NUMERICAL ANALYSIS OF THIRD ORDER ADVECTION-VISCOUS WAVE EQUATION USING FINITE DIFFERENCE METHOD

BY

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Declaration

This thesis is my own work and has not been presented for a degree award in any other institution.

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Abstract

Wave equation is a linear hyperbolic partial differential equation (PDE) which describes the propagation of a variety of waves arising in physical situations. In its simplest form, the one dimension wave equation refers to a scalar function $u = u(x,t)$, which satisfy the PDE $u_{tt} = C^2 u_{xx}$. When the wave propagates in complex media, form of the governing wave equation changes, so in particular a viscous loss in term μu_{xxt} and the advection term αu_x are added to the right hand side of the equation thus: $u_{tt} = C^2 u_{xx} + \mu u_{xxt} + \alpha u_x$. Many researchers have tried to solve this equation analytically and numerically without advection component using laplace transform or finite difference method. In our study we solved viscous wave equation with advection component included. The numerical approach we have used is finite difference method to discretize the equation, its associated initial and boundary conditions by their finite difference analogue. We managed to analyze third order advection viscous eave equations by developing two numerical schemes namely Central Crank Nicolson and Forward Crank Nicolson Difference Schemes analyzed their stability and consistency to ascertain convergence of the schemes to unique solutions of the original equation. We used MATLAB to find the solutions of matrices developed from the algebraic equations and to draw tables and graphs. The trend of graphs shows that the amplitude of wave decreases with increase in distance. We hope that the solution we obtained may be useful in the analysis and prediction of viscous advection wave equation problems which have great application in sound instruments, construction engineering and medicine. 1.544

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PDE	Partial Differential Equation	1
u	function of one spatial dimension and time	2
x	spatial coordinate	2
t	time	2
α	Velocity of the fluid	2
u	dependent variable	2
u_x	partial derivative with respect x	2
u_{xx}	second order partial derivative with respect to x	3
u_{xxt}	Third order partial derivativetwice with respect to x and once t	3
DS	Difference Scheme	3
C	speed of sound propagation	3
μ	kinetic viscosity coefficient	3
u_t	first order partial derivative with respect to t	6
u_{tt}	second order partial derivative with respect to t	6
ODE	Ordinary Differential Equations	10
DBC	Dirichlet boundary conditions	13
C-CNDS	Central-Crank Nicolson Difference Scheme	15
F-CNDS	Forward-Crank Nicolson Difference Scheme	17
OT	other Accumulated Truncation error terms	30
IC	Initial Conditions	35
BC	Boundary Conditions	35

Chapter 1

INTRODUCTION

1.1 Background to the study

Taylor introduced Finite difference which was then studied as abstract self-standing mathematical objects in works by Boole, Thomson and Jordan [26]. In numerical analysis Finite Difference Method is a class of numerical techniques for solving differential equations by approximating derivatives with finite differences. Both spatial domain and time are discretized, or broken into a finite number of steps and the values of the solution at these discrete points is approximated by solving algebraic equations containing finite difference and values from nearby points.

Finite difference method converts ordinary differential equations or partial differential equations which may be linear or non-linear into a system of linear algebraic equations that can be solved by matrix algebraic techniques. Modern computers can perform these linear algebraic computations efficiently which, along with their relative ease of implementation has led to widespread use of finite difference method. In modern numerical analysis, finite difference method is one of the most common approaches to the numerical solution of partial differential equations along with finite element method. Finite different technique is based on the approximation that permits replacing differential equation by a discretisation [21]. The resulting system will be a linear algebraic equations reduced from the PDE in the equation. Sound and music are part of our everyday sensory experience just as eyes are for the detection of light and color, so are ears for the detection of sounds. We seldom take time to ponder the characteristics and behaviors of sound and the mechanisms by which sounds are produced, propagated, and detected. Sound is created by vibrating objects and propagated in form of a wave through a medium from one location to another [5]. A wave can be described as a disturbance that travels through a medium, transporting energy from one location to another. The medium is the material that facilitates propagation of wave energy; it can be thought of as a series of interconnected and interacting particles. The media can be gases, liquids, solids and plasmas.

Waves are placed in two categories: Electromagnetic and mechanical waves. Electromagnetic waves have an electric and magnetic nature and are capable of being transmitted through a vacuum while mechanical waves require a material medium for propagation because they rely on particle interaction for transmission. Sound waves are mechanical

waves which cannot travel through a vacuum. The science of sound including its production, transmission and effects is called acoustics. Acoustic waves are pressure disturbances in the form of vibrational waves that propagate through a compressible medium. These vibrational waves displace the molecules of the medium from their quiescent point, after which a restoring elastic force pulls the molecules back.This elastic force along with inertia causes the molecules to oscillate, allowing acoustic waves to propagate [30].
Sound waves are propagated through a material by the transfer of mechanical energy from one molecule to another; in solids and liquids by intermolecular forces and collisions while in gases by intermolecular collisions alone. From these two facts, Keith [18] concluded that the velocity of sound is greater in solids and liquids than in gases. According to Serway and Faughn [11], a sound wave propagates through an elastic medium as a longitudinal wave as exhibited by compressions and rare factions in the material medium. The speed of the sound wave depends on the value of the Bulk modulus, an elastic property and the equilibrium density, an inertial property of the material through which it is traveling. One of the first phenomena to be modeled with a PDE was wave motion. Waves occur in a variety of physical situations which include vibrating strings, vibrating membranes, waves traveling through a solid media among other situations.
The simplest form of a hyperbolic equation is the advection given by: $u_t = -\alpha u_x$ where $u(x,t)$ is a function of one spatial dimension and time, and α is a velocity that is constant in space and time. This is called advection equation, as it describes the time dependant shifting of function $u(x)$ along x axis with a velocity α. In the field of physics, engineering and Earth science advection is the transport of a substance or quantity by bulk motion. The properties of that substance is carried with it. Generally the majority of the advected substances is a fluid. The properties of the advected substances are conserved example is energy [6]. Reynolds number is the ratio of inertial forces to viscous forces. It is used to determine whether the fluid flow is laminar ($R_n < 2100$) or turbulent ($R_n > 2100$). The central difference approximation works well for low Reynolds Number. The inertial term in the momentum equation becomes bigger and dominant over viscous terms it is therefore necessary to introduce biased differentiating of the advection term to suppress oscillations in the solution [6].

1.2 Basic Concept

1.2.1 Partial Differential Equations

The general second order PDE is of the form $Hu_{xx}+Pu_{xy}+Ju_{yy}+Ku_x+Lu_y+Mu=Q$. Where $H, P, J, K, L, Mand\ Q$ are functions of x and y. The PDE can be classified into three different types: Elliptic, parabolic and hyperbolic. These classifications are done by computing the discriminant $\triangle = P^2 - 4HJ$. The equation $Hu_{xx} + Pu_{xy} + Ju_{yy} + Ku_x + Lu_y + Mu = Q$ is either elliptic, parabolic or hyperbolic depending on the value of $\triangle$ at any point $(x, y) whetheritis <, = or > 0$, respectively.

(i) Elliptic PDEs are either Poisson:

$$u_{xx} + u_{yy} = f(x, y) \tag{1.1}$$

or Laplace:

$$u_{xx} + u_{yy} = 0 \tag{1.2}$$

(ii) Parabolic :

$$u_t = u_{xx} \tag{1.3}$$

(iii) Hyperbolic:

$$u_{tt} = u_{xx} \tag{1.4}$$

which is the equation under our study with a viscous term μu_{xxt} and advection term αu_x:

$$u_{tt} = C^2 u_{xx} + \mu u_{xxt} \tag{1.5}$$

$$u_{tt} = C^2 u_{xx} + \mu u_{xxt} + \alpha u_x \tag{1.6}$$

1.2.2 Finite Difference Method

(a) **Difference scheme** [27]:
A difference scheme is an approximation of a derivative at a point using the collective values of the nearby points. There are three difference schemes.

(i) Central Difference Scheme: This is a numerical scheme which is symmetric about the point at which the derivative is being approximated.

(ii) Forward Difference Scheme: This is the scheme which uses the point "a head" of the point being approximated.

(iii) Backward Difference Scheme: This is the scheme which uses the point "behind" of the point being approximated.

(b) **Grid**
A grid is a set of points called mesh points, in which the solution of a difference equation is approximated. A grid can take any of the following two forms, namely.

(i) A uniform Grid: The points here are uniformly spaced.

(ii) A non-uniform grid: The points here are not uniformly spaced.

1.2.3 Discretization

Continuum models must often be discretized in order to obtain a numerical solution. Discretization is the process of transferring continuum functions, models, variables into discrete counterparts. This process is usually carried out as the first step towards making them suitable for numerical evaluation and implementation on digital computers [32].

1.2.4 A Well posed problem

A system of PDE's is well posed if it satisfies the following conditions [37]

(i) Existance: The problem has a solution.

(ii) Uniqueness: There is no more than one solution.

(iii) Stability: A small change of equation give rise to a small change in solution.

Otherwise it is ill posed.

1.2.5 Closed form solution

(i) In mathematics an expression is said to be a closed form expression if it can be expressed analytically in terms of a finite number of certain 'well known' functions, typically these well known functions are defined to be elementary functions or constants, one variable or elementary arithmetic operations $(\times, \div, -, +)$, n^{th} root, exponential and algorithms.

(ii) An equation is said to be of closed form solution if it solves a given problem in terms of functions and mathematical operations from a given generally accepted set.
An infinite sum would not be considered closed form.

(iii) An equation is considered closed form because it is expressed as a finite number of operations.
closed form solution and numerical solutions are similar in that they both can be evaluated with
a finite number of standard operations.

(iv) Non-Closed form: This is well known infinite series which converges to a sine function [15].

