(y'')^{3} + 5(y')^{-1}y''y''' - y^{-2}(y')^{2}y'' - 3y^{-1}y'y''' \tag{5}$$

Since the equation is fourth order differential equation, we use the fourth extension of G which from the n^{th} extension of the form [4]:

$$G^{(n)} = G\sum_{i=1}^{n}\left\{\beta^{(i)} - \sum_{j=1}^{i}\binom{i}{j}y^{i+1-j}\alpha^{(j)}\right\}\frac{\partial}{\partial y^{(i)}}$$ is given by:

$$G^{(4)} = \alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y} + (\beta' - \alpha' y') \frac{\partial}{\partial y'} + (\beta'' - 2y''\alpha' - y'\alpha'') \frac{\partial}{\partial y''}$$
$$+ (\beta''' - 3y'''\alpha' - 3y''\alpha'' - y'\alpha'') \frac{\partial}{\partial y'''} + \beta^{(4)} - 4y^{(4)}\alpha' - 4y''\alpha''' - y'\alpha^{(4)}) \frac{\partial}{\partial y^{(4)}} \quad (6)$$

When $G^{(4)}$ acts on the differential equation, we have:

$$G^{(4)}[y^{(4)} - 4y^{-1}(y'')^2 + 4y'^{-2}(y'')^3 - 5(y'^{-1})y''y''' + y^{-2}(y')^2 y'' + 3y^{-1}y'y'''] = 0$$

On expanding, we obtain

$$\alpha[(y^{(5)}) + 4y^{-2}(y')(y'')^2 - 8y^{-1}(y'')(y''') - 8(y^{-1})(y'')(y''') - 8(y')^{-3}(y'')^4$$
$$+ 12(y')^{-2}(y'')^2(y''') + 5(y')^{-2}(y'')^2(y''') - 5(y')^{-1}(y''') + 5(y')(y''^2(y''') - 5(y')(y''')^2$$
$$- 5(y')^{-1}(y'')(y^{(4)}) - 2y^{-3}(y')^3(y'') + 2y^{-2}(y')(y'')^2 + y^{-2}(y')^2(y''' - 3y^{-2}(y')^2(y''')$$
$$+ 3y^{-1}(y'')()y'''] + 3y^{-1}(y')(y^{(4)})] + \beta[4y^{-2}(y'')^2 - 2y^{-3}(y')^2(y'') - 3y^{-2}(y')(y''')]$$
$$+ [\beta' - \alpha' y'][-8(y')^{-3}(y'')^4 + 5(y')^{-2}(y'')^2(y''') + 2y^{-2}(y')(y'')^2 + 3y^{-1}(y''')]$$
$$+ [\beta'' - 2y''\alpha' - y'\alpha''][-8y^{-1}(y'')(y''') + 12(y')^{-2}(y'')^2(y''') - 5(y')^{-1}(y''') + y^{-2}(y')^2]$$
$$+ [\beta''' - 3y'''\alpha' - 3y''\alpha'' - y'\alpha'''][-5(y')^{-1}(y'') + 3y^{-1}(y')] + [\beta^{(4)} - 4y^{(4)}\alpha' - 6y'''\alpha'' - 4y''\alpha'''$$
$$- y'\alpha^{(4)}][1] = 0 \quad (7)$$

But we notice

$$y^{(5)} = (y^{(4)})'$$
$$= (4y^{-1}(y'')^2 - 4(y')^{-2}(y'')^3 + 5(y')^{-1}y''y''' - y^{-2}(y')^2 y'' - 3y^{-1}y'y''')'$$
$$= -4y^{-2}(y')(y'') + 8y^{-1}(y'')(y''') + 8(y')^{-3}(y'')^4 - 12(y')^{-2}(y'')^2(y''')$$
$$- 5(y')^{-2}(y'')^2(y''') + 5(y')^{-1}(y''')^2 + 5(y')^{-1}(y'')(y^{(4)}) + 2y^{-3}(y')^3(y'')$$
$$- 2y^{-2}(y')(y'')^2 - y^{-2}(y')^2(y''') + 3y^{-2}(y')^2(y''') - 3y^{-1}(y'')(y''') - 3y^{-1}(y')(y^{(4)}) \quad (8)$$

Substituting (8) in (7) gives:

$$\alpha[(-4y^{-2}(y')(y'')^2+8y^{-1}(y'')(y''')+8(y')^{-3}(y'')^4-12(y')^{-2}(y'')^2(y''')$$
$$-5(y')^{-2}(y'')^2(y''')+5(y)^{-1}(y''')^2+5(y')^{-1}(y'')(y^{(4)})+2y^{-3}(y')^3(y'')$$
$$-2y^{-2}(y')(y'')^2-y^{-2}(y')^2(y''')+3y^{-2}(y')^2(y''')-3y^{-1}(y'')(y''')$$
$$-3y^{-1}(y')(y^{(4)})+4y^{-2}(y')(y'')^2-8y^{-1}(y'')(y''')-8(y')^{-3}(y'')^4+12(y')^{-2}(y'')^2(y''')$$
$$+5(y')^{-2}(y'')^2(y''')-5(y')^{-1}(y''')^2-5(y')^{-1}(y'')(y^{(4)})-2y^{-3}(y')^3(y'')+2y^{-2}(y')(y'')^2$$
$$+y^{-2}(y')^2(y''')-3y^{-2}(y')^2(y''')+3y^{-1}(y'')(y''')+3y^{-1}(y')(y^{(4)})]$$
$$+\beta[4y^{-2}(y'')^2-2y^{-3}(y')^2(y'')-3y^{-2}(y')(y''')]$$
$$+[\beta'-\alpha'y'][-8(y')^{-3}(y'')^4+5(y')^{-2}(y'')^2(y''')+2y^{-2}(y')(y'')^2+3y^{-1}(y''')]$$
$$+[\beta''-2y''\alpha'-y'\alpha''][-8y^{-1}(y'')(y''')+12(y')^{-2}(y'')^2(y''')-5(y)^{-1}(y''')+y^{-2}(y')^2]$$
$$+[\beta'''-3y'''\alpha'-3y''\alpha''-y'\alpha'''][-5(y')^{-1}(y'')+3y^{-1}(y')]$$
$$+[\beta^{(4)}-4y^{(4)}\alpha'-6y'''\alpha''-4y''\alpha'''-y'\alpha^{(4)}]=0$$

