

OPERATORS WITH SLOWLY GROWING RESOLVENTS
TOWARDS THE SPECTRUM

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Abstract: A closed densely defined operator H , on a Banach space \mathcal{X} , whose spectrum is contained in \mathbb{R} and satisfies

$$\|(z - H)^{-1}\| \leq c \frac{\langle z \rangle^\alpha}{|\Im z|^\beta} \quad \forall z \notin \mathbb{R} \quad (0.1)$$

for some $\alpha, \beta \geq 0$; $c > 0$, is said to be of (α, β) -type \mathbb{R} . If instead of (0.1) we have

$$\|(z - H)^{-1}\| \leq c \frac{|z|^\alpha}{|\Im z|^\beta} \quad \forall z \notin \mathbb{R}, \quad (0.2)$$

then H is of $(\alpha, \beta)'$ -type \mathbb{R} .

Examples of such operators include self-adjoint operators, Laplacian on $L^1(\mathbb{R})$, Schrödinger operators on $L^p(\mathbb{R}^n)$ and operators H whose spectra lie in \mathbb{R} and permit some control on $\|e^{iHt}\|$.

In this paper we will characterise the (α, β) -type \mathbb{R} operators. In particular we show that property (0.1) is stable under dialation by real numbers in the interval $(0,1)$ and perturbation by positive reals. We will also show that is H is of (α, β) -type \mathbb{R} then so is H^2 .

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1. Preliminaries

Suppose H is a closed densely defined operator on a Banach space \mathcal{X} whose spectrum is contained in \mathbb{R} and there exists $c > 0$ such that

$$\|(z - H)^{-1}\| \leq c \frac{\langle z \rangle^\alpha}{|\Im z|^\beta} \tag{1.1}$$

for all $z \notin \mathbb{R}$ and some $\alpha, \beta \geq 0$. We will say that H is of (α, β) -type \mathbb{R} . Here, we define $\langle \cdot \rangle$ by $\langle z \rangle^2 := 1 + |z|^2$ and $\Im z$ denotes the *imaginary part of z* (the *real part of z* will be denoted by $\Re z$). If instead of (1.1) we have

$$\|(z - H)^{-1}\| \leq c \frac{|z|^\alpha}{|\Im z|^\beta} \tag{1.2}$$

for all $z \notin \mathbb{R}$ and some $\alpha, \beta \geq 0$, we will say that H is of $(\alpha, \beta)'$ -type \mathbb{R} .

1.1. Examples of (α, β) -type \mathbb{R} Operators

Proposition 1.1. *Let H be a self-adjoint operator on a Hilbert space \mathcal{H} . Then H is of $(0, 1)$ -type \mathbb{R} .*

Proof. If $S \subset \mathcal{H}$, denote the closure of S by \overline{S} and its orthogonal complement by S^\perp . Let the adjoint of an operator H be denoted by H^* , its *kernel* be $\ker(H) := \{f \in \mathcal{D}(H) \text{ such that } Hf = 0\}$ and suppose $z \in \mathbb{C} \setminus \mathbb{R}$, then

$$\begin{aligned} 2i\Im \langle (z - H)f, f \rangle &= \langle (z - H)f, f \rangle - \overline{\langle (z - H)f, f \rangle} \\ &= 2i\Im z \|f\|^2 \end{aligned}$$

$$\text{if and only if } \Im \langle f, (z - H)f \rangle = \Im z \|f\|^2.$$

$$\text{Which implies } |\Im z| \|f\| \leq \|(z - H)f\|. \tag{1.3}$$

That is, $z - H$ is bounded from below and hence $z - H$ is injective, and so $\ker(z - H) = \{0\}$. Since H is self adjoint, we have

$$\ker((z - H)^*) = \ker(\bar{z} - H) = \{0\}. \tag{1.4}$$

But because H is closed densely defined,

$$\overline{\Re(z - H)} = \ker((z - H)^*)^\perp.$$

Therefore using (1.4),

$$\overline{\Re(z - H)} = \{0\}^\perp = \mathcal{H}.$$

Conclusion:

1. $(z - H)^{-1}$ exists and is bounded,
2. $\Re(z - H)$ is dense in \mathcal{H}

thus $z \in \rho(H)$.