1.2.6 Taylors series expansion

Taylors series expansion of $u(x+h,t)$ and $u(x-h,t)$ about a selected grid point(m,n) as discussed below [21, 26]. Lets assume that the function and its partial derivatives are continuous. Then the finite difference approximation to derivatives can be obtained from Taylors series expansion of two variables [12] to get:

$$u(x+h,t)-u(x,t) = hu_x + \frac{1}{2!}h^2u_{xx} + \frac{1}{3!}h^3u_{xxx} + \frac{1}{4!}h^4u_{xxxx} \tag{1.7}$$

$$+\frac{1}{5!}h^5u_{xxxxx} + ... + OT$$

$$u(x-h,t)-u(x,t)=-hu_x+\frac{1}{2!}h^2u_{xx}+\frac{1}{3!}h^3u_{xxx}+ \tag{1.8}$$

$$\frac{1}{4!}h^4u_{xxxx}+\frac{1}{5!}h^5u_{xxxxx}+...+OT$$

From Equation (1.1) neglecting terms $O(h^2)$ and above and rearranging will give:

$$u_x=\frac{1}{h}\left[u(x+h,t)-u(x,t)\right];O(h) \tag{1.9}$$

Which is a forward difference FD approximation to u_x with an error of order h. Similarly, from Equation (1.2.5.2) neglecting $O(h^2)$ and above terms and rearranging will give.

$$u_x=\frac{1}{h}\left[u(x,t)-(x-h,t)\right];O(h) \tag{1.10}$$

Which is a backward difference BD approximation to u_x with an error of order h. Similarly, subtracting (1.1) from equation (1.2) neglecting $O(h^3)$ and above terms and rearranging will give.

$$u_x=\frac{1}{2h}\left[u(x+h,t)-u(x-h,t)\right];O(h^2) \tag{1.11}$$

This is the central difference CD approximation to u_x and is second order accurate. Similarly adding Equation (1.1) and Equation (1.2) neglecting terms of order $O(h^3)$ and above and rearranging will give.

$$u_{xx}=\frac{1}{h^2}[u(x+h,t)-2u(x,t)+(x-h,t)];O(h^2) \tag{1.12}$$

This is the central difference CD approximation to u_{xx} and is second order accurate. writing $u(x,t)$ as $U_{mh,nk}=U_{m,n}$. Equation (1.9), Equation (1.10), Equation (1.11) and Equation (1.12) will be stated as :

$$u_x=\frac{U_{m+1,n}-Um,n}{h}+O(h)...FD \tag{1.13}$$

$$u_x=\frac{U_{m,n}-Um-1,n}{h}+O(h)...BD \tag{1.14}$$

$$u_x=\frac{U_{m+1,n}-Um-1,n}{2h}+O(h)...CD \tag{1.15}$$

$$u_{xx}=\frac{U_{m+1,n}-2Um,n+U_{m-1,n}}{h^2}+O(h^2)...CD \tag{1.16}$$

Similarly we have approximation for the partial derivatives with respect to time t as listed below:

$$u_t=\frac{U_{m,n+1}-Um,n}{k}+O(k)...FD \tag{1.17}$$

$$u_t=\frac{U_{m,n}-Um,n-1}{k}+O(k)...BD \tag{1.18}$$

$$u_t = \frac{U_{m,n+1} - Um,n-1}{2k} + O(k)...CD \tag{1.19}$$

$$u_{tt} = \frac{U_{m,n+1} - 2Um,n + U_{m,n-1}}{k^2} + O(k^2)...CD \tag{1.20}$$

Which are called Forward Difference, Backward Difference, Central difference and another Central Difference respectively.

1.2.7 Stability of a scheme

Stability considerations are very important in getting the numerical solution of a differential equation, using finite difference methods. The solution of the finite difference equation is said to be stable, if the error do not grow exponentially as we progress from one time step to another. We have employed matrix methods in our analysis. The theory behind matrix method is that the modulus of the eigenvalues of the amplification matrix should be less than unity while the Analyze Stability Of Numerical Scheme For Solving Third order Advection Viscous Wave Equation. Von Numann uses Fourier modes[**?**].

1.3 Statement of the problem

Dean [8] and Rop [35] solved the equation $u_{tt} = C^2 u_{xx} + \mu u_{xxt}$ analytically and numerically respectively without advection component. However, in our study we have analyzed viscous wave equation with advection component included. In our approach we used finite difference method which involves replacing the equation, its associated initial and boundary conditions by their finite difference analogue.

1.4 Objectives of the study

1.4.1 General Objective

The objective of this study was to analyze third order advection-viscous wave equation using finite difference method.

1.4.2 Specific Objectives

Specifically the study was to;

(i) Develop numerical schemes for third order advection viscous wave equation.

(ii) Analyze stability of numerical schemes for solving third order advection viscous wave equation.

(iii) Numerically solve third order advection viscous wave equation.

1.5 Significance of the study

We expect that the results obtained in our study are useful in construction of musical instruments and hearing gadgets, detection and monitoring under water objects and marine animals, in medicine (ultrasound and doppler effect) and in developing further research in the field of finite difference approximations of advection-viscous wave equation problems.

Chapter 2

LITERATURE REVIEW

In this chapter, we reviewed related literature by detailing what related researches have been done in relation to advection-viscous wave equation in terms of when they were done, their associated findings and critique and the advection teerm which is not studied in their work. We then divided them into sections in line with the specific objectives. There are several people who have attempted to analyze wave equations, here are some of them:

2.1 Development of numerical schemes for wave equations

Dean [8] solved third order viscous wave equation analytically using Laplace Transform and obtained the solution:

$$u_{tt} + C^2 u_{xx} = \mu u_{xxt}, 0 < x < 1, 0 < t < T, -\infty < u < \infty \tag{2.1}$$

$$u(x,t) = \frac{2x}{\pi t}\int_0^1 e^{-2t\omega^2}\left[2t(1-\omega^2)^{\frac{1}{2}} - 2x\omega\right] d\omega + \frac{e^{-2t}}{\pi}\int_0^\infty e^{-t\omega}\left[\sin(\frac{2+\omega}{\sqrt{1+\omega}}x) - \sin(2x)\right] d\omega \tag{2.2}$$

This solution obtained has no physical interpretation. Here we saw that there is need for a result that can be interpreted in physical situations and in a more clear way. The equation solved has no advection component, which is part of the equation under our study.

Rao [31] used a variable separable method to solve a second order wave equation;

$$\begin{aligned} u_{tt} + C^2 u_{xx} &= 0, \qquad (2.3) \\ 0 < x < L, t &> 0 \end{aligned}$$

where C is a constant (the velocity of the wave) subject to boundary conditions:

$$u(0,t) = 0, u(L,t) = 0 : t > 0 \tag{2.4}$$

$$u(x,0) = f(x), u_t(x,0) = g(x) \tag{2.5}$$

The analytical solution of the equation has no physical interpretation. In our study we solved the same equation numerically using finite different method with advection and viscous components included.

Verwer [39] studied the numerical time integration of a class of viscous wave equations using Runge-Kutta methods:

$$u_{tt} + C^2 u_{xx} = \mu u_{xxt} + s(x,t), 0 < x < 1, 0 < t < T, -\infty < u < \infty \tag{2.6}$$

where $s(x,t)$ is the source term, subject to Dirichlet boundary conditions and got the non closed solution:

$$u(x,t) = x^2 \sin(t) \tag{2.7}$$

The study performed a general semi-discrete approximation by following the method of lines approach. The authors also discussed a number of stability properties of the semi-discrete approximation. The author introduced the Runge-Kutta method and discussed stability properties such as the algebraic stability and A-stability as well as the idea of the stability region. The results revealed that A-Stable RungeKutta methods are very good candidates for time integration, in particular, diagonally implicit ones. Special attention was paid to the question on how the A-Stability property can be translated to this non-standard class of viscous wave equations. However, it can be noted that the viscous wave equation could be very steep so that for time integration, traditional explicit methods are no longer efficient; it was stated. The current study used the finite-difference numerical method to solve the related advection-viscous wave equation, where the source term, s(x,t) is zero. Muhalia [25]used finite element method on one dimension unsteady heat conduction equation:

$$\varphi \frac{\partial \phi}{\partial t} + k \frac{\partial^2 \phi}{dx^2} = e^{-t}, 0 \leq x \leq 1 \tag{2.8}$$

Cheniguel and Ayandi [7], used the finite elements method by discretized the domain using four-nodded rectangular finite elements which assumed simple and regular region and which could not be applied to complex and irregular regions. Buckingham [24], Solved the equation on the sound field from a moving source in a viscous medium.Zienkiwizc [41] worked on the finite element method in engineering science. Genutchen [38], analyzed the steady-state unconfined flow using the Van Genutchen permeability equation again in a regular rectangular domain. Liu and Ligget [23] used the dual boundary integral equation methods. Arjmand [9] worked on Highly accurate difference scheme for the numerical solution of third order ordinary differential equations .

2.2 Stability analysis of the numerical scheme for solving wave equations

Haixia [13] in their study, performed an analysis for the numerical dispersion and Von Neumann stability criteria of the diffusive-viscous wave equation for second-order FD scheme. They considered the diffusive-viscous wave equation in a 1-D medium that can be mathematically described as follows:

$$u_{tt} + \gamma u_t + \eta u_{xxt} + \nu^2 u_{xx} = 0 : (x,t) \epsilon (-\infty, \infty) \tag{2.9}$$

Where u is the wave field; $\eta \geq 0$ and $\nu \geq 0$ are diffusive and viscous attenuation parameters, respectively. These are the functions of the porosity and the permeability of reservoir rocks together with the viscosity and the density of the fluid;v is the wave propagation velocity coefficient in a non-dispersive medium. The second term in the equation characterizes a diffusively dispersive force, whereas the third term describes the viscosity. In this case, t is the time, and x is the space variables. They also presented it in such a way that the equation being extended to the two-dimensional case:

$$u_{tt} + \gamma u_t + \eta u_{xxt} + u_{zzt} + \nu^2 u_{xx} + u_{zz} = 0 : (x,t) \in (-\infty, \infty) \tag{2.10}$$

The definitions of the variables are the same as Equation (2.9), and the x, z field are the Cartesian coordinates and will vary from negative to positive infinity. They observed that this equation is used to explain the relationship between the frequency dependence of reflections and the liquid saturation in a reservoir which can be quite complex. Von Neumann approach was used to analyze for stability. The results showed that the stability limits and time step vary with the wave equation being considered and also the type of media itself.