Or equivalently

$$-4\alpha y^{-2}(y')(y'')^2+8\alpha y^{-1}(y'')(y''')+8\alpha(y')^{-3}(y'')^4-12\alpha(y')^{-2}(y'')^2(y''')$$
$$-5\alpha(y')^{-2}(y'')^2(y''')+5\alpha(y')^{-1}(y''')^2+5\alpha(y')^{-1}(y'')(y^{(4)})+2\alpha y^{-3}(y')^3(y'')$$
$$-2\alpha y^{-2}(y')(y'')^2-\alpha y^{-2}(y')^2(y')^2(y''')+3\alpha y^{-2}(y')^2(y''')-3\alpha y^{-1}(y'')(y''')$$
$$-3\alpha y^{-1}(y')(y^{\{4\}})+4\alpha y^{-2}(y')(y'')^2-8\alpha y^{-1}(y'')(y''')-8(y')^{-3}(y'')^4$$
$$+12\alpha(y')^{-2}(y'')^2(y''')+5\alpha(y')^{-2}(y'')^2(y''')-5\alpha(y')^{-1}(y''')^2-5\alpha(y')^{-1}(y'')(y^{(4)})$$
$$-2\alpha y^{-3}(y')^3(y'')+2\alpha y^{-2}(y')(y'')^2+\alpha y^{-2}(y')^2(y''')-3\alpha y^{-2}(y')^2(y''')$$
$$+3\alpha y^{-1}(y'')(y''')+3\alpha y^{-1}(y^{(4)})+4\beta y^{-2}(y'')^2-2\beta y^{-3}(y')^2(y'')-3\beta y^{-2}(y')(y''')$$
$$-8\beta'(y')^{-3}(y'')^4+5\beta'(y')^{-2}(y'')^2(y''')+2\beta'y^{-2}(y')(y'')^2+3\beta'y^{-1}(y''')$$
$$+8\alpha'(y')^{-2}(y'')^4-5\alpha'(y')^{-1}(y'')^2(y''')-2\alpha'y^{-2}(y')^2(y'')^2-3\alpha'y^{-1}(y')(y''')$$
$$-8\beta''y^{-1}(y'')(y''')+12\beta''(y')^{-2}(y'')^2(y''')-5\beta''(y')^{-1}(y''')+\beta''y^{-2}(y')^2$$
$$+16\alpha'y^{-1}(y'')^2(y''')-24\alpha'(y')^{-2}(y'')^3(y''')+5\alpha''(y''')-\alpha''y^{-2}(y')^1$$
$$-5\beta'''(y')^{-1}(y'')+15\alpha'(y')^{-1}(y'')(y''')+15\alpha''(y')^{-1}(y'')^2+5\alpha'''(y'')$$
$$3\beta'''y^{-1}(y')-27\alpha'y^{-1}(y')(y''')-9\alpha''y^{-1}(y')(y'')-3\alpha'''y^{-1}(y')^2$$
$$\beta^{(4)}-4\alpha'y^{(4)}-6\alpha''y'''-4\alpha'''y''-\alpha^{(4)}y'=0$$