The conclusion of the proposition now follows from (1.3). □

Proposition 1.2. *Let $H_0 = -\frac{d^2}{dx^2}$ on $L^1(\mathbb{R})$ where $\mathfrak{D}(H_0) = \{f \in L^1(\mathbb{R}) : f'' \in L^1(\mathbb{R}), f' \text{ absolutely continuous}\}$. Then*

1. $(z - H_0)^{-1}$ is a convolution operator, for each $z \notin \mathbb{R}$.
2. H_0 is of $(0, 1)$ -type \mathbb{R} .

Proof. Let $z \in \rho(H_0)$. Then

$$(z - \widehat{H_0})^{-1}f(\zeta) = (z - a(\zeta))^{-1}\widehat{f}(\zeta), \quad \zeta \in \mathbb{R}$$

where \widehat{g} denotes Fourier transform of g and the symbol of H_0 , $a(\zeta) = \zeta^2$. So that $(z - \widehat{H_0})^{-1}f(\zeta) = \widehat{g * f}(\zeta)$ where $g * f$ denotes the convolution of g and f . Observe that

$$\widehat{g}(\zeta) := \frac{(2\pi)^{-1/2}}{z - \zeta^2}$$

decays rapidly enough as $|\zeta| \rightarrow \infty$, $\widehat{g} \in L^1(\mathbb{R})$.

$$\text{Thus } (z - H_0)^{-1}f(x) = g * f(x), \quad f \in L^1(\mathbb{R})$$

where $g \in C_0(\mathbb{R})$. In fact $g(x) = \frac{i}{2\sqrt{z}}e^{-i\sqrt{z}|x|}$ explicitly.

This proves the first part of the proposition.

Next

$$\begin{aligned} \|(z - H_0)^{-1}\| &= \sup \{ \|g * f\|_1 : f \in L^1, \|f\|_1 = 1 \} \\ &\leq \|g\|_1 \\ &\quad (\text{since } \|g * f\|_1 \leq \|g\|_1 \|f\|_1, f, g \in L^1) \\ &= \int_{-\infty}^{\infty} \left| \frac{i}{2\sqrt{z}}e^{-i\sqrt{z}|x|} \right| dx. \end{aligned}$$

By means of a change of variable and reflectional symmetry we conclude that

$$\|(z - H_0)^{-1}\| = 2 \int_0^{\infty} \left| \frac{i}{2\sqrt{z}}e^{-i\sqrt{z}|x|} \right| dx \leq \frac{2}{|\Im z|}. \quad \square$$

2. Main Results

Theorem 2.1. *Let H be a bounded operator with $\sigma(H) \subseteq \mathbb{R}$, and*

$$\|e^{iHt}\| \leq C(1 + |t|)^\alpha, \tag{2.1}$$

where α is a non-negative integer. Then H is of $(\alpha, \alpha + 1)$ -type \mathbb{R} .

Proof. Let $z \notin \mathbb{R}$. Since $(z - iH)^{-1} = \begin{cases} \int_0^\infty e^{-zt} e^{iHt} dt & \Re z > 0 \\ -\int_{-\infty}^0 e^{-zt} e^{iHt} dt & \Re z < 0 \end{cases}$

(see Bratteli and Robinson Proposition 3.16, [1]). Therefore for z with $\Re z > 0$ we have

$$\|(z - iH)^{-1} f\| \leq \int_0^\infty e^{-|\Re z|t} \|e^{iHt} f\| dt \leq C \|f\| \int_0^\infty e^{-|\Re z|t} (1+t)^\alpha dt.$$

If $\Re z < 0$, put $s = -t$ then

$$\begin{aligned} \|-(z - iH)^{-1} f\| &\leq \|f\| \int_0^\infty |e^{zs}| \|e^{-iHs}\| ds \leq \int_0^\infty e^{-|\Re z|s} \|e^{-iHs} f\| ds \\ &\leq C \|f\| \int_0^\infty e^{-|\Re z|s} (1+s)^\alpha ds. \end{aligned}$$

But

$$\begin{aligned} \int_0^\infty e^{-|\Re z|t} (1+t)^\alpha dt &= \left[-\frac{1}{|\Re z|} e^{-|\Re z|t} (1+t)^\alpha \right]_0^\infty \\ -\int_0^\infty -\frac{1}{|\Re z|} e^{-|\Re z|t} \alpha (1+t)^{\alpha-1} dt &= \frac{1}{|\Re z|} + \frac{\alpha}{|\Re z|^2} + \frac{\alpha(\alpha-1)}{|\Re z|^3} + \dots + \frac{\alpha!}{|\Re z|^{\alpha+1}}. \end{aligned}$$