Variations also arise from boundary conditions, irregular grids, or properties in the medium. The numerical solution to the viscous type of wave equation is needed in practical applications because it is difficult to obtain the analytical solution in complex media. The authors suggested that the stability condition remains to be solved for the diffusive-viscous wave equation using Finite difference method. This is exactly where the current study came in once more to use the suggested method to solve a related wave equation which is of third order but with a viscous term, test for stability and hence convergence but using eigenvalue matrix approach. The model equation being handled is a variation of Equation (2.10) above. Finally, it was stated that Finite difference is most commonly used in numerical modeling. This wave equation was proposed recently in the field of oil and gas exploration. The low-frequency seismic anomalies related to hydrocarbon reservoirs have lately attracted wide attention. The finite difference method is the most straight forward numerical approach in seismic modeling, and it is also becoming increasingly more important in the seismic industry and structural modeling due to its relative accuracy and computational efficiency. The current study branched to prove for consistency and hence convergence of the solution schemes of the equation.

Jachimaviciene [16] carried out stability using the weak linear stability theory for im-

plicit Von Neumann method with non local boundary conditions. The solution obtained implied convergence to exact solution. Cheniguel [7] solved heat equation using Adomian Decomposition method. This investigation mainly concentrated on the problem of approximation on one dimension wave equation with non-local boundary conditions. The solution converges and exact though it did not consider the advection and viscous components. Rodolf [34] worked on stability of numerical schemes for partial differential equations. Arnold [4] worked on Stability, Consistency and Convergence of numerical discretization.
He concluded that the equation converges to a unique solution.

2.3 Solution of wave equation

Letinen [22] presented an overview of the acoustic wave equation and the common time-domain numerical solution strategies in closed environments. The study used a time-domain numerical solution approach to obtain the solution of the wave equation:

$$u_{tt} = C^2 u_{xx} + f \tag{2.11}$$

Where u is the pressure field, c is the velocity of sound assumed taken as the constant in the surrounding atmosphere, environment and u_{tt} is the second spatial derivative concerning time. The function $f = f(t,x)$, is the force term which is time-dependent. The solution of the wave equation is a time-dependent pressure field:

$$f(t,x); x\epsilon\tau$$

Where τ, denotes the set of points inside the environment to be simulated. This work was mainly on the acoustic wave equation, and the common time-domain numerical solution strategies in closed environments were considered. The equation was presented, and its qualities analyzed. Common principles of numerical approximation of derivatives were then reviewed, computations by Finite Differences and discretisation of the derivatives was done which gave related practical considerations. The study also investigated the stability of the schemes and methods. The method was developed hierarchically but did not present those more advanced concepts which are a limitation to the method used. When larger time steps were taken, the solution was soon ruined by high-magnitude noise as calculation errors built up uncontrollably. The study found that the implicit methods were unconditionally stable. However, despite the precision of the algorithms, direct application of these methods does not yield a practical system for solving for the impulse responses, at least if the whole frequency range of human hearing is to be simulated.
Evericnk [10] in his study, numerically solved a wave equation using the finite element method. This equation is used to model a different kind of waves, for example, sound waves and the oscillation of strings. A special case is the one-dimensional wave equation which can be more simply written as;

$$u_{tt} = C^2 u_{xx}. \tag{2.12}$$

With Dirichlet boundary conditions (DBC) . The function u can be the displacement of a string or membrane of a drum, the height of a water wave or air pressure for sound waves, dependant on the field the equation is being used. The constant c defines the speed at which the wave "moves". The study used a one-dimensional wave equation to illustrate other conditions using the solutions when the function is twice differentiable. A general solution to the one dimensional wave equation derived by dAlemberts was first presented. The study focused on one-and two-dimensional wave equations over a bounded domain D where the spatial boundary of the domain, partial D, was fixed at zero with an initial value in time and initial boundary conditions specified. Finite element method was used to solve the wave equation numerically and interestingly used a hybrid approach of the finite element method to discretize in space and a finite difference method to discretize in time using application of second-order finite-difference approximations and finally solve the resulting system of equations. It was observed that this method allows much freedom with both how to discretize the domain and how to approximate the solutions. The results of the numerical analysis done on the one-dimensional systems revealed that there are restrictions on what kind of waves, the solutions to the wave equation and the numerical method used. When small values of h and t are used, the numerical restrictions approximate those of the wave equation in question. The author recommended that many things could be done differently. In this thesis, a related wave equation with a third-order advection viscous term was solved using finite difference method.

Chapter 3

RESEARCH METHODOLOGY

We use numerical method to analyze third order advection-viscous wave equation. The finite difference method is applied to function approximations to approximate solutions of differential equations. We have applied forward difference and crank Nicolson schemes which have been analyzed for stability and consistency.

3.1 Development of numerical schemes for third order advection viscous wave equation

There are four basic steps of finite difference method:

(i) Discretization of the domain:
The region is discretized to form the grid points, the approximate solution to the problem will be found.

(ii) The dependant variable and its derivatives are approximated by the finite difference approximation, this will lead to either a single explicit equation or a system of linear differential equations. Non-linear problems gives non-linear system of equations which cannot be solved directly [19]. Consequently, the equations are solved in a one dimensional form, though the principles of the method are also applicable to multi dimensional solutions [2].

(iii) Where necessary or/and available, the values of the dependant variables are initialized. These initial values may be exact as to a real solution for the transient problem, or they may be approximate (rough guess) for a steady solution.

(iv) the computation cycle begins.The approximations are used to calculate values of the dependant variables at all the solution region. Normally for the evaluation problem we calculate the solution from one time level to the next time level. For the steady state solutions the rough guess gives the improved solution which will then be used as input for the next iteration until values remains unchanged when computed at successive time level. In the latter state a steady state solution is assumed to have been reached [17]. Convergence criterion is specified in order to decide when sufficiently accurate solution has been reached [1].

3.1.1 Central-Crank Nicolson difference scheme

(i) Centered-forward difference approximation for u_{xxt}
Centered difference approximation was used foru_{xx} and a forward difference approximation in time t was used for u_t to obtain the approximation for third order viscous term $u_{xxt} = \frac{\partial}{\partial t}(u_{xx})$. This yields:

$$u_{xx} = \frac{1}{h^2}\left[U_{m+1,n} - 2U_{m,n} + U_{m-1,n}\right] + O(h^2) \tag{3.1}$$

and the first derivative in time t yield:

$$u_t = \frac{1}{k}\left[U_{m,n+1} - U_{m,n}\right] + O(k) \tag{3.2}$$

Therefore the third order term becomes.

$$u_{xxt} = \frac{1}{k}\left[(U_{xx})_{m,n+1} - (U_{xx})_{m,n}\right]; O(h^2) + O(k) \tag{3.3}$$

This becomes:

$$\begin{aligned} u_{xxt} = \frac{1}{k}[\frac{1}{h_2}[U_{m+1,n+1} - 2Um,n+1 + U_{m-1,n+1}] \\ -\frac{1}{h_2}[U_{m+1,n} - 2Um,n + U_{m-1,n}]] \\ +O(h^2) + O(k) \end{aligned} \tag{3.4}$$

When simplified and ignoring other higher terms we get:

$$\begin{aligned} u_{xxt} = \frac{1}{kh^2}[U_{m+1,n+1} - 2Um,n+1 + U_{m-1,n+1} \\ -U_{m+1,n} + 2Um,n - U_{m-1,n}] + O(h^2) + O(k) \end{aligned} \tag{3.5}$$

Simplifying further.

$$\begin{aligned} u_{xxt} = \frac{1}{kh^2}[U_{m+1,n+1} + Um-1,n+1- \\ (U_{m-1,n+1} + U_{m-1,n}) + 2(Um,n - U_{m,n+1})] + O(h^2) + O(k) \end{aligned} \tag{3.6}$$

(iii) Crank Nicolson implicit difference approximation for u_{xx}
Although the explicit method is computationally simple, it has one serious drawback The time step $\delta t = k$ is necessarily very small because the process is valid only for $0 < \frac{k}{h^2} \leq \frac{1}{2}$ i.e $k \leq \frac{1}{2}h^2$, and $h = \delta x$ must be kept small in order to attain a reasonable accuracy. Crank and Nicolson(1947) proposed and used a method that reduces the total value of calculation and is valid (i.e convergent and stable), for all finite values of r. they considered partial differential equations as being satisfied at the midpoint $\{ih, (j+\frac{1}{2})k\}$ and replaced $\frac{\partial^2 u}{\partial x^2}$ by the mean of it's finite difference approximations at the n^{th} and $(n+1)^{th}$ time levels.