Which can progressively be expressed as

$$
\begin{aligned}
&-4\alpha y^{-2}(y')(y'')^2+8\alpha y^{-1}(y'')(y''')+8\alpha(y')^{-3}(y'')^4-12\alpha(y')^{-2}(y'')^2(y''')\\
&-5\alpha(y')^{-2}(y'')^2(y''')+5\alpha(y')^{-1}(y''')^2+2\alpha y^{-3}(y')^3(y'')\\
&-2\alpha y^{-2}(y')(y'')^2-\alpha y^{-2}(y')^2(y')^2(y''')+3\alpha y^{-2}(y')^2(y''')-3\alpha y^{-1}(y'')(y''')\\
&+4\alpha y^{-2}(y')(y'')^2-8\alpha y^{-1}(y'')(y''')-8(y')^{-3}(y'')^4\\
&+12\alpha(y')^{-2}(y'')^2(y''')+5\alpha(y')^{-2}(y'')^2(y''')-5\alpha(y')^{-1}(y''')^2\\
&-2\alpha y^{-3}(y')^3(y'')+2\alpha y^{-2}(y')(y'')^2+\alpha y^{-2}(y')^2(y''')-3\alpha y^{-2}(y')^2(y''')\\
&+3\alpha y^{-1}(y'')(y''')+4\beta y^{-2}(y'')^2-2\beta y^{-3}(y')^2(y'')-3\beta y^{-2}(y')(y''')\\
&-8\beta'(y')^{-3}(y'')^4+5\beta'(y')^{-2}(y'')^2(y''')+2\beta' y^{-2}(y')(y'')^2+3\beta' y^{-1}(y''')\\
&+8\alpha'(y')^{-2}(y'')^4-5\alpha'(y')^{-1}(y'')^2(y''')-2\alpha' y^{-2}(y')^2(y'')^2-3\alpha' y^{-1}(y')(y''')\\
&-8\beta'' y^{-1}(y'')(y''')+12\beta''(y')^{-2}(y'')^2(y''')-5\beta''(y')^{-1}(y''')+\beta'' y^{-2}(y')^2\\
&+16\alpha' y^{-1}(y'')^2(y''')-24\alpha'(y')^{-2}(y'')^3(y''')+10\alpha'(y')^{-1}(y'')(y''')-2\alpha' y^{-2}(y')^2(y'')\\
&+8\alpha'' y^{-1}(y')(y'')(y''')-12\alpha'' y^{-1}(y'')^2(y''')+5\alpha''(y''')-\alpha'' y^{-2}(y')^1\\
&-5\beta'''(y')^{-1}(y'')+15\alpha'(y')^{-1}(y'')(y''')+15\alpha''(y')^{-1}(y'')^2+5\alpha'''(y'')\\
&+3\beta''' y^{-1}(y')-27\alpha' y^{-1}(y')(y''')-9\alpha'' y^{-1}(y')(y'')-3\alpha''' y^{-1}(y')^2+\beta^{(4)}\\
&-16\alpha' y^{-1}(y'')^2+16\alpha'(y')^{-2}(y'')^3-20\alpha'(y')^{-1}y''y'''+4\alpha' y^{-2}(y')^2y''+12\alpha' y^{-1}y'y'''\\
&-6\alpha'' y''-4\alpha''' y''-\alpha^{(4)}y'=0
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&4\beta y^{-2}(y'')^2-2\beta y^{-3}(y')^2(y'')-3\beta y^{-2}(y')(y''')\\
&-8\beta'(y')^{-3}(y'')^4+5\beta'(y')^{-2}(y'')^2(y''')+2\beta' y^{-2}(y')(y'')^2+3\beta' y^{-1}(y''')\\
&+8\alpha'(y')^{-2}(y'')^4-5\alpha'(y')^{-1}(y'')^2(y''')-2\alpha' y^{-2}(y')^2(y'')^2-3\alpha' y^{-1}(y')(y''')\\
&-8\beta'' y^{-1}(y'')(y''')+12\beta''(y')^{-2}(y'')^2(y''')-5\beta''(y')^{-1}(y''')+\beta'' y^{-2}(y')^2\\
&+16\alpha' y^{-1}(y'')^2(y''')-24\alpha'(y')^{-2}(y'')^3(y''')+10\alpha'(y')^{-1}(y'')(y''')-2\alpha' y^{-2}(y')^2(y'')\\
&+8\alpha'' y^{-1}(y')(y'')(y''')-12\alpha'' y^{-1}(y'')^2(y''')+5\alpha''(y''')-\alpha'' y^{-2}(y')^1\\
&-5\beta'''(y')^{-1}(y'')+15\alpha'(y')^{-1}(y'')(y''')+15\alpha''(y')^{-1}(y'')^2+5\alpha'''(y'')\\
&+3\beta''' y^{-1}(y')-27\alpha' y^{-1}(y')(y''')-9\alpha'' y^{-1}(y')(y'')-3\alpha''' y^{-1}(y')^2+\beta^{(4)}\\
&-16\alpha' y^{-1}(y'')^2+16\alpha'(y')^{-2}(y'')^3-20\alpha'(y')^{-1}y''y'''+4\alpha' y^{-2}(y')^2y''+12\alpha' y^{-1}y'y'''\\
&-6\alpha'' y'''-4\alpha''' y''-\alpha^{(4)}y'=0
\end{aligned}
\tag{9}
$$

We recall that primes in (9) refer to total derivatives and so the first, the second, the third and fourth total derivatives of α and β can be expressed in terms of partial derivatives as follows:

$$\alpha' = \frac{\partial \alpha}{\partial x} + y'\frac{\partial \beta}{\partial y}$$

$$\alpha'' = \frac{\partial^2 \alpha}{\partial x^2} + 2y'\frac{\partial^2 \alpha}{\partial x \partial y} + y'^2\frac{\partial^2 \alpha}{\partial y^2} + y''\frac{\partial \alpha}{\partial y}$$

$$\alpha''' = \frac{\partial^3 \alpha}{\partial x^3} + 3y'\frac{\partial^3 \alpha}{\partial x^2 \partial y} + 3y''\frac{\partial^2 \alpha}{\partial x \partial y} + y'''\frac{\partial \alpha}{\partial y} + 3y'^2\frac{\partial^3 \alpha}{\partial x \partial y^2} + 3y'y''\frac{\partial^2 \alpha}{\partial y^2} + y'^3\frac{\partial^3 \alpha}{\partial y^3}$$

$$\alpha^4 = \frac{\partial^4 \alpha}{\partial x^4} + 4y'\frac{\partial^4 \alpha}{\partial x^3 \partial y} + 6y''\frac{\partial^3 \alpha}{\partial x^2 \partial y} + 4y'''\frac{\partial^2 \alpha}{\partial x \partial y} + y^{(4)}\frac{\partial \alpha}{\partial y}$$
$$+ 3y'^2\frac{\partial^4 \alpha}{\partial x^2 \partial y^2} + 9y'y''\frac{\partial^3 \alpha}{\partial x \partial y^2} 4y'y'''\frac{\partial^2 \alpha}{\partial y^2} + 3y''^2\frac{\partial^2 \alpha}{\partial y^2} + 4y'^3\frac{\partial^4 \alpha}{\partial x \partial y^3}$$
$$+ 6y'^2 y''\frac{\partial^3 \alpha}{\partial y^3} + 3y'^2\frac{\partial^4 \alpha}{\partial x^2 \partial y^2} + 3y'y''\frac{\partial^3 \alpha}{\partial x \partial y^2} + y'^4\frac{\partial^4 \alpha}{\partial y^4}$$

And so is true for β.