Consequently we may conclude that

$$\begin{aligned} \|(z - iH)^{-1} f\| &\leq C \|f\| \frac{1}{|\Re z|} \left[1 + \frac{\alpha}{|\Re z|^1} + \frac{\alpha(\alpha-1)}{|\Re z|^2} + \dots + \frac{\alpha!}{|\Re z|^\alpha} \right] \\ &\leq C \|f\| \frac{\alpha!}{|\Re z|} \left[1 + \frac{\alpha}{1! |\Re z|} + \frac{\alpha(\alpha-1)}{2! |\Re z|^2} + \dots + \frac{\alpha!}{\alpha! |\Re z|^\alpha} \right] \\ &= C \|f\| \frac{\alpha!}{|\Re z|} \left(1 + \frac{1}{|\Re z|} \right)^\alpha \leq 2^{\alpha/2} C \alpha! \frac{\langle z \rangle^\alpha}{|\Re z|^{\alpha+1}} \|f\|, \end{aligned}$$

where we have used Hölder's inequality to obtain

$$1 + |\Re z| \leq \sqrt{2}(1 + |\Re z|^2)^{1/2} \leq \sqrt{2} \langle z \rangle.$$

Now put $w := i^{-1}z$. Then $w \notin \mathbb{R}$ and

$$\begin{aligned} \|(w - H)^{-1} f\| &= \|i(z - iH)^{-1} f\| \leq 2^{\alpha/2} C \alpha! \frac{\langle z \rangle^\alpha}{|\Re z|^{\alpha+1}} \|f\| \\ &= 2^{\alpha/2} C \alpha! \frac{\langle w \rangle^\alpha}{|\Im w|^{\alpha+1}} \|f\|. \quad \square \end{aligned}$$

Remark 2.2. 1. Note that the converse of Theorem 2.1 is false. A counter example is the following:

Let $H_0 = -\frac{d^2}{dx^2}$ on $L^1(\mathbb{R})$. Then by Proposition 1.2, $(z - H_0)^{-1}$ is a convolution operator, for each $z \notin \mathbb{R}$ and is of $(0, 1)$ -type \mathbb{R} .

However, operators e^{iH_0t} are unbounded for all $t \neq 0$, see for example, Brenner et al [2, p. 27].

2. Since the map $z \rightarrow e^{itz}$ is in $H^\infty(\Omega_\epsilon)$, where $\Omega_\epsilon := \{z \in \mathbb{C} : |\Im z| < \epsilon\}$ for some $\epsilon > 0$, one may conjecture that the conclusion of theorem 2.1 holds even for unbounded operators whose spectra lie in Ω_ϵ and admit the bound (2.1). This problem as far as we can tell, remains open.

A Schrödinger operator operator, $H := -\Delta + V$ on $L^p(\mathbb{R}^N)$, $1 \leq p \leq \infty$ considered in this study, has potential V such that $V_+ \in K_N^{loc}$ and $V_- \in K_N$, see the footnote¹. The spectrum of the Schrödinger operator with potential chosen as outlined, is real and independent of p (see, Hempel and Voigt [5], Proposition 4.3(a)). The operators $e^{i\Delta t}$ are however unbounded on $L^p(\mathbb{R}^N)$ for all $t \neq 0$, $N \geq 1$ and for all $p \neq 2$ in the range $[1, \infty]$, Brenner et al p. 27 in [2].

Theorem 2.3. *Let H be a Schrödinger operator on $L^p(\mathbb{R}^N)$ then $H + \lambda$ is $(N, N + 1)'$ - type \mathbb{R} for $\lambda > 0$ large enough.*

Proof. See Pang [8]. □

Theorem 2.4. *Let H be a Schrödinger operator on $L^p(\mathbb{R}^N)$ then H is $(\alpha, \alpha + 1)'$ - type \mathbb{R} for $\alpha := N \left| \frac{1}{p} - \frac{1}{2} \right|$.*