This is the average of central difference approximation at n^{th} and $(n+1)^{th}$ time levels

$$\begin{aligned} u_{xx} = \frac{1}{2}[\frac{1}{h^2}[U_{m+1,n+1} - 2U_{m,n+1} + U_{m-1,n+1}] \\ +\frac{1}{h^2}[U_{m+1,n} - 2U_{m,n} + U_{m-1,n}]] \end{aligned} \qquad (3.7)$$

(iv) Central-Crank Nicolson difference scheme

We replace u_{tt} with it's Centered difference approximation, u_{xx}
with it's Crank Nicolson approximation, u_{xxt} with it's Centered forward
difference approximation and u_x with it's central
difference approximation to yield:

$$\begin{aligned} \frac{1}{k^2}[U_{m,n+1} - 2U_{m,n} + U_{m,n-1}] &= \\ \frac{C^2}{2}[\frac{1}{h^2}[U_{m+1,n+1}- \\ 2U_{m,n+1} + U_{m-1,n+1}] + \frac{1}{h^2}[U_{m+1,n}- \\ 2U_{m,n} + U_{m-1,n}]] + \frac{\mu}{kh^2}[U_{m+1,n+1} \\ -2U_{m,n+1} + U_{m-1,n+1} - U_{m+1,n} + 2U_{m,n} - U_{m-1,n}]+ \\ \frac{\alpha}{2h}[U_{m+1,n} - U_{m-1,n}] + O(h + h^2 + k^2) \end{aligned} \qquad (3.8)$$

Let the mesh ratios be ,

$$\begin{aligned} r &= \frac{k^2 C^2}{2h^2} \\ \beta &= \frac{\mu k}{h^2} \\ and\, \xi &= \frac{k^2 \alpha}{2h} \end{aligned} \qquad (3.9)$$

$$\begin{aligned}
U_{m,n+1} - 2U_{m,n} + U_{m,n-1} =& \\
r[U_{m+1,n+1} - 2U_{m,n+1} + U_{m-1,n+1}& \\
+U_{m+1,n} - 2U_{m,n} + U_{m-1,n}]+& \\
\beta[U_{m+1,n+1} - 2U_{m,n+1} + U_{m-1,n+1} - U_{m+1,n} + 2U_{m,n} - U_{m-1,n}]+& \\
\xi[U_{m+1,n} - U_{m-1,n}]; O(h^2), O(k^2)&
\end{aligned} \qquad (3.10)$$

Removing the brackets we get:

$$\begin{aligned}
U_{m,n+1} - 2U_{m,n} + U_{m,n-1} =& \\
rU_{m+1,n+1} - 2rU_{m,n+1} + rU_{m-1,n+1}& \\
+rU_{m+1,n} - 2rU_{m,n} + rU_{m-1,n}& \\
+\beta U_{m+1,n+1} - 2\beta U_{m,n+1} + \beta U_{m-1,n}& \\
-\beta U_{m+1,n} + 2\beta U_{m,n} - \beta U_{m-1,n}& \\
+\xi U_{m+1,n} - \xi U_{m-1,n}; O(h^2), O(k^2)&
\end{aligned} \qquad (3.11)$$

On rearranging we get:

$$\begin{aligned}
-rU_{m-1,n+1} - \beta U_{m-1,n+1} + U_{m,n+1}+& \\
2rU_{m,n+1} + 2\beta U_{m,n+1} - rU_{m+1,n+1} - \beta U_{m+1,n+1}& \\
= rU_{m-1,n} - \beta U_{m-1,n} - \xi U_{m-1,n} + 2U_{m,n} - 2rU_{m,n}& \\
+2\beta U_{m,n} + rU_{m+1,n} - \beta U_{m+1,n} + \xi U_{m+1,n} - U_{m,n-1}&
\end{aligned} \qquad (3.12)$$

Factorizing we get;

$$\begin{aligned}
-(r+\beta)U_{m-1,n+1} + (1+2r+2\beta)U_{m,n+1} - (r+\beta)U_{m+1,n+1} =& \\
(r-\beta-\xi)U_{m-1,n} + (2-2r+2\beta)U_{m,n} + (r-\beta+\xi)U_{m+1,n} - U_{m,n-1}&
\end{aligned} \qquad (3.13)$$

Which is the Central-Crank Nicolson Difference Scheme.

3.1.2 Forward-Crank Nicolson difference scheme

We replace u_{tt} with it's Centered difference approximation, u_{xx} with it's Crank Nicolson approximation, u_{xxt} with it's Centered forward difference approximation and u_x with it's forward difference approximation to yield:

$$\frac{1}{k^2}\left[U_{m,n+1} - 2U_{m,n} + U_{m,n-1}\right] = \frac{C^2}{2}\left[\frac{1}{h^2}[U_{m+1,n+1} - 2U_{m,n+1} + U_{m-1,n+1}] + \frac{1}{h^2}[U_{m+1,n} - 2U_{m,n} +\right.$$

$$+\frac{\mu}{kh^2}\left[U_{m+1,n+1}-2U_{m,n+1}+U_{m-1,n+1}-U_{m+1,n}+2U_{m,n}-U_{m-1,n}\right]+\frac{\alpha}{h}\left[U_{m+1,n}-U_{m,n}\right]+O(h+$$

Let the mesh ratios be

$$\begin{aligned} r &= \frac{k^2C^2}{2h^2} \\ \beta &= \frac{\mu k}{h^2} \\ and\, \xi &= \frac{k^2\alpha}{h} \end{aligned} \tag{3.16}$$

$$\begin{aligned} U_{m,n+1}-2U_{m,n}+U_{m,n-1} &= \\ r[U_{m+1,n+1}-2U_{m,n+1}+U_{m-1,n+1}+U_{m+1,n}-2U_{m,n} & \\ +U_{m-1,n}] & \\ +\beta[U_{m+1,n+1}-2U_{m,n+1}+U_{m-1,n+1} & \\ -U_{m+1,n}-2U_{m,n}+U_{m-1,n}] & \\ +\xi[U_{m+1,n}-U_{m,n}]; O(h^2), O(k^2) & \end{aligned} \tag{3.17}$$

Removing the brackets we get:

$$\begin{aligned} U_{m,n+1}-2U_{m,n}+U_{m,n-1} & \\ = rU_{m+1,n+1}-2rU_{m,n+1}+rU_{m-1,n+1}+rU_{m+1,n} & \\ -2rU_{m,n}+rU_{m-1,n} & \quad , \\ +\beta U_{m+1,n+1} & \end{aligned} \tag{3.18}$$

$$\begin{aligned} -2\beta U_{m,n+1}+\beta U_{m-1,n+1} & \\ -\beta U_{m+1,n}+2\beta U_{m,n}+\beta U_{m-1,n} & \\ +\xi U_{m+1,n}-\xi U_{m,n}; O(h^2), O(k^2) & \end{aligned} \tag{3.19}$$

On rearranging we get:

$$\begin{aligned} -rU_{m+1,n+1}-\beta U_{m+1,n+1}+2rU_{m,n+1} & \\ +2\beta U_{m,n+1}+U_{m,n+1} & \\ -rU_{m-1,n+1}-\beta U_{m-1,n+1} & \end{aligned} \tag{3.20}$$

$$
\begin{aligned}
&= rU_{m+1,n} - \beta U_{m+1,n} + \\
&+\xi U_{m+1,n} + 2U_{m,n} - 2rU_{m,n} - \xi U_{m,n} + \\
&rU_{m-1,n} - \\
&\beta U_{m-1,n} - U_{m,n-1}
\end{aligned}
\tag{3.21}
$$

On factorizing, the scheme becomes:

$$
\begin{aligned}
-(r+\beta)U_{m+1,n+1} + (1+2r+2\beta)U_{m,n+1} - (r+\beta)U_{m-1,n+1} \\
= (r+\beta+\xi)U_{m+1,n} + (2-2r+2\beta-\xi)U_{m,n} + (r-\beta)U_{m-1,n} - U_{m,n-1}
\end{aligned}
\tag{3.22}
$$

3.2 Stability analysis of the schemes

Stability of a scheme refers to a qualitative measure of well-posedness of the discrete problem. It is considered a very important feature in getting the numerical solutions of a differential equation using finite difference method. We have analyzed the stability of the numerical scheme for the third order advection-viscous wave equation [29, 34].

Example 1

Consider third order viscous wave Equation [35]

$$u_{tt} + C^2 u_{xx} = \mu u_{xxt} \tag{3.23}$$

Whose Centered Difference Scheme is:

$$-\beta U_{m-1,n+1} + (2\beta+1)U_{m,n+1} - \beta U_{m+1,n+1} = (r-\beta)U_{m-1,n} + \tag{3.24}$$

$$(2\beta+2+2r)U_{m,n} + (r-\beta)U_{m+1,n} - U_{m,n-1}; O(h^2) + O(k^2)$$

Where $r = \frac{k^2}{h^2}v$ and $\beta = \frac{kc^2}{h^2}$. The stability analysis using matrix method shows that scheme (3.33) for Equation (3.3.2) satisfies the stability requirement given as;

$$\left| \frac{1+4\beta\sin^2(\frac{xm}{m+1})}{2+4(\beta-r)\sin^2(\frac{xm}{m+1})} \right| \leq 1 \tag{3.25}$$

Implying that $r > 0$,$\beta > 0$, $r \geq \frac{1}{4}$, $r < \beta$ and $0 \leq x \leq 5$ The condition achieved is the required Stability criterion. Thus the centered forward difference is unconditionally stable.

3.2.1 Stability analysis for central Crank Nicolson difference scheme

We use the matrix method to analyze stability of the scheme (3.22) Expanding this scheme by taking $m = 1, 2, 3, ...(E-1)$, we get the system of equations

$$\begin{aligned} -(r+\beta)U_{m-1,n+1} + (1+2r+2\beta)U_{m,n+1} - (r+\beta)U_{m+1,n+1} = \\ (r-\beta-\xi)U_{m-1,n} + (2-2r+2\beta)U_{m,n} + (r-\beta+\xi)U_{m+1,n} + U_{m,n-1} \end{aligned}$$

$$\begin{aligned} -(r+\beta)U_{0,n+1} + (1+2r+2\beta)U_{1,n+1} - (r+\beta)U_{2,n+1} = \\ (r-\beta-\xi)U_{0,n} + (2-2r+2\beta)U_{1,n} + (r-\beta+\xi)U_{2,n} + U_{1,n-1} \end{aligned}$$

$$\begin{aligned} -(r+\beta)U_{1,n+1} + (1+2r+2\beta)U_{2,n+1} - (r+\beta)U_{3,n+1} = \\ (r-\beta-\xi)U_{1,n} + (2-2r+2\beta)U_{2,n} + (r-\beta+\xi)U_{3,n} + U_{2,n-1} \end{aligned}$$

$$\begin{aligned} -(r+\beta)U_{2,n+1} + (1+2r+2\beta)U_{3,n+1} - (r+\beta)U_{4,n+1} = \\ (r-\beta-\xi)U_{2,n} + (2-2r+2\beta)U_{3,n} + (r-\beta+\xi)U_{4,n} + U_{3,n-1} \end{aligned}$$

$$\begin{aligned} -(r+\beta)U_{3,n+1} + (1+2r+2\beta)U_{4,n+1} - (r+\beta)U_{5,n+1} = \\ (r-\beta-\xi)U_{3,n} + (2-2r+2\beta)U_{4,n} + (r-\beta+\xi)U_{5,n} + U_{4,n-1} \end{aligned}$$