Therefore, we express equation (9) as

$$4\beta y^{-2}(y'')^2-2\beta y^{-3}(y')^2(y'')-3\beta y^{-2}(y')(y''')-8\left[\frac{\partial\beta}{\partial x}+y'\frac{\partial\beta}{\partial y}\right](y')^{-3}(y'')^4+$$

$$5\left[\frac{\partial\beta}{\partial x}+y'\frac{\partial\beta}{\partial y}\right](y')^{-2}(y'')^2(y''')+2\left[\frac{\partial\beta}{\partial x}+y'\frac{\partial\beta}{\partial y}\right]y^{-2}(y')(y'')^2+3\left[\frac{\partial\beta}{\partial x}+y'\frac{\partial\beta}{\partial y}\right]y^{-1}(y''')$$

$$+8\left[\frac{\partial\alpha}{\partial x}+y'\frac{\partial\alpha}{\partial y}\right](y')^{-2}(y'')^4-5\left[\frac{\partial\alpha}{\partial x}+y'\frac{\partial\alpha}{\partial y}\right](y')^{-1}(y'')^2(y''')-2\left[\frac{\partial\alpha}{\partial x}+y'\frac{\partial\alpha}{\partial y}\right]y^{-2}(y')^2(y'')^2$$

$$-3\left[\frac{\partial\alpha}{\partial x}+y'\frac{\partial\alpha}{\partial y}\right]y^{-1}(y')(y''')-8\left[\frac{\partial^2\beta}{\partial x^2}+2y'\frac{\partial^2\beta}{\partial x\partial y}+y'^2\frac{\partial^2\beta}{\partial y^2}+y''\frac{\partial\beta}{\partial y}\right]y^{-1}(y'')(y''')$$

$$+12\left[\frac{\partial^2\beta}{\partial x^2}+2y'\frac{\partial^2\beta}{\partial x\partial y}+y'^2\frac{\partial^2\beta}{\partial y^2}+y''\frac{\partial\beta}{\partial y}\right](y')^{-2}(y'')^2(y''')$$

$$-5\left[\frac{\partial^2\beta}{\partial x^2}+2y'\frac{\partial^2\beta}{\partial x\partial y}+y'^2\frac{\partial^2\beta}{\partial y^2}+y''\frac{\partial\beta}{\partial y}\right](y')^{-1}(y''')$$

$$+\left[\frac{\partial^2\beta}{\partial x^2}+2y'\frac{\partial^2\beta}{\partial x\partial y}+y'^2\frac{\partial^2\beta}{\partial y^2}+y''\frac{\partial\beta}{\partial y}\right]y^{-2}(y')^2+16\left[\frac{\partial\alpha}{\partial x}+y'\frac{\partial\alpha}{\partial y}\right]y^{-1}(y'')^2(y''')$$

$$-24\left[\frac{\partial\alpha}{\partial x}+y'\frac{\partial\alpha}{\partial y}\right](y')^{-2}(y'')^3(y''')+10\left[\frac{\partial\alpha}{\partial x}+y'\frac{\partial\alpha}{\partial y}\right](y')^{-1}(y'')(y''')-2\left[\frac{\partial\alpha}{\partial x}+y'\frac{\partial\alpha}{\partial y}\right]y^{-2}(y')^2(y'')$$

$$+8\left[\frac{\partial^2\alpha}{\partial x^2}+2y'\frac{\partial^2\alpha}{\partial x\partial y}+y'^2\frac{\partial^2\alpha}{\partial y^2}+y''\frac{\partial\alpha}{\partial y}\right]y^{-1}(y')(y'')(y''')-12\left[\frac{\partial^2\alpha}{\partial x^2}+2y'\frac{\partial^2\alpha}{\partial x\partial y}+y'^2\frac{\partial^2\alpha}{\partial y^2}+y''\frac{\partial\alpha}{\partial y}\right]$$

$$y^{-1}(y'')^2(y''')+5\left[\frac{\partial^2\alpha}{\partial x^2}+2y'\frac{\partial^2\alpha}{\partial x\partial y}+y'^2\frac{\partial^2\alpha}{\partial y^2}+y''\frac{\partial\alpha}{\partial y}\right](y''')$$

$$-\left[\frac{\partial^2\alpha}{\partial x^2}+2y'\frac{\partial^2\alpha}{\partial x\partial y}+y'^2\frac{\partial^2\alpha}{\partial y^2}+y''\frac{\partial\alpha}{\partial y}\right]y^{-2}(y')^1$$