Proof. See S. Nakamura Theorem 8, [7]. □

Theorem 2.5. *Let H be an $n \times n$ matrix with real eigenvalues. Then H is of $(n - 1, n)$ - type \mathbb{R} . H is of $(0, 1)$ - type \mathbb{R} if and only if it is diagonalizable.*

Proof. Consider an $n \times n$ tridiagonal matrix :

$$A = \begin{pmatrix} a_1 & b_1 & & & 0 \\ c_1 & a_2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & a_{n-1} & b_{n-1} \\ 0 & & & c_{n-1} & a_n \end{pmatrix} \tag{2.2}$$

with either $c_i = 0$ for all i or $b_i = 0$ for all i . Then

$$\det(A) = \prod_{i=1}^n a_i. \tag{2.3}$$

¹ $V_-(x) := \max\{0, V(x)\}$ and $V_+(x) := \min\{0, V(x)\}$.

Next, suppose $\lambda_1, \dots, \lambda_n$ are the real eigenvalues of H and $z \notin \mathbb{R}$. Then H has the Jordan canonical form

$$H = \begin{pmatrix} \lambda_1 & b_1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & b_{n-1} \\ 0 & & & \lambda_n \end{pmatrix},$$

where $b_\nu = 0$ or 1 for all ν . Thus

$$(z - H) = \begin{pmatrix} z - \lambda_1 & -b_1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & -b_{n-1} \\ 0 & & & z - \lambda_n \end{pmatrix}.$$

This is a special case of (2.2) with $c_1 = c_2 = \dots = c_{n-1} = 0$. Hence

$$\begin{aligned} \det(z - H) &= \prod_{\nu=1}^n (z - \lambda_\nu) \quad (\text{by (2.3)}) \\ &\neq 0 \end{aligned}$$

(since $z \neq \lambda_\nu$ for all ν). Thus $\text{cof}(z - H) = (d_{ij})$, where

$$d_{ij} = \begin{cases} (-1)^{i+j} (z - \lambda_1) \dots (z - \lambda_{j-1}) (-b_j) \dots (-b_{i-1}) (z - \lambda_{i+1}) \dots (z - \lambda_n), & i > j, \\ \prod_{\nu \neq i}^n (z - \lambda_\nu), & i = j, \\ 0, & i < j. \end{cases}$$

If $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm and A^T is the transpose of the matrix A , then by the above together with the fact that $(z - H)^{-1} = (\det(z - H))^{-1} (\text{cof}(z - H))^T$ we see that

$$\begin{aligned} \|(z - H)^{-1}\|_{HS}^2 &= \left\| [(z - H)^{-1}]^T \right\|_{HS}^2 = \left\| \frac{\text{cof}(z - H)}{\det(z - H)} \right\|_{HS}^2 \\ &= \sum_{i=1}^n \sum_{j=1}^i \left| \frac{\prod_{\substack{1 \leq \nu \leq j-1 \\ i+1 \leq \nu \leq n}} (z - \lambda_\nu) \cdot \prod_{j \leq \nu \leq i-1} (-b_\nu)}{\prod_{\nu=1}^n (z - \lambda_\nu)} \right|^2 \\ &\quad (\text{since } d_{ij} = 0 \text{ for all } i < j). \end{aligned}$$

Here we have used the fact that $\|A\|_{HS}^2 = \sum_1^n \|Ae_i\|^2$ for any orthonormal system e_1, \dots, e_n .

Thus

$$\|(z - H)^{-1}\|_{HS}^2 = \sum_{i=1}^n \sum_{j=1}^i \left| \prod_{j \leq \nu \leq i} \frac{1}{(z - \lambda_\nu)} \cdot \prod_{j \leq \nu \leq i-1} (-b_\nu) \right|^2 \tag{2.4}$$

$$\leq \sum_{i=1}^n \sum_{j=1}^i \prod_{j \leq \nu \leq i} \left| \frac{1}{z - \lambda_\nu} \right|^2 \tag{2.5}$$

(since $|-b_\nu| \leq 1$ for all ν).