$$\begin{aligned} -(r+\beta)U_{4,n+1} + (1+2r+2\beta)U_{5,n+1} - (r+\beta)U_{6,n+1} = \\ (r-\beta-\xi)U_{4,n} + (2-2r+2\beta)U_{5,n} + (r-\beta+\xi)U_{6,n} + U_{5,n-1} \end{aligned}$$

$$\vdots$$

$$\vdots$$

$$\vdots$$

$$\begin{aligned}-(r+\beta)U_{E-3,n+1}+(1+2r+2\beta)U_{E-2,n+1}-(r+\beta)U_{E,n+1}=\\(r-\beta-\xi)U_{E-3,n}+(2-2r+2\beta)U_{E-2,n}+(r-\beta+\xi)U_{E-1,n}+U_{m,n-1}\end{aligned}$$

$$\begin{aligned}-(r+\beta)U_{E-2,n+1}+(1+2r+2\beta)U_{E-1,n+1}-(r+\beta)U_{E,n+1}=\\(r-\beta-\xi)U_{E-2,n}+(2-2r+2\beta)U_{E-1,n}+(r-\beta+\xi)U_{E,n}+U_{E-1,n-1}\end{aligned}$$

$$\begin{bmatrix}(1+2r+2\beta) & -(r+\beta) & \cdots & 0 & 0\\ -(r+\beta) & (1+2r+2\beta) & \cdots & 0 & 0\\ 0 & -(r+\beta) & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & -(r+\beta) & 0\\ 0 & 0 & \cdots & (1+2r+2\beta) & -(r+\beta)\\ 0 & 0 & \cdots & -(r+\beta) & (1+2r+2\beta)\end{bmatrix}\begin{bmatrix}U_{1,n+1}\\ U_{2,n+1}\\ U_{3,n+1}\\ \vdots\\ U_{E-3,n+1}\\ U_{E-2,n+1}\\ U_{E-1,n+1}\end{bmatrix}=$$

$$\begin{bmatrix}(2-2r+2\beta) & (r-\beta-\xi) & \cdots & 0 & 0\\ (r-\beta+\xi) & (2-2r+2\beta) & \cdots & 0 & 0\\ 0 & (r-\beta+\xi) & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & (r-\beta-\xi) & 0\\ 0 & 0 & \cdots & (2-2r+2\beta) & (r-\beta-\xi)\\ 0 & 0 & \cdots & (r-\beta+\xi) & (2-2r+2\beta)\end{bmatrix}\begin{bmatrix}U_{1,n}\\ U_{2,n}\\ U_{3,n}\\ \vdots\\ U_{E-3,n}\\ U_{E-2,n}\\ U_{E-1,n}\end{bmatrix}$$

$$+\begin{bmatrix}(r+\beta)U_{0,n+1}+(r-\beta-\xi)U_{0,n}-U_{1,n-1}\\ \vdots\\ \vdots\\ \vdots\\ (r+\beta)U_{E,n+1}+(r-\beta-\xi)U_{E,n}-U_{E+1,n-1}\end{bmatrix}$$

Let matrix K be given as: $K=$

$$\begin{bmatrix}(1+2r+2\beta) & -(r+\beta) & \cdots & 0 & 0\\ -(r+\beta) & (1+2r+2\beta) & \cdots & 0 & 0\\ 0 & -(r+\beta) & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & -(r+\beta) & 0\\ 0 & 0 & \cdots & (1+2r+2\beta) & -(r+\beta)\\ 0 & 0 & \cdots & -(r+\beta) & (1+2r+2\beta)\end{bmatrix}$$

And matrix L be given as:

$L=$

$$\begin{bmatrix} (2-2r+2\beta) & (r-\beta-\xi) & \cdots & 0 & 0 \\ (r-\beta+\xi) & (2-2r+2\beta) & \cdots & 0 & 0 \\ 0 & (r-\beta+\xi) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & (r-\beta-\xi) & 0 \\ 0 & 0 & \cdots & (2-2r+2\beta) & (r-\beta-\xi) \\ 0 & 0 & \cdots & (r-\beta+\xi) & (2-2r+2\beta) \end{bmatrix}$$

And matrix b_1 be given as:

$$\begin{bmatrix} (r+\beta)U_{0,n+1} + (r-\beta-\xi)U_{0,n} - U_{1,n-1} \\ \vdots \\ \vdots \\ \vdots \\ (r+\beta)U_{E,n+1} + (r-\beta-\xi)U_{E,n} - U_{E+1,n-1} \end{bmatrix}$$

The matrix can be written as:

$$KU_{m,n+1} = LU_{m,n} + b_1 \tag{3.26}$$

or

$$U_{m,n+1} = K^{-1}LU_{m,n} + K^{-1}b \tag{3.27}$$

Substituting for K and L we get

$$U_{m,n+1} = [(I-(r+\beta))P_{E-1}]^{-1}[(2I+(r-\beta-\xi))Q_{E-1}]U_{m,n} \tag{3.28}$$

$$+[(I-(r+\beta))P_{E-1}]^{-1}b_1 \tag{3.29}$$

$$U_{m,n+1} = BU_{m,n} + D \tag{3.30}$$

Matrix $B = [(I-(r+\beta))P_{E-1}]^{-1}[(2I+(r-\beta-\xi))Q_{E-1}]$
Is the amplification Matrix.

The condition for stability is that each of the eigenvalue of the amplification matrix should be less than 1 General formula for eigenvalues for tri-diagonal matrix λ is:

$$\lambda = a_{11} + 2a_{12}(1 - 2sin^2(\frac{xm}{m+1})) \tag{3.31}$$

Eigenvalue for Matrix K

$$\begin{aligned} a_{11} &= 1+2r+2\beta & (3.32)\\ a_{12} &= -(r+\beta) & (3.33)\end{aligned}$$

$$\lambda = (1+2r+2\beta)+2(-r-\beta)(1-2sin^2(\frac{xm}{m+1})) \quad (3.34)$$

Opening brackets and simplifying yields.

$$\lambda_1 = 1+4(r+\beta)sin^2(\frac{xm}{m+1}) \quad (3.35)$$

Eigenvalue for Matrix L

$$\begin{aligned} b_{11} &= 2-2r+2\beta & (3.36)\\ b_{12} &= r-\beta-\xi & (3.37)\\ b_{21} &= r-\beta+\xi & (3.38)\end{aligned}$$

$$\lambda_2 = (2-2r+2\beta)+2\sqrt{(r-\beta-\xi)(r-\beta+\xi)}(1-2sin^2(\frac{xm}{m+1})) \quad (3.39)$$

Eigenvalue for amplification matrix B is given by:

$$\frac{\lambda_1}{\lambda_2} = \frac{1+4(r+\beta)sin^2(\frac{xm}{m+1})}{(2-2r+2\beta)+2\sqrt{(r-\beta-\xi)(r-\beta+\xi)}(1-2sin^2(\frac{xm}{m+1}))} \quad (3.40)$$

For Central Crank Nicolson Difference Scheme to be stable

$$\frac{1}{2}\left|\frac{1+4(r+\beta)sin^2(\frac{xm}{m+1})}{(1-r+\beta)+\sqrt{(r-\beta-\xi)(r-\beta+\xi)}(1-2sin^2(\frac{xm}{m+1}))}\right| \leq 1 \quad (3.41)$$

Simplifying Further:

$$\frac{1}{2}\left|\frac{1+4(r+\beta)sin^2(\frac{xm}{m+1})}{(1-r+\beta)+\sqrt{(r^2-2r\beta+\beta^2-\xi^2)}(1-2sin^2(\frac{xm}{m+1}))}\right| \leq 1 \tag{3.42}$$

Implying that:

$$r>0,\beta>0,\xi>0,r\geq\frac{1}{4},r<\beta,\beta<\xi and\, 0\leq x\leq 5 \tag{3.43}$$

The condition achieved shows that the required stability criterion has been satisfied. Thus C-CNDS is unconditionally stable.

3.2.2 Stability analysis for forward - Crank Nicolson difference scheme

We use the matrix method to analyze stability of the scheme (3.22) Expanding this scheme by taking $m = 1, 2, 3, ..(E-1)$, we get the system of equations.

$$\begin{gathered}-(r+\beta)U_{m-1,n+1}+(1+2r+2\beta)U_{m,n+1}-(r+\beta)U_{m+1,n+1}\\=(r-\beta)U_{m-1,n}+(2-2r+2\beta-\xi)U_{m,n}+(r+\beta+\xi)U_{m+1,n}-U_{m,n-1}\end{gathered}$$

$$\begin{gathered}-(r+\beta)U_{0,n+1}+(1+2r+2\beta)U_{1,n+1}-(r+\beta)U_{2,n+1}\\=(r-\beta)U_{0,n}+(2-2r+2\beta-\xi)U_{1,n}+(r+\beta+\xi)U_{2,n}-U_{1,n-1}\end{gathered}$$

$$\begin{gathered}-(r+\beta)U_{1,n+1}+(1+2r+2\beta)U_{2,n+1}-(r+\beta)U_{3,n+1}\\=(r-\beta)U_{1,n}+(2-2r+2\beta-\xi)U_{2,n}+(r+\beta+\xi)U_{3,n}-U_{2,n-1}\end{gathered}$$

$$\begin{gathered}-(r+\beta)U_{2,n+1}+(1+2r+2\beta)U_{3,n+1}-(r+\beta)U_{4,n+1}\\=(r-\beta)U_{2,n}+(2-2r+2\beta-\xi)U_{3,n}+(r+\beta+\xi)U_{4,n}-U_{3,n-1}\end{gathered}$$

$$\begin{gathered}-(r+\beta)U_{3,n+1}+(1+2r+2\beta)U_{4,n+1}-(r+\beta)U_{5,n+1}\\=(r-\beta)U_{3,n}+(2-2r+2\beta-\xi)U_{4,n}+(r+\beta+\xi)U_{5,n}-U_{4,n-1}\end{gathered}$$