$$-5\left[\frac{\partial^3\beta}{\partial x^3}+3y'\frac{\partial^3\beta}{\partial x^2\partial y}+3y''\frac{\partial^2\beta}{\partial x\partial y}+y'''\frac{\partial\beta}{\partial y}+3y'^2\frac{\partial^3\beta}{\partial x\partial y^2}+3y'y''\frac{\partial^2\beta}{\partial y^2}+y'^3\frac{\partial^3\beta}{\partial y^3}\right](y')^{-1}(y'')$$

$$+15\left[\frac{\partial\alpha}{\partial x}+y'\frac{\partial\alpha}{\partial y}\right](y')^{-1}(y'')(y''')+15\left[\frac{\partial^2\alpha}{\partial x^2}+2y'\frac{\partial^2\alpha}{\partial x\partial y}+y'^2\frac{\partial^2\alpha}{\partial y^2}+y''\frac{\partial\alpha}{\partial y}\right](y')^{-1}(y'')^2$$

$$+5\left[\frac{\partial^3\alpha}{\partial x^3}+3y'\frac{\partial^3\alpha}{\partial x^2\partial y}+3y''\frac{\partial^2\alpha}{\partial x\partial y}+y'''\frac{\partial\alpha}{\partial y}+3y'^2\frac{\partial^3\alpha}{\partial x\partial y^2}+3y'y''\frac{\partial^2\alpha}{\partial y^2}+y'^3\frac{\partial^3\alpha}{\partial y^3}\right](y'')$$

$$+3\left[\frac{\partial^3\beta}{\partial x^3}+3y'\frac{\partial^3\beta}{\partial x^2\partial y}+3y''\frac{\partial^2\beta}{\partial x\partial y}+y'''\frac{\partial\beta}{\partial y}+3y'^2\frac{\partial^3\beta}{\partial x\partial y^2}+3y'y''\frac{\partial^2\beta}{\partial y^2}+y'^3\frac{\partial^3\beta}{\partial y^3}\right]y^{-1}(y')$$

$$-27\left[\frac{\partial\alpha}{\partial x}+y'\frac{\partial\alpha}{\partial y}\right]y^{-1}(y')(y''')-9\left[\frac{\partial^2\alpha}{\partial x^2}+2y'\frac{\partial^2\alpha}{\partial x\partial y}+y'^2\frac{\partial^2\alpha}{\partial y^2}+y''\frac{\partial\alpha}{\partial y}\right]y^{-1}(y')(y'')2$$

$$-3\left[\frac{\partial^3\alpha}{\partial x^3}+3y'\frac{\partial^3\alpha}{\partial x^2\partial y}+y'''\frac{\partial\alpha}{\partial y}+3y'^2\frac{\partial^3\alpha}{\partial x\partial y^2}+3y'y''\frac{\partial^2\alpha}{\partial y^2}+y'^3\frac{\partial^3\alpha}{\partial y^3}\right]y^{-1}(y')^2$$

$$+\frac{\partial^4\beta}{\partial x^4}+4y'\frac{\partial^4\beta}{\partial x^3\partial y}+6y''\frac{\partial^3\beta}{\partial x^2\partial y}+4y'''\frac{\partial^2\beta}{\partial x\partial y}+y^{(4)}\frac{\partial\beta}{\partial y}+3y'^2\frac{\partial^4\beta}{\partial x^2\partial y^2}+9y'y''\frac{\partial^3\beta}{\partial x\partial y^2}$$

$$+4y'y'''\frac{\partial^2\beta}{\partial y^2}+3y''^2\frac{\partial^2\beta}{\partial y^2}+4y'^3\frac{\partial^4\beta}{\partial x\partial y^3}+6y'^2y''\frac{\partial^3\beta}{\partial y^3}+3y'^2\frac{\partial^4\beta}{\partial x^2\partial y^2}$$

$$-16\left[\frac{\partial\alpha}{\partial x}+y'\frac{\partial\alpha}{\partial y}\right]y^{-1}(y'')^2+16\left[\frac{\partial\alpha}{\partial x}+y'\frac{\partial\alpha}{\partial y}\right](y')^{-2}(y'')^3-20\left[\frac{\partial\alpha}{\partial x}+y'\frac{\partial\alpha}{\partial y}\right](y')^{-1}y''y'''$$

$$+4\left[\frac{\partial\alpha}{\partial x}+y'\frac{\partial\alpha}{\partial y}\right]y^{-2}(y')^2y''+12\left[\frac{\partial\alpha}{\partial x}+y'\frac{\partial\alpha}{\partial y}\right]y^{-1}y'y'''-6\left[\frac{\partial^2\alpha}{\partial x^2}+2y'\frac{\partial^2\alpha}{\partial x\partial y}+y'^2\frac{\partial^2\alpha}{\partial y^2}+y''\frac{\partial\alpha}{\partial y}\right]y'''$$

$$-4\left[\frac{\partial^3\alpha}{\partial x^3}+3y'\frac{\partial^3\alpha}{\partial x^2\partial y}+3y''\frac{\partial^2\alpha}{\partial x\partial y}+y'''\frac{\partial\alpha}{\partial y}+3y'^2\frac{\partial^3\alpha}{\partial x\partial y^2}+3y'y''\frac{\partial^2\alpha}{\partial y^2}+y'^3\frac{\partial^3\alpha}{\partial y^3}\right]y''$$

$$-[\frac{\partial^4\alpha}{\partial x^4}+4y'\frac{\partial^4\alpha}{\partial x^3\partial y}+6y''\frac{\partial^3\alpha}{\partial x^2\partial y}+4y'''\frac{\partial^2\alpha}{\partial x\partial y}+y^{(4)}\frac{\partial\alpha}{\partial y}$$