In this case $\sigma(H) = \{\lambda_1, \dots, \lambda_n\}$, and we can find $\lambda \in \sigma(H)$ such that

$$\text{dist}(z, \sigma(H)) = |z - \lambda|. \tag{2.6}$$

So, $|z - \lambda| \leq |z - \lambda_i|$ for all i and

$$\|(z - H)^{-1}\|_{HS}^2 \leq \sum_{i=1}^n \sum_{j=1}^i \left| \frac{1}{z - \lambda} \right|^{2(i-j+1)}. \tag{2.7}$$

We observe that

$$\|(z - H)^{-1}\|_{HS} \leq \sqrt{\frac{n(n+1)}{2}} \cdot \frac{\langle z \rangle^{n-1}}{|\Im z|^n} \text{ for all } z \notin \mathbb{R}. \tag{2.8}$$

Since all norms in a finite dimensional space are equivalent, we conclude that

$$\|(z - H)^{-1}\| \leq c \frac{\langle z \rangle^{n-1}}{|\Im z|^n} \text{ for all } z \notin \mathbb{R} \text{ and some } c > 0. \tag{2.9}$$

If H is diagonalizable, then

$$(z - H) = \begin{pmatrix} z - \lambda_1 & & 0 \\ & \ddots & \\ 0 & & z - \lambda_n \end{pmatrix}.$$

This is a special case of (2.2) with $c_i = b_i = 0$ for all $i = 2, 1 \dots n - 1$. In this case $\text{cof}(z - H) = (d_{ij})$ where

$$d_{ij} = \begin{cases} 0 & , \quad i > j, \\ \prod_{\nu \neq i}^n (z - \lambda_\nu) & , \quad i = j, \\ 0 & , \quad i < j, \end{cases}$$

and

$$\|(z - H)^{-1}\|_{HS}^2 \leq \sum_{i=1}^n \left| \frac{\prod_{\nu \neq i} (z - \lambda_\nu)}{\prod_{\nu=1}^n (z - \lambda_\nu)} \right|^2 = \sum_{i=1}^n \left| \frac{1}{z - \lambda_i} \right|^2 \leq \left| \frac{1}{z - \lambda} \right|^2 n$$

with λ chosen as in (2.6).

Therefore

$$\|(z - H)^{-1}\|_{HS} \leq \frac{\sqrt{n}}{|z - \lambda|} \leq \frac{\sqrt{n}}{|\Im z|}. \tag{2.10}$$

Thus,

$$\|(z - H)^{-1}\| \leq \frac{c}{|\Im z|} \text{ for some } c > 0. \tag{2.11}$$

Conversely, if H is not diagonalizable, then from (2.4) we have

$$\|(z - H)^{-1}\|_{HS}^2 = \sum_{i=1}^n \sum_{j=1}^i \left| \prod_{j \leq \nu \leq i} \frac{1}{(z - \lambda_\nu)} \cdot \prod_{j \leq \nu \leq i-1} (-b_\nu) \right|^2$$

with $b_k \neq 0$ for some $k \in \{1, \dots, n - 1\}$. Thus

$$\|(z - H)^{-1}\|_{HS}^2 = \sum_{i=1}^n \left| \frac{1}{z - \lambda_i} \right|^2 + \sum_{i=2}^n \sum_{j=k_i}^{i-1} \left| \prod_{j \leq \nu \leq i-1} \frac{-b}{(z - \lambda_\nu)} \right|^2,$$

where $k_i := \min\{j : b_j, b_{j+1}, \dots, b_{i-1} \neq 0\}$. But $|\Im z| = |\Im(z - \lambda_i)|$ for all $i = 1, \dots, n$ and $|-b_j| = 1$ for all $k_i \leq j \leq i - 1$. So if we set $\Re(z + \lambda) := \max\{\Re(z - \lambda_i) : i = 1, \dots, n\}$ then

$$\begin{aligned} |\Im z|^2 \|(z - H)^{-1}\|_{HS}^2 &> \sum_{i=1}^n \frac{|\Im z|^2}{[|\Im z| + |\Re(z - \lambda)|]^2} \\ &+ \sum_{i=2}^n \frac{|\Im z|^2}{[|\Im z| + |\Re(z - \lambda)|]^2} \sum_{j=k_i}^{i-1} \frac{1}{[|\Im z| + |\Re(z + \lambda)|]^{a_j}} \\ &\text{with } a_j \geq 2 \text{ for all } j. \\ &= nK + \sum_{i=2}^n K \sum_{j=k_i}^{i-1} \frac{1}{[|\Im z| + |\Re(z + \lambda)|]^{a_j}} \end{aligned} \tag{2.12}$$

where $K := \frac{|\Im z|^2}{[|\Im z| + |\Re(z - \lambda)|]^2}$, $K \rightarrow 0$ as $\Im z \rightarrow 0$ but $\frac{1}{[|\Im z| + |\Re(z + \lambda)|]^{a_j}} \rightarrow \infty$ a_i times faster, as $\Im z \rightarrow 0$ for any fixed $\Re(z + \lambda)$. Therefore it follows from (2.12) that there is no $D > 0$ such that $|\Im z|^2 \|(z - H)^{-1}\|_{HS}^2 \leq D$ for all $z \notin \mathbb{R}$. \square