$$\vdots$$

$$\vdots$$

$$\vdots$$

$$\begin{aligned}&-(r+\beta)U_{E-3,n+1}+(1+2r+2\beta)U_{E-2,n+1}-(r+\beta)U_{E-1,n+1}\\ &=(r-\beta)U_{E-3,n}+(2-2r+2\beta-\xi)U_{E-2,n}+(r+\beta+\xi)U_{E,n}-U_{E-2,n-1}\end{aligned}$$

$$\begin{aligned}&-(r+\beta)U_{E-2,n+1}+(1+2r+2\beta)U_{E-1,n+1}-(r+\beta)U_{E,n+1}\\ &=(r-\beta)U_{E-2,n}+(2-2r+2\beta-\xi)U_{E-1,n}+(r+\beta+\xi)U_{E,n}-U_{E-1,n-1}\end{aligned}$$

$$\begin{bmatrix}(1+2r+2\beta) & -(r+\beta) & \cdots & 0 & 0\\ -(r+\beta) & (1+2r+2\beta) & \cdots & 0 & 0\\ 0 & -(r+\beta) & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & -(r+\beta) & 0\\ 0 & 0 & \cdots & (1+2r+2\beta) & -(r+\beta)\\ 0 & 0 & \cdots & -(r+\beta) & (1+2r+2\beta)\end{bmatrix}\begin{bmatrix}U_{1,n+1}\\ U_{2,n+1}\\ U_{3,n+1}\\ \vdots\\ U_{E-3,n+1}\\ U_{E-2,n+1}\\ U_{E-1,n+1}\end{bmatrix}=$$

$$\begin{bmatrix}(2-2r+2\beta-\xi) & (r+\beta+\xi) & \cdots & 0 & 0\\ (r-\beta) & (2-2r+2\beta-\xi) & \cdots & 0 & 0\\ 0 & (r-\beta) & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & (r+\beta+\xi) & 0\\ 0 & 0 & \cdots & (2-2r+2\beta-\xi) & (r+\beta+\xi)\\ 0 & 0 & \cdots & (r-\beta) & (2-2r+2\beta-\xi)\end{bmatrix}\begin{bmatrix}U_{1,n}\\ U_{2,n}\\ U_{3,n}\\ \vdots\\ U_{E-3,n}\\ U_{E-2,n}\\ U_{E-1,n}\end{bmatrix}$$

$$-\begin{bmatrix}-(r+\beta)U_{0,n+1}+(r-\beta)U_{0,n}-U_{1,n-1}\\ \vdots\\ \vdots\\ \vdots\\ -(r+\beta)U_{E,n+1}+(r-\beta)U_{E,n}-U_{E+1,n-1}\end{bmatrix}$$

Let R represent matrix

$$\begin{bmatrix} (1+2r+2\beta) & -(r+\beta) & \cdots & 0 & 0 \\ -(r+\beta) & (1+2r+2\beta) & \cdots & 0 & 0 \\ 0 & -(r+\beta) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -(r+\beta) & 0 \\ 0 & 0 & \cdots & (1+2r+2\beta) & -(r+\beta) \\ 0 & 0 & \cdots & -(r+\beta) & (1+2r+2\beta) \end{bmatrix}$$

Let T represent:

$$\begin{bmatrix} (2-2r+2\beta-\xi) & (r+\beta+\xi) & \cdots & 0 & 0 \\ (r-\beta) & (2-2r+2\beta-\xi) & \cdots & 0 & 0 \\ 0 & (r-\beta) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & (r+\beta+\xi) & 0 \\ 0 & 0 & \cdots & (2-2r+2\beta-\xi) & (r+\beta+\xi) \\ 0 & 0 & \cdots & (r-\beta) & (2-2r+2\beta-\xi) \end{bmatrix}$$

Matrix b_2 to represent:

$$-\begin{bmatrix} -(r+\beta)U_{0,n+1}+(r-\beta)U_{0,n}-U_{1,n-1} \\ \vdots \\ \vdots \\ \vdots \\ -(r+\beta)U_{E,n+1}+(r-\beta)U_{E,n}-U_{E+1,n-1} \end{bmatrix}$$

The matrix can be written as:

$$RU_{m,n+1} \quad = \quad TU_{m,n}+b_1 \tag{3.44}$$

or

$$U_{m,n+1} \quad = \quad R^{-1}TU_{m,n}+R^{-1}b \tag{3.45}$$

Substituting for R and T we get

$$U_{m,n+1} = [(I-(r+\beta))P_{E-1}]^{-1}[(2I+(r-\beta-\xi))Q_{E-1}]U_{m,n} \quad (3.46)$$
$$+[(I-(r+\beta))P_{E-1}]^{-1}b \quad (3.47)$$

$$U_{m,n+1} = BU_{m,n} + D \quad (3.48)$$

Matrix $B = [(I-(r+\beta))P_{E-1}]^{-1}[(2I+(r-\beta-\xi))Q_{E-1}]$
Is the amplification Matrix.

The condition for stability is that each of the eigenvalue of the amplification matrix should be less than 1 General formula for eigenvalues for tri-diagonal matrix λ is:

$$\lambda = a_{11} + 2a_{12}(1 - 2sin^2(\frac{xm}{m+1})) \quad (3.49)$$

Eigenvalue for Matrix R

$$a_{11} = 1 + 2r + 2\beta \quad (3.50)$$
$$a_{12} = -(r+\beta) \quad (3.51)$$

$$\lambda_1 = (1 + 2r + 2\beta) + 2(-r-\beta)(1 - 2sin^2(\frac{xm}{m+1})) \quad (3.52)$$

Opening brackets and simplifying yields.

$$\lambda_1 = 1 + 4(r+\beta)sin^2(\frac{xm}{m+1}) \quad (3.53)$$

Eigenvalue for Matrix T

$$\begin{aligned} b_{11} &= 2 - 2r + 2\beta - \xi \\ b_{12} &= r + \beta + \xi \\ b_{21} &= r - \beta \end{aligned} \quad (3.54)$$

$$\lambda_2 \quad = \quad (2-2r+2\beta-\xi)+2\sqrt{(r+\beta+\xi)(r-\beta)}(1-2sin^2(\frac{xm}{m+1})) \tag{3.55}$$

Eigenvalue for amplification matrix B is given by:

$$\frac{\lambda_1}{\lambda_2} \quad = \quad \frac{1+4(r+\beta)sin^2(\frac{xm}{m+1})}{(2-2r+2\beta-\xi)+2\sqrt{(r+\beta+\xi)(r-\beta)}(1-2sin^2(\frac{xm}{m+1}))} \tag{3.56}$$

For Forward Crank Nicolson Difference Scheme to be stable

$$\left|\frac{1+4(r+\beta)sin^2(\frac{xm}{m+1})}{(2-2r+2\beta-\xi)-2\sqrt{(r+\beta+\xi)(r-\beta)}(1-2sin^2(\frac{xm}{m+1}))}\right| \leq 1 \tag{3.57}$$

Simplifying further.

$$\frac{1}{2}\left|\frac{1+4(r+\beta)sin^2(\frac{xm}{m+1})}{(1-r+\beta-\xi)-\sqrt{(r^2-2r\beta+\beta2-\xi^2)}(1-2sin^2(\frac{xm}{m+1}))}\right| \leq 1 \tag{3.58}$$

Implying that:

$$r>0, \beta>0, \xi>0, r \geq \frac{1}{4}, r<\beta, \beta<\xi and\, 0 \leq x \leq 5 \tag{3.59}$$

The condition achieved shows that the required stability criterion has been satisfied. Thus F-CNDS is unconditionally stable.

3.3 Consistency of the Schemes

The original equation has to be recovered from the algebraic equations to ascertain consistency. This is considered as the minimum requirement for any discretisation[27]. Consistency has been demonstrated in terms of a Taylors expansion of the discretized equations for C-CNDS and F-CNDS.

Consistency simply means that as δt and δx vanishes, the solution of the equation must satisfy the scheme with errors that vanish. Consistency guarantees that the scheme truly approximates the equation we intend to solve with it(not something else)[34].

3.3.1 Consistency of Central Crank Nicolson Difference Scheme

The order of a differential equation is derived from the term with the highest derivative in our case the viscous term is the third order derivative, two in space and one in time. Consistency dictates that the original equation is recovered from the final scheme, This was done using Taylor's series. Expansion of the equation was done as discussed in chapter one. This involves combining this application with Pascals Triangle to ease the expansion. The finite difference schemes were used to analyze consistency with the original equation. Centered difference schemes (CD) were used for both temporal and spatial second derivatives. While for first derivative still in time,FD scheme was used. This is discussed below by first re-writing the equation in numerical notations thus: We have developed scheme for C-CNDS as:

$$-(r+\beta)U_{m-1,n+1}+(1+2r+2\beta)U_{m,n+1}-(r+\beta)U_{m+1,n+1}=(r-\beta-\xi)U_{m-1,n}+(2-2r+2\beta)$$

$$U_{m+1,n+1} = u(x,t)+(hu_x+ku_t)+\frac{1}{2!}(h^2u_{xx}+2h_{x,t}+k^2u_{tt})+\frac{1}{3!}(h^3u_{xxx}+3kh^2u_{xxt}+3k^2hu_{xtt}$$

$$U_{m,n+1} = u(x,t)+(ku_t)+\frac{1}{2!}(k^2u_{tt})\frac{1}{3!}(k^3u_{ttt})+...+OT \tag{3.62}$$

$$U_{m-1,n+1} = u(x,t)+(-hu_x+ku_t)+\frac{1}{2!}(h^2u_{xx}-2h_{x,t}+k^2u_{tt})+\frac{1}{3!}(-h^3u_{xxx}+3kh^2u_{xxt}-3k^2h$$

$$U_{m+1,n} = u(x,t)+(hu_x)+\frac{1}{2!}(h^2u_{xx})+\frac{1}{3!}(h^3u_{xxx})+...+OT \tag{3.64}$$

$$U_{m,n} = u(x,t) \tag{3.65}$$

$$U_{m-1,n} = u(x,t)-(hu_x)+\frac{1}{2!}(h^2u_{xx})-\frac{1}{3!}(h^3u_{xxx})+...+OT \tag{3.66}$$

$$U_{m,n-1} = u(x,t)-(ku_t)+\frac{1}{2!}(k^2u_{tt})$$

$$-\frac{1}{3!}(k^3u_{ttt})...+OT \tag{3.67}$$

Combining Terms

$$U_{m+1,n+1} + U_{m-1,n+1} \quad = \quad 2[u(x,t) + ku_t + \frac{1}{2}h^2u_{xx} +$$

$$\frac{1}{2}k^2u_{tt} + \frac{3}{6}h^2ku_{xxt} \qquad (3.68)$$

$$U_{m+1,n} + U_{m-1,n} \quad = \quad 2[u(x,t) + \frac{1}{2}h^2u_{xx}$$

$$+\frac{1}{24}k^4u_{xxxx} + ... + OT \qquad (3.69)$$

Subtracting and Doubling Terms

$$U_{m,n} - U_{m,n+1} \quad = \quad -2[ku_t + \frac{1}{2!}k^2u_{tt} + \frac{1}{3!}k^3u_{ttt} + \frac{1}{4!}k^4u_{tttt}...... + OT \qquad (3.70)$$