$$+3y'^2\frac{\partial^4\alpha}{\partial x^2\partial y^2}+9y'y''\frac{\partial^3\alpha}{\partial x\partial y^2}4y'y'''\frac{\partial^2\alpha}{\partial y^2}+3y''^2\frac{\partial^2\alpha}{\partial y^2}+4y'^3\frac{\partial^4\alpha}{\partial x\partial y^3}$$

$$+6y'^2y''\frac{\partial^3\alpha}{\partial y^3}+3y'^2\frac{\partial^4\alpha}{\partial x^2\partial y^2}+3y'y''\frac{\partial^3\alpha}{\partial x\partial y^2}+y'^4\frac{\partial^4\alpha}{\partial y^4}]][y']=0 \tag{10}$$

Expanding the equation (10) we have

$$4\beta y^{-2}(y'')^2-2\beta y^{-3}(y')^2(y'')-3\beta y^{-2}(y')(y''')-8(y')^{-3}(y'')^4\frac{\partial\beta}{\partial x}$$

$$-8(y')^{-2}(y'')^4\frac{\partial\beta}{\partial y}+5(y')^{-2}(y'')^2(y''')\frac{\partial\beta}{\partial x}+(y')^{-1}(y'')^2(y''')\frac{\partial\beta}{\partial y}$$

$$+2y^{-2}(y')(y'')^2\frac{\partial\beta}{\partial x}+2y^{-2}(y')^2(y'')^2\frac{\partial\beta}{\partial y}+3y^{-1}(y''')\frac{\partial\beta}{\partial x}+3y^{-1}(y')(y''')\frac{\partial\beta}{\partial y}$$

$$+8(y')^{-2}(y'')^4\frac{\partial\alpha}{\partial x}+8(y')^{-1}(y'')^4\frac{\partial\alpha}{\partial y}-5(y')(y'')^2(y''')\frac{\partial\alpha}{\partial x}-5(y'')^2(y''')\frac{\partial\alpha}{\partial y}$$

$$-2y^{-2}(y')^2(y'')^2\frac{\partial\alpha}{\partial x}-2y^{-2}(y')^3(y'')^2\frac{\partial\alpha}{\partial y}-3y^{-1}(y')(y''')\frac{\partial\alpha}{\partial x}-3y^{-1}(y')^2(y''')\frac{\partial\alpha}{\partial y}$$

$$-8y^{-1}(y'')(y''')\frac{\partial^2\beta}{\partial x^2}-16y^{-1}(y')(y'')(y''')\frac{\partial^2\beta}{\partial x\partial y}-8y^{-1}(y')^2(y'')(y''')\frac{\partial^2\beta}{\partial y^2}$$

$$-8y^{-1}(y'')^2(y''')\frac{\partial\beta}{\partial y}+12(y')^{-2}(y'')^3(y''')\frac{\partial\beta}{\partial y}-5(y')^{-1}(y''')\frac{\partial^2\beta}{\partial x^2}$$

$$-10(y''')\frac{\partial^2\beta}{\partial x\partial y}-5(y')^1(y''')\frac{\partial^2\beta}{\partial y^2}-5(y)^{-1}(y'')(y''')\frac{\partial\beta}{\partial y}+y^{-2}(y')^2\frac{\partial^2\beta}{\partial x^2}$$

$$+2y^{-2}(y')^3\frac{\partial^2\beta}{\partial x\partial y}+y^{-2}(y')^4\frac{\partial^2\beta}{\partial y^2}+y^{-2}(y')^2(y'')\frac{\partial\beta}{\partial y}+16y^{-1}(y'')^2(y''')\frac{\partial\alpha}{\partial x}$$

$$+16y^{-1}(y')(y'')^2(y''')\frac{\partial\alpha}{\partial y}-24(y')^{-2}(y'')^3(y''')\frac{\partial\alpha}{\partial x}-24(y')^{-1}(y'')^3(y''')\frac{\partial\alpha}{\partial y}$$

$$+10(y')^{-1}(y'')(y''')\frac{\partial\alpha}{\partial x}+10(y'')(y''')\frac{\partial\alpha}{\partial y}-2y^{-2}(y')^2(y'')\frac{\partial\alpha}{\partial x}-2y^{-1}(y')^3(y'')\frac{\partial\alpha}{\partial y}$$

$$+8y^{-1}(y')(y'')(y''')\frac{\partial^2\alpha}{\partial x^2}+16y^{-1}(y')^2(y'')(y''')\frac{\partial^2\alpha}{\partial x\partial y}+8y^{-1}(y')^3(y'')(y''')\frac{\partial^2\alpha}{\partial y^2}$$

$$+8y^{-1}(y')(y'')^2(y''')\frac{\partial\alpha}{\partial y}-12(y')^{-1}(y'')^2(y''')\frac{\partial^2\alpha}{\partial x^2}-24(y'')^2(y''')\frac{\partial^2\alpha}{\partial x\partial y}$$

$$-12(y')^1(y'')^2(y''')\frac{\partial^2\alpha}{\partial y^2}-12(y')^{-1}(y'')^3(y''')\frac{\partial\alpha}{\partial y}+5(y''')\frac{\partial^2\alpha}{\partial x^2}+10(y')(y''')\frac{\partial^2\alpha}{\partial x\partial y}$$