3. Conclusions and Applications

Theorem 3.1. H is of $(0, 1)$ -type \mathbb{R} with the constant $c = 1$ if and only if iH is a generator of a one-parameter group of isometries on \mathcal{X} .

Proof. Necessity. Assume H is of $(0, 1) - type \mathbb{R}$.

Clearly $\pm iH$ are closed densely defined (by the hypothesis on H). Suppose $\lambda > 0$. Then $\lambda \in \rho(\pm iH)$ since $\sigma(H) \subset \mathbb{R}$, and

$$\|(\lambda \pm iH)^{-1}\| = \left\| \frac{1}{i} \left(\frac{\lambda}{i} \pm H \right)^{-1} \right\| = \|(-i\lambda \pm H)^{-1}\| \leq |\Im(\pm i\lambda)|^{-1} = \lambda^{-1}.$$

Thus by Hille-Yosida theorem (See Goldstein [4], p. 16-17), $\pm iH$ are generators of contraction semigroups. Finally, the conclusion follows by invoking Prop. 1.14 of Davies [3].

Sufficiency. Suppose iH is a generator of a group of isometries $\{T(t)\}$.

Then for all $w \in \mathbb{C}$ with $\Re w \neq 0$, $w \in \rho(iH)$ and

$$(\lambda - iH)^{-1} = \begin{cases} \int_0^\infty T(t)e^{-\lambda t} dt & \text{if } \Re \lambda > 0 \\ -\int_\infty^0 T(t)e^{-\lambda t} dt & \text{if } \Re \lambda < 0 \end{cases}$$

(see Bratteli and Robinson, [1, Proposition 3.16]).

From this

$$\|(\lambda - iH)^{-1}\| \leq \int_0^\infty e^{-|\Re \lambda|t} \|T(t)\| dt \leq \int_0^\infty e^{-|\Re \lambda|t} dt = |\Re(\lambda)|^{-1}.$$

Now put $z := \frac{\lambda}{i}$. □

Theorem 3.2. *If H is of $(\alpha, \alpha + 1)' - type \mathbb{R}$ then λH is also of $(\alpha, \alpha + 1)' - type \mathbb{R}$ with the same constant c for all $\lambda > 0$.*

Proof. Let $z \notin \mathbb{R}$, then

$$\begin{aligned} \|(z - \lambda H)^{-1}\| &= \left\| \lambda^{-1} \left(\frac{z}{\lambda} - H \right)^{-1} \right\| \\ &\leq |\lambda^{-1}| c \left| \Im \frac{z}{\lambda} \right|^{-1} \left(\frac{\left| \frac{z}{\lambda} \right|}{\left| \Im \frac{z}{\lambda} \right|} \right)^\alpha \quad (\text{hypothesis}) \\ &= c |\Im z|^{-1} \left(\frac{|z|}{|\Im z|} \right)^\alpha. \end{aligned}$$

Remark 3.3. The type of stability shown in Theorem 3.2 will be called *scale invariance*.

Theorem 3.4. *If H is of $(\alpha, \alpha + 1) - type \mathbb{R}$ then λH is also of $(\alpha, \alpha + 1) - type \mathbb{R}$ with the same constant c for all $0 < \lambda < 1$.*

Proof. Let $z \in \mathbb{C} \setminus \mathbb{R}$, then

$$\|(z - \lambda H)^{-1}\| = \left\| \lambda^{-1} \left(\frac{z}{\lambda} - H \right)^{-1} \right\| \leq |\lambda^{-1}| c \left| \Im \frac{z}{\lambda} \right|^{-1} \left(\frac{\left\langle \frac{z}{\lambda} \right\rangle}{\left| \Im \frac{z}{\lambda} \right|} \right)^\alpha \quad (\text{hypothesis}).$$