Difference of The Two Sums

$$(U_{m+1,n+1} - U_{m-1,n+1}) - (U_{m+1,n} + U_{m-1,n}) =$$

$$2[ku_t + \frac{1}{2!}k^2u_{tt} + \frac{3}{3!}h^2ku_{xxt} + \frac{1}{3!}k^2u_{ttt} + ... + OT \qquad (3.71)$$

Adding and Expressing Equations (3.79) and (3.80) in Taylors Series we get

$$(U_{m,n} - U_{m,n+1}) + [(U_{m+1,n+1} - U_{m-1,n+1}) - (U_{m+1,n} + U_{m-1,n})] =$$

$$\begin{aligned}
&-2[ku_t + \frac{1}{2!}k^2u_{tt} + \frac{1}{3!}k^3u_{ttt} + \frac{1}{4!}k^4u_{tttt} \\
&2[u(x,t) + ku_t + \frac{1}{2!}h^2u_{xx}\frac{1}{2!}k^2u_{tt} + \frac{3}{3!}h^2ku_{xxt} + \\
&\frac{1}{3!}k^3u_{ttt} + \frac{1}{4!}h^4u_{xxxx} + \frac{6}{4!}h^2k^2u_{xxtt}] \\
&-2[u(x,t) + \frac{1}{2}h^2u_{xx} + \frac{1}{24}h^4u_{xxxx}] + ... + OT \qquad (3.72)
\end{aligned}$$

Re-arranging and simplifying equation (3.81) we get:

$$\begin{aligned} u_{xxt} &= \frac{1}{kh^2}[h^2ku_{xtt}+\frac{1}{2}h^2k^2u_{xxtt}+ \\ \frac{1}{12}h^4k^{u}_{xxxxt}+\frac{1}{6}^{2}h^2k^3u_{xxttt}+...+OT & \end{aligned} \tag{3.73}$$

Multiplying both sides of equation (3.82) by μ we get:

$$\begin{aligned} \mu u_{xxt} &= \frac{\mu}{kh^2}[h^2ku_{xtt}+\frac{1}{2}h^2k^2u_{xxtt}+ \\ \frac{1}{12}h^4k^{u}_{xxxxt}+\frac{1}{6}^{2}h^2k^3u_{xxttt}+...+OT & \end{aligned} \tag{3.74}$$

$$\mu u_{xxt} = \frac{\mu}{kh^2}[h^2ku_{xtt}]+O(k)+O(h^2) \tag{3.75}$$

This implies that original third order term has been recovered.

Recovering term of second order in time u_{tt}

Combining equations (3.76), - twice (3.73) and (3.71) we get:

$$\begin{aligned} U_{m,n-1}-2U_{m,n}+U_{m,n+1} &= u(x,t) \\ -ku_t+\frac{1}{2!}ku_{tt}-\frac{1}{3!}k^3u_{ttt} & \\ -2u(x,t) & \\ u(x,t)+ku_t+\frac{1}{2!}ku_{tt}+\frac{1}{3!}k^3u_{ttt} & \end{aligned} \tag{3.76}$$

This Yields.

$$\begin{aligned} U_{m,n-1}-2U_{m,n}+U_{m,n+1} &= k^2u_{tt} \\ \frac{1}{k^2}[U_{m,n-1}-2U_{m,n}+U_{m,n+1}] &= u_{tt} \end{aligned} \tag{3.77}$$

The first term has been recovered.

Recovering term of second order in space u_{xx}

Combining Approximations:

$$\begin{array}{r}
U_{m+1,n+1} - 2U_{m,n+1} + U_{m-1,n+1} + U_{m+1,n} - 2U_{m,n} + U_{m-1,n} = \\
u(x,t) + (hu_x + ku_t) \\
+\frac{1}{2!}(h^2u_{xx} + 2h_{x,t} + k^2u_{tt}) \\
+\frac{1}{3!}(h^3u_{xxx} + 3kh^2u_{xxt} + 3k^2hu_{xtt} + k^3u_{ttt}) \\
-2[u(x,t) + (ku_t) + \frac{1}{2!}(k^2u_{tt})\frac{1}{3!}(k^3u_{ttt})] \\
u(x,t) + (-hu_x + ku_t) \\
+\frac{1}{2!}(h^2u_{xx} - 2h_{x,t} + k^2u_{tt}) \\
+\frac{1}{3!}(-h^3u_{xxx} + 3kh^2u_{xxt} - 3k^2hu_{xtt} + k^3u_{ttt}) \\
+[u(x,t) + (ku_t) + \frac{1}{2!}(k^2u_{tt}) + \\
\frac{1}{3!}(k^3u_{ttt})] \\
-2u(x,t) \\
\frac{1}{3!}(h^3u_{xxx})]
\end{array}$$

$$\begin{aligned}
U_{m+1,n+1} - 2U_{m,n+1} + U_{m-1,n+1} + U_{m+1,n} - 2U_{m,n} & \\
+U_{m-1,n} &= 2h^2u_{xx}
\end{aligned} \tag{3.78}$$

Further simplification yields

$$\begin{aligned}
\frac{1}{2h^2}[U_{m+1,n+1} - 2U_{m,n+1} + U_{m-1,n+1} & \\
+U_{m+1,n} - 2U_{m,n} + U_{m-1,n}] &= u_{xx}
\end{aligned} \tag{3.79}$$

Multiplying both sides by C^2 we get:

$$\begin{aligned}
\frac{C^2}{2h^2}[U_{m+1,n+1} - 2U_{m,n+1} + U_{m-1,n+1} & \\
+U_{m+1,n} - 2U_{m,n} + U_{m-1,n}] &= C^2u_{xx}
\end{aligned} \tag{3.80}$$

Recovering term of first order in space u_x

Subtracting Equation (3.75) from Equation (3.73) we get:

$$U_{m+1,n} = u(x,t) + (hu_x) + \frac{1}{2!}(h^2u_{xx}) + \frac{1}{3!}(h^3u_{xxx}) + ... + OT \tag{3.81}$$

$$U_{m-1,n} = u(x,t) - (hu_x) + \frac{1}{2!}(h^2u_{xx}) - \frac{1}{3!}(h^3u_{xxx}) + ... + OT \tag{3.82}$$

$$U_{m+1,n} - U_{m-1,n} \quad = \quad [u(x,t) + (hu_x) + \frac{1}{2!}(h^2u_{xx}) + \frac{1}{3!}(h^3u_{xxx})]$$
$$-[u(x,t) + (hu_x) + \frac{1}{2!}(h^2u_{xx}) + \frac{1}{3!}(h^3u_{xxx})]$$

$$U_{m+1,n} - U_{m-1,n} = (2hu_x) + ... + OT \tag{3.83}$$

Dividing both sides by $2h$ and multiplying by α yields:

$$\frac{\alpha}{2h}[U_{m+1,n} - U_{m-1,n}] = \alpha u_x) + ... + OT \tag{3.84}$$

This is the recovered advection term.

Finally combining all recovered terms (3.86),(3.89),(3.84) and (3.89) we get:

$$u_{tt} = C^2u_{xx} + \mu u_{xxt} + \alpha u_x) + ... + OT \tag{3.85}$$

Hence the C-CNDS shows consistency to the original third order advection viscous wave equation.

Chapter 4

RESULTS AND DISCUSSION

4.1 Introduction

The third order advection viscous wave equation solution is presented and an analysis of effects of varying the mesh ratio parameters. The Finite Difference numerical schemes are quoted as developed in chapter three namely Equation (3.22) and Equation (3.31) respectively. The finite difference schemes are used to solve equation (1.6). The numerical results obtained are presented for varying values of h,k, t and also m, n iteratively, displayed and discussed in one-dimensional manner. The subscripts m and n-designates the grid displacements along the x-direction and along the t-direction respectively. Therefore m is varied as $m = 1, 2, 3, ..., 10$ with m and n fixed as $n = 1$ and $m = 1, 2, 3, ..., 10$. Linear algebraic equations obtained were discussed below. In finding the solutions of equation (1.6), we used C-CNDS first to get the solution results with the suitable initial and boundary conditions. F-CNDS followed a similar procedure.