$$+5(y')^2(y''')\frac{\partial^2\alpha}{\partial y^2}+5(y'')(y''')\frac{\partial\alpha}{\partial y}-y^{-2}(y')^1\frac{\partial^2\alpha}{\partial x^2}-2y^{-2}(y')^2\frac{\partial^2\alpha}{\partial x\partial y}$$

$$-y^{-2}(y')^3\frac{\partial^2\alpha}{\partial y^2}-y^{-2}(y')^1(y'')\frac{\partial\alpha}{\partial y}-5(y')^{-1}(y'')\frac{\partial^3\beta}{\partial x^3}-15(y'')\frac{\partial^3\beta}{\partial x^2\partial y}$$

$$-5(y')^{-1}(y'')^2\frac{\partial^2\beta}{\partial x\partial y}+5(y')^{-1}(y'')(y''')\frac{\partial\beta}{\partial y}-15(y')(y'')\frac{\partial^3\beta}{\partial x\partial y^2}-15(y'')^2\frac{\partial^2\beta}{\partial y^2}$$

$$-5(y')^2(y'')\frac{\partial^3\beta}{\partial y^3}+15(y')^{-1}(y'')(y''')\frac{\partial\alpha}{\partial x}+15(y'')(y''')\frac{\partial\alpha}{\partial y}+15(y')(y'')^2\frac{\partial^2\alpha}{\partial x^2}$$

$$+30(y'')^2\frac{\partial^2\alpha}{\partial x\partial y}+15(y')^1(y'')^2\frac{\partial^2\alpha}{\partial y^2}+15(y')^{-1}(y'')^3\frac{\partial\alpha}{\partial y}+5(y'')\frac{\partial^3\alpha}{\partial x^3}+15(y')(y'')\frac{\partial^3\alpha}{\partial x^2\partial y}$$

$$+15(y'')^2\frac{\partial^2\alpha}{\partial x\partial y}+5(y'')(y''')\frac{\partial\alpha}{\partial y}+15(y')^2(y'')\frac{\partial^3\alpha}{\partial x\partial y^2}+15(y')(y'')^2\frac{\partial^2\alpha}{\partial y^2}+15(y')^3(y'')\frac{\partial^3\alpha}{\partial y^3}$$

$$+3y^{-1}(y')\frac{\partial^3\beta}{\partial x^3}+9y^{-1}(y')^2\frac{\partial^3\beta}{\partial x^2\partial y}+9y^{-1}(y')(y'')\frac{\partial^2\beta}{\partial x\partial y}+3y^{-1}(y')(y''')\frac{\partial\beta}{\partial y}$$

$$+9y^{-1}(y')^3\frac{\partial^3\beta}{\partial x\partial y^2}+9y^{-1}(y')^2(y'')\frac{\partial^2\beta}{\partial y^2}+3y^{-1}(y')^4\frac{\partial^3\beta}{\partial y^3}-27y^{-1}(y')(y''')\frac{\partial\alpha}{\partial x}$$

$$-27y^{-1}(y')^2(y''')\frac{\partial\alpha}{\partial y}-9y^{-1}(y')(y'')\frac{\partial^2\alpha}{\partial x^2}-18y^{-1}(y')^2(y'')\frac{\partial^2\alpha}{\partial x\partial y}-9y^{-1}(y')^3(y'')\frac{\partial^2\alpha}{\partial y^2}$$

$$-9y^{-1}(y')(y'')^2\frac{\partial\alpha}{\partial y}-3y^{-1}(y')^2\frac{\partial^3\alpha}{\partial x^3}-9y^{-1}(y')^3\frac{\partial^3\alpha}{\partial x^2\partial y}-9y^{-1}(y')^2(y'')\frac{\partial^2\alpha}{\partial x\partial y}$$

$$-3y^{-1}(y')^2(y''')\frac{\partial\alpha}{\partial y}-9y^{-1}(y')^4\frac{\partial^3\alpha}{\partial x\partial y^2}+9y^{-1}(y')^3(y'')\frac{\partial^2\alpha}{\partial y^2}-3y^{-1}(y')^5\frac{\partial^3\alpha}{\partial y^3}$$

$$\frac{\partial^4\beta}{\partial x^4}+4y'\frac{\partial^4\beta}{\partial x^3\partial y}+6y''\frac{\partial^3\beta}{\partial x^2\partial y}+4y'''\frac{\partial^2\beta}{\partial x\partial y}+4y^{-1}(y'')^2\frac{\partial\beta}{\partial y}-4(y')(y'')^3\frac{\partial\beta}{\partial y}+5(y')^{-1}(y'')(y''')\frac{\partial\beta}{\partial y}$$

$$-y^{-2}(y')^2(y'')\frac{\partial\beta}{\partial y}-3y^{-1}(y')(y''')\frac{\partial\beta}{\partial y}+3(y')^2\frac{\partial^4\beta}{\partial x^2\partial y^2}+9(y')(y'')\frac{\partial^3\beta}{\partial x\partial y^2}+4(y')(y''')\frac{\partial^2\beta}{\partial y^2}$$

$$+3(y'')^2\frac{\partial^2\beta}{\partial y^2}+4(y')^3\frac{\partial^4\beta}{\partial x\partial y^3}+6(y')^2y''\frac{\partial^3\beta}{\partial y^3}+3(y')^2\frac{\partial^4\beta}{\partial x^2\partial y^2}+3(y')(y'')\frac{\partial^3\beta}{\partial x\partial y^2}+4(y')^3\frac{\partial^4\beta}{\partial x\partial}$$