Thus

$$\begin{aligned} \|(z - \lambda H)^{-1}\| &\leq |\lambda^{-1}| c |\Im z|^{-1} |\lambda| \left(\frac{\sqrt{\lambda^2 + |z|^2}}{|\Im z|} \right)^\alpha \leq c |\Im z|^{-1} \left(\frac{\sqrt{1 + |z|^2}}{|\Im z|} \right)^\alpha \\ &= c |\Im z|^{-1} \left(\frac{\langle z \rangle}{|\Im z|} \right)^\alpha. \end{aligned}$$

Remark 3.5. The type of stability shown in Theorem 3.4 will be called *scale sub-invariance*.

Theorem 3.6. If H is of (α, β) - type \mathbb{R} then $H + \lambda$ is also of (α, β) - type \mathbb{R} for all $\lambda \in \mathbb{R}$.

Proof. If $\lambda \in \mathbb{R}$ and $z \notin \mathbb{R}$ then $\Im(z - \lambda) = \Im z$. Therefore we get

$$\begin{aligned} \|(z - (H + \lambda))^{-1}\| &= \left\| [(z - \lambda) - H]^{-1} \right\| \leq c \frac{\langle z - \lambda \rangle^\alpha}{|\Im(z - \lambda)|^\beta} \quad (\text{hypothesis}) \\ &\leq c_1 \frac{\langle z \rangle^\alpha}{|\Im z|^\beta}, \end{aligned}$$

where $c_1 = c2^{\alpha/2} \langle \lambda \rangle^\alpha$. □

If H is of $(\alpha, \beta)'$ - type \mathbb{R} then it is also of (α, β) - type \mathbb{R} with the converse not true in general. However the following result provides some sort of converse to this.

Theorem 3.7. If λH is of $(\alpha, \alpha + 1)$ - type \mathbb{R} for all $\lambda > 0$, then H is of $(\alpha, \alpha + 1)'$ - type \mathbb{R} with the same constant c .

Proof. Let $z \notin \mathbb{R}$, then

$$\begin{aligned} \|(z - H)^{-1}\| &= \left\| \lambda (\lambda z - \lambda H)^{-1} \right\| \\ &\leq |\lambda| c |\Im(\lambda z)|^{-1} \left(\frac{\langle \lambda z \rangle}{|\Im(\lambda z)|} \right)^\alpha \quad (\text{hypothesis}) \\ &= c |\Im z|^{-1} \left(\frac{\sqrt{\lambda^{-2} + |z|^2}}{|\Im z|} \right)^\alpha \quad \text{for all } \lambda > 0. \end{aligned}$$

Letting $\lambda \rightarrow \infty$, we observe that

$$\|(z - H)^{-1}\| \leq c |\Im z|^{-1} \left(\frac{|z|}{|\Im z|} \right)^\alpha. \quad \square$$

Conjecture 3.8. If H is of (α, β) - type \mathbb{R} , $\beta \geq \alpha$ and $0 \notin \sigma(H)$ then

H is of $(\alpha + 1, \beta)'$ - type \mathbb{R} .

Theorem 3.9. Let H be of (α, β) - type \mathbb{R} . Then H^2 is of $(\frac{\alpha+\beta-1}{2}, \beta)$ - type \mathbb{R} .

Proof. Given $z \notin \mathbb{R}$, we have

$$\frac{1}{|\Im \sqrt{z}|^2} = \frac{1}{\frac{1}{2} \|z\| - \Re z} \leq \frac{4|z|}{|\Im z|^2}.$$

Also since $\langle \sqrt{z} \rangle^2 \leq \sqrt{2} \langle z \rangle$ and $2\sqrt{z}(z - H^2)^{-1} = (\sqrt{z} - H)^{-1} - (-\sqrt{z} - H)^{-1}$, we get

$$\begin{aligned} \|(z - H^2)^{-1}\| &\leq \frac{1}{2} \left| \frac{1}{\sqrt{z}} \right| \{ \|(\sqrt{z} - H)^{-1}\| + \|(-\sqrt{z} - H)^{-1}\| \} \\ &\leq c \left| \frac{1}{\sqrt{z}} \right| \frac{\langle \sqrt{z} \rangle^\alpha}{|\Im \sqrt{z}|^\beta} \quad (\text{hypothesis}) \\ &\leq \frac{c 2^{\alpha/4+\beta} \langle z \rangle^{(\alpha+\beta-1)/2}}{|\Im z|^\beta}. \quad \square \end{aligned}$$

