JARAMOGI OGINGA ODINGA UNIVERSITY OF SCIENCE AND TECHNOLOGY

SCHOOL OF MATHEMATICS AND ACTUARIAL SCIENCE

FIRST YEAR FIRST SEMESTER EXAMINATION FOR THE DEGREE OF MASTERS OF SCIENCE IN PURE AND APPLIED MATHEMATICS

COURSE CODE: SMA 803

COURSE TITLE: FUNCTIONAL ANALYSIS

DATE: 1/03/2013 **TIME: 9.00 AM -12.00 PM**

<u>INSTRUCTIONS</u>

1. This examination paper contains **FIVE** questions.
2. Answer **any THREE questions**.

QUESTION ONE (20 marks)

a) State and prove

   i. Hölder's inequality for $\ell^p(\mathrm{k})$ spaces (4 marks)

   ii. Minkowski inequality for $\ell^p(\mathrm{k})$ spaces (5 marks)

b) Show that the metric spaces $(\ell^p(\mathrm{k}),\rho_p)$ are complete for $1 \leq p < \infty$ (8 marks)

c) In case $p=2$, show that $\ell^p(\mathrm{k})$, i.e. $\ell^2(\mathrm{k})$ has inner product function compatible with its norm (3 marks)

QUESTION TWO (20 marks)

a) Let (X,ρ) be a compatible metric space and $f:(X,\rho)\to(X,\rho)$ be a contradiction mapping. show that f has a unique fixed point. If the mapping $f:(X,\rho)\to(X,\rho)$, where (X,ρ) is a complete metric space , is such that f^p (i.e. f composed with itself p times) where p is some positive integer ≥ 2, is a contradiction mapping, show that f has exactly one fixed point (10 marks)

b) Let A, B be nonvoid subsets of a metric space (X,ρ)

   i. If A is compact, show that there exists an $x_0 \in A$ such that $\text{dist.}(\mathrm{x}_0, B) = \rho(A,B)$ (the distance between A and B)

   ii. If A and B are both compact, show that there exists an $x_0 \in A$ and $y_0 \in B$ such that $\rho(\mathrm{x}_0, y_0) = \rho(A,B)$ (10 marks)

QUESTION THREE (20 marks)

a) Let $(X,\|\cdot\|)$ be a n.l.s. If every absolutely convergent series $\sum_n X_n$ of elements $x_n \in X$ is strongly convergent, show that $(X,\|\cdot\|)$ is a Banach space. (10 marks)

b) Let $(X,\|\cdot\|)$ be a Banach space, show that a family $\{x_\alpha : \alpha \in \Lambda\}$ of elements of X is summable if and only if for each real number $\varepsilon > 0$ there exists a finite subset $\prod_\varepsilon$ of Λ such that $\left\|\sum_{\alpha\in\Delta} x_\alpha\right\| < \varepsilon$ whenever Δ is a finite subset of Λ satisfying the condition $\Delta \cap \prod_\varepsilon = \phi$. (10 marks)

QUESTION FOUR (20 marks)

a) Let X, Y be n.l.spaces over k and $T: x \to y$ be a linear transformation which is continuous at a point $x_n \in X$ with respect to the strong (norm) topologies in X, Y. Show that T is bounded. (5 marks)

b) Let X be a n.l.s and Y a Banach space. Let T be a bounded linear transformation defined on a linear subspace D of X into Y. Show that there is a unique linear transformation $\widehat{T}$ defined on the closure $\overline{D}$ of D such that $\widehat{T}$ extends T and that $\|\widehat{T}\| = \|T\|$. (10 marks)

c) If m is a proper closed linear subspace of a n.l,s. $(X,\|\cdot\|)$ and ε is any real number satisfying $0<\varepsilon<1$, show that there exists an element $x_\varepsilon \in X$ such that $\|x_\varepsilon\| = 1$ and $\text{dist.}(\mathrm{x}_\varepsilon, m) \geq \varepsilon$ (5 marks)

QUESTION FIVE (20 marks)

a) Let X be a n.l.s and let m be a linear subspace of X. Suppose that $z \in X$ and $\text{dist.}(z, m) = d > 0$. Show that there exists a $q \in X^*$ such that $q\left(m = \{\bar{o}\}, q(z) = d\right)$, and $\|q\| = 1$. show also that if $m = \{\bar{o}\}$, then we have $q(z) = \|z\|$. (8 marks)

b) Let X and Y be n.l.spaces over k .show that $B(x, y)$ is a linear space over k under the usual operations of sum and scalar multiplication. Show that the mapping $T \mapsto \|T\|, \left(T \in B(x, y)\right)$ is a norm in $B(x, y)$. If Y is a Banach space, show that $\left(B(x, y), \|.\|\right)$ is a Banach space. (12 marks)