$$+6(y')^2y''\frac{\partial^3\beta}{\partial y^3}+3(y')^2\frac{\partial^4\beta}{\partial x^2\partial y^2}+3(y')(y'')\frac{\partial^3\beta}{\partial x\partial y^2}+(y')^4\frac{\partial^4\beta}{\partial y^4}-16y^{-1}(y'')^2\frac{\partial\alpha}{\partial x}-16y^{-1}(y')(y'')^2\frac{\partial\alpha}{\partial y}$$

$$+16(y')^{-2}(y'')^3\frac{\partial\alpha}{\partial x}+16(y)^{-1}(y'')^3\frac{\partial\alpha}{\partial y}-20(y')^{-1}(y'')(y''')\frac{\partial\alpha}{\partial x}+20(y'')(y''')\frac{\partial\alpha}{\partial y}$$

$$+4y^{-2}(y')^2(y'')\frac{\partial\alpha}{\partial x}+4y^{-2}(y')^3(y'')\frac{\partial\alpha}{\partial y}+12y^{-1}(y')(y''')\frac{\partial\alpha}{\partial x}+12y^{-1}(y')^2(y''')\frac{\partial\alpha}{\partial y}$$

$$-6(y''')\frac{\partial^2\alpha}{\partial x^2}-12(y')(y''')\frac{\partial^2\alpha}{\partial x\partial y}-6(y')^2(y''')\frac{\partial^2\alpha}{\partial y^2}-6(y'')(y''')\frac{\partial\alpha}{\partial y}-4(y'')\frac{\partial^3\alpha}{\partial x^3}-12(y')(y'')\frac{\partial^3\alpha}{\partial x^2\partial y}$$

$$-12(y'')^2\frac{\partial^2\alpha}{\partial x\partial y}-4(y'')(y''')\frac{\partial\alpha}{\partial y}-12(y')^2(y'')\frac{\partial^3\alpha}{\partial x\partial y^2}-12(y')(y'')^2\frac{\partial^2\alpha}{\partial y^2}-4(y')^3(y'')\frac{\partial^3\alpha}{\partial y^3}-(y')\frac{\partial^4\alpha}{\partial x^4}$$

$$-4(y')^2\frac{\partial^4\alpha}{\partial x^3\partial y}-6(y')(y'')\frac{\partial^3\alpha}{\partial x^2\partial y}-4(y')(y''')\frac{\partial^2\alpha}{\partial x\partial y}-4y^{-1}(y')(y'')^2\frac{\partial\beta}{\partial y}+4(y')^{-1}(y'')^3\frac{\partial\beta}{\partial y}$$

$$-5(y'')(y''')\frac{\partial\beta}{\partial y}+y^{-2}(y')^3\frac{\partial\beta}{\partial y}+3y^{-1}(y')^2(y''')\frac{\partial\beta}{\partial y}-3(y')^3\frac{\partial^4\alpha}{\partial x^2\partial y^2}-9(y')^2(y'')\frac{\partial^3\alpha}{\partial x\partial y^2}$$

$$-4(y')^2(y''')\frac{\partial^2\alpha}{\partial y^2}-3(y')(y'')^2\frac{\partial^2\alpha}{\partial y^2}-4(y')^4\frac{\partial^4\alpha}{\partial x\partial y^3}-6(y')^3(y'')\frac{\partial^3\alpha}{\partial y^3}-3(y')^3\frac{\partial^4\alpha}{\partial x^2\partial y^2}-3(y')^2(y'')\frac{\partial^3\alpha}{\partial x\partial y^2}$$

$$-(y')^5\frac{\partial^4\alpha}{\partial y^4}=0 \tag{11}$$

The equation is an identity in $x, y, y', y'', y,'''$ that is it holds for any arbitrary choices of $x, y, y', y'', y,'''$ [3]. Since α and β are functions of x and y only, we must equate the coefficients of the powers of $y', y'', y,'''$ and their combinations to zero. We obtain the following systems of partial differential equations known as Determining Equations [3], [4]:

$$(y')^3 y'' y''': 8y^{-1} \frac{\partial^2 \alpha}{\partial y^2} = 0 \tag{12}$$

$$(y')^2 y'' y''': -8y^{-1} \frac{\partial^2 \beta}{\partial y^2} + 16y^{-1} \frac{\partial^2 \alpha}{\partial x \partial y} = 0 \tag{13}$$

$$(y')^1 y'' y''': -16y^{-1} \frac{\partial^2 \beta}{\partial x \partial y} + 8y^{-1} \frac{\partial^2 \alpha}{\partial x^2} = 0 \tag{14}$$

$$(y')^0 y'' y''': -8y^{-1} \frac{\partial^2 \beta}{\partial x^2} + 10\frac{\partial \alpha}{\partial y} + 5\frac{\partial \alpha}{\partial y} - 5\frac{\partial \alpha}{\partial y} + 5\frac{\partial \alpha}{\partial y} - 20\frac{\partial \alpha}{\partial y} - 6\frac{\partial \alpha}{\partial y} - 4\frac{\partial \alpha}{\partial y} = 0$$

$$\Rightarrow -8y^{-1} \frac{\partial^2 \beta}{\partial x^2} + 5\frac{\partial \alpha}{\partial y} - 5\frac{\partial \beta}{\partial y} = 0 \tag{15}$$

II. CONCLUSION

In this paper, we have subjected the Nonlinear Wave Equation to extended generator and constructed the Determining Equations.

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