Proposition 3.10. Let A be of $(\alpha, \alpha + 1)$ - type \mathbb{R} with

$$\|(z - A)^{-1}\| \leq c_1 \frac{\langle z \rangle^\alpha}{|\Im z|^{\alpha+1}} \quad \text{for some } c_1 > 0 \text{ and } \alpha \geq 0$$

and B of $(\beta, \beta + 1)$ - type \mathbb{R} with

$$\|(z - B)^{-1}\| \leq c_2 \frac{\langle z \rangle^\beta}{|\Im z|^{\beta+1}} \quad \text{for some } c_2 > 0 \text{ and } \beta \geq 0.$$

Then

$$\begin{aligned} \|(z - A)^{-1} - (z - B)^{-1}\| \\ \leq (1 + \sqrt{2}c_1)(1 + \sqrt{2}c_2) \|(i + A)^{-1} - (i + B)^{-1}\| \frac{\langle z \rangle^{\alpha+\beta+2}}{|\Im z|^{\alpha+\beta+2}}. \end{aligned}$$

Proof. $(z - A)$ and $(z - A)^{-1}$ commute on $\mathfrak{D}(A)$ and hence $(i + A)$ and $(z - A)^{-1}$ also commute on $\mathfrak{D}(A)$ since by linearity of $(z - A)^{-1}$, for all $g \in \mathfrak{D}(A)$, we have

$$\begin{aligned} (z - A)^{-1}(i + A)g &= (z - A)^{-1} \{ (z + i) - (z - A) \} g \\ &= (z - i)(z - A)^{-1}g - (z - A)^{-1}(z - A)g \\ &= (z - i)(z - A)^{-1}g - (z - A)(z - A)^{-1}g = (i + A)(z - A)^{-1}g. \end{aligned}$$

Also, $(i + A)^{-1}$ maps into $\mathfrak{D}(A)$ and hence $(i + A)$ and $(z - A)^{-1}$ commute on $\mathfrak{R}((i + A)^{-1})$.

The operators $D := (i + A)(z - A)^{-1}$, $H := [(i + A)^{-1} - (i + B)^{-1}]$ and $E := (i + B)(z - B)^{-1}$ are defined everywhere and bounded on \mathcal{X} as can be seen by writing $D = (i + z)(z - A)^{-1} - I$ and $E = (i + z)(z - B)^{-1} - I$.

Now for $f \in \mathcal{X}$ we have

$$\begin{aligned} D [(i + A)^{-1} - (i + B)^{-1}] Ef &= (i + A)(z - A)^{-1} [(i + A)^{-1} - (i + B)^{-1}] (i + B)(z - B)^{-1} f \\ &= - [(z - A)^{-1} - (z - B)^{-1}] f. \end{aligned}$$

Thus

$$\begin{aligned} \|D\| = \|(i + z)(z - A)^{-1} - I\| &\leq 1 + (1 + |z|) \|(z - A)^{-1}\| \\ &\leq (1 + \sqrt{2}c_1) \frac{\langle z \rangle^{\alpha+1}}{|\Im z|^{\alpha+1}}. \end{aligned}$$

Similarly, $\|E\| \leq (1 + \sqrt{2}c_2) \frac{\langle z \rangle^{\beta+1}}{|\Im z|^{\beta+1}}$. Therefore

$$\|(z - A)^{-1} - (z - B)^{-1}\| \leq (1 + \sqrt{2}c_1)(1 + \sqrt{2}c_2) \|H\| \frac{\langle z \rangle^{\alpha+\beta+2}}{|\Im z|^{\alpha+\beta+2}}.$$

Theorem 3.11. *If H is of $(\alpha, \alpha + 1)'$ -type \mathbb{R} , for some $\alpha > 0$, then H admits $C_0^\infty(\mathbb{R})$ functional calculus.*

Proof. See Balabane et al [6], Theorem 4.11. □

In fact we believe that $(\alpha, \alpha + 1)$ -type \mathbb{R} and $(\alpha, \alpha + 1)'$ -type \mathbb{R} operators do admit a much larger functional calculus than $C_0^\infty(\mathbb{R})$.

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