Boundary and initial conditions

The initial and boundary conditions given below includes elements within the domain which were used in case one, two and three.

$$u_t(x,0) = Sin(x_m)\ x \in (0,L)$$

$$u(x,0) = 0, x \in (0,L)$$

$$u(0,t) = 0, t \in (0,T)$$

$$u(0,t) = Sin(t) : t > 0$$

$$u(L,t) = 0 : t > 0, t\epsilon(0,T)$$

Case one:

To find the solution of the Third Order Advection-Viscous Wave Equation at four levels namely $n = 1,2,3,4$. Difference Schemes and Matlab software were used. An approximate temporal mesh size considered for the calculation is taken for different values of $h, k, r, \beta, \xi andt$ we get the results as follows for given values thus:

$$h = \frac{1}{2}, k = \frac{1}{2}, r = \frac{1}{2}, \beta = 2and\xi = \frac{1}{2}$$

The schemes become:

(i) **C-CNDS**

$$\begin{aligned} -(r+\beta)U_{m-1,n+1} + (1+2r+2\beta)U_{m,n+1} - (r+\beta)U_{m+1,n+1} \\ = (r-\beta+\xi)U_{m-1,n} + (2-2r+2\beta)U_{m,n} + (r-\beta+\xi)U_{m+1,n} - U_{m,n-1} \end{aligned}$$

$$\begin{aligned} -2.5U_{m-1,n+1} + 6U_{m,n+1} - 2.5U_{m+1,n+1} \\ = -2U_{m-1,n} + 5U_{m,n} - U_{m+1,n} - U_{m,n-1} \end{aligned}$$

(ii) **F-CNDS**

$$\begin{aligned} -(r+\beta)U_{m-1,n+1} + (1+2r+2\beta)U_{m,n+1} - (r+\beta)U_{m+1,n+1} \\ = (r+\beta+\xi)U_{m-1,n} + (2-2r+2\beta-\xi)U_{m,n} + (r-\beta)U_{m+1,n} - U_{m,n-1} \end{aligned}$$

$$\begin{aligned}&-2.5U_{m-1,n+1}+6U_{m,n+1}-2.5U_{m+1,n+1}\\&=3U_{m-1,n}+4.5U_{m,n}-1.5U_{m+1,n}-U_{m,n-1}\end{aligned}$$

The solution at $t=0$ and $t=1$ are known and referred as the zero and the first time level solutions respectively. We fix n(from $n=1$) and vary m iteratively as follows:

4.2 Central Crank Nicholson difference scheme: Case One

Using the C-CNDS $h=k=0.5, r=\xi=0.5 and \beta=2$ to obtain a system of solutions as shown below:

n=1(t=0.5)

$$\begin{aligned}
m=1: -2.5U_{0,2}+6U_{1,2}-2.5U_{2,2} &= -2U_{0,1}+5U_{1,1}-U_{2,1}-U_{1,0}\\
m=2: -2.5U_{1,2}+6U_{2,2}-2.5U_{3,2} &= -2U_{1,1}+5U_{2,1}-U_{3,1}-U_{2,0}\\
m=3: -2.5U_{2,2}+6U_{3,2}-2.5U_{4,2} &= -2U_{2,1}+5U_{3,1}-U_{4,1}-U_{3,0}\\
m=4: -2.5U_{3,2}+6U_{4,2}-2.5U_{5,2} &= -2U_{3,1}+5U_{4,1}-U_{5,1}-U_{4,0}\\
m=5: -2.5U_{4,2}+6U_{5,2}-2.5U_{6,2} &= -2U_{4,1}+5U_{5,1}-U_{6,1}-U_{5,0}\\
m=6: -2.5U_{5,2}+6U_{6,2}-2.5U_{7,2} &= -2U_{5,1}+5U_{6,1}-U_{7,1}-U_{6,0}\\
m=7: -2.5U_{6,2}+6U_{7,2}-2.5U_{8,2} &= -2U_{6,1}+5U_{7,1}-U_{8,1}-U_{7,0}\\
m=8: -2.5U_{7,2}+6U_{8,2}-2.5U_{9,2} &= -2U_{7,1}+5U_{8,1}-U_{9,1}-U_{8,0}\\
m=9: -2.5U_{8,2}+6U_{9,2}-2.5U_{10,2} &= -2U_{8,1}+5U_{9,1}-U_{10,1}-U_{9,0}\\
m=10: -2.5U_{9,2}+6U_{10,2}-2.5U_{11,2} &= -2U_{9,1}+5U_{10,1}-U_{11,1}-U_{10,0}
\end{aligned}$$

This is a system of equations obtained using the C-CNDS algorithm when $n=1$ with the initial and boundary conditions, the system of equations can be written in matrix vector form as shown:

$$\begin{bmatrix}
6 & -2.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-2.5 & 6 & -2.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -2.5 & 6 & -2.5 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -2.5 & 6 & -2.5 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -2.5 & 6 & -2.5 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -2.5 & 6 & -2.5 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -2.5 & 6 & -2.5 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -2.5 & 6 & -2.5 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.5 & 6 & -2.5\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.5 & 6
\end{bmatrix}
\begin{bmatrix}
U_{1,2}\\ U_{2,2}\\ U_{3,2}\\ U_{4,2}\\ U_{5,2}\\ U_{6,2}\\ U_{7,2}\\ U_{8,2}\\ U_{9,2}\\ U_{10,2}
\end{bmatrix}$$

$$= \begin{bmatrix}
1.144826385\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0
\end{bmatrix}$$

The solution to $u(x,t)$ obtained using MATLAB software when $m=1,2,3,...,10$ and

$n = 1$ are as follows:
$U_{1,2} = 0.04170, U_{2,2} = 0.07006, U_{3,2} = 0.07609, U_{4,2} = 0.06669, U_{5,2} = 0.05818,$
$U_{6,2} = 0.04909, U_{7,2} = 0.04154, U_{8,2} = 0.03351, U_{9,2} = 0.02351, U_{10,2} = 0.01610.$

n=2(t=1)

$$\begin{aligned}
m=1: -2.5U_{0,3}+6U_{1,3}-2.5U_{2,3} &= -2U_{0,2}+5U_{1,2}-U_{2,2}-U_{1,1}\\
m=2: -2.5U_{1,3}+6U_{2,3}-2.5U_{3,3} &= -2U_{1,2}+5U_{2,2}-U_{3,2}-U_{2,1}\\
m=3: -2.5U_{2,3}+6U_{3,3}-2.5U_{4,3} &= -2U_{2,2}+5U_{3,2}-U_{4,2}-U_{3,1}\\
m=4: -2.5U_{3,3}+6U_{4,3}-2.5U_{5,3} &= -2U_{3,2}+5U_{4,2}-U_{5,2}-U_{4,1}\\
m=5: -2.5U_{4,3}+6U_{5,3}-2.5U_{6,3} &= -2U_{4,2}+5U_{5,2}-U_{6,2}-U_{5,1}\\
m=6: -2.5U_{5,3}+6U_{6,3}-2.5U_{7,3} &= -2U_{5,2}+5U_{6,2}-U_{7,2}-U_{6,1}\\
m=7: -2.5U_{6,3}+6U_{7,3}-2.5U_{8,3} &= -2U_{6,2}+5U_{7,2}-U_{8,2}-U_{7,1}\\
m=8: -2.5U_{7,3}+6U_{8,3}-2.5U_{9,3} &= -2U_{7,2}+5U_{8,2}-U_{9,2}-U_{8,1}\\
m=9: -2.5U_{8,3}+6U_{9,3}-2.5U_{10,3} &= -2U_{8,2}+5U_{9,2}-U_{10,2}-U_{9,1}\\
m=10: -2.5U_{932}+6U_{10,3}-2.5U_{11,3} &= -2U_{9,2}+5U_{10,2}-U_{11,2}-U_{10,1}
\end{aligned}$$

This is a system of equations obtained using the C-CNDS algorithm when $n=2$ with the initial and boundary conditions, the system of equations can be written in matrix vector form as shown:

$$\begin{bmatrix}
6 & -2.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-2.5 & 6 & -2.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -2.5 & 6 & -2.5 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -2.5 & 6 & -2.5 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -2.5 & 6 & -2.5 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -2.5 & 6 & -2.5 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -2.5 & 6 & -2.5 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -2.5 & 6 & -2.5 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.5 & 6 & -2.5\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.5 & 6
\end{bmatrix}
\begin{bmatrix}
U_{1,3}\\ U_{2,3}\\ U_{3,3}\\ U_{4,3}\\ U_{5,3}\\ U_{6,3}\\ U_{7,3}\\ U_{8,3}\\ U_{9,3}\\ U_{10,3}
\end{bmatrix}$$

$$= \begin{bmatrix}
1.089355\\ 0.34299\\ 0.30702\\ 0.23945\\ 0.20661\\ 0.17063\\ 0.14303\\ 0.10798\\ 0.06663\\ 0.03384
\end{bmatrix}$$

The solution to $u(x,t)$ obtained using MATLAB software when $m = 1,2,3,...,10$ and

$n = 2$ are as follows:
$U_{1,3} = 0.0317, U_{2,3} = 0.06106, U_{3,3} = 0.06909, U_{4,3} = 0.0569, U_{5,3} = 0.05118$,
$U_{6,3} = 0.03909, U_{7,3} = 0.03454, U_{8,3} = 0.02551, U_{9,3} = 0.0141, U_{10,4} = 0.0101.$

n=3(t=1.5)

$$\begin{aligned}
m=1: -2.5U_{0,4}+6U_{1,4}-2.5U_{2,4} &= -2U_{0,3}+5U_{1,3}-U_{2,3}-U_{1,2}\\
m=2: -2.5U_{1,4}+6U_{2,4}-2.5U_{3,4} &= -2U_{1,3}+5U_{2,3}-U_{3,3}-U_{2,2}\\
m=3: -2.5U_{2,4}+6U_{3,4}-2.5U_{4,4} &= -2U_{2,3}+5U_{3,3}-U_{4,3}-U_{3,2}\\
m=4: -2.5U_{3,4}+6U_{4,4}-2.5U_{5,4} &= -2U_{3,3}+5U_{4,3}-U_{5,3}-U_{4,2}\\
m=5: -2.5U_{4,4}+6U_{5,4}-2.5U_{6,4} &= -2U_{4,3}+5U_{5,3}-U_{6,3}-U_{5,2}\\
m=6: -2.5U_{5,4}+6U_{6,4}-2.5U_{7,4} &= -2U_{5,3}+5U_{6,3}-U_{7,3}-U_{6,2}\\
m=7: -2.5U_{6,4}+6U_{7,4}-2.5U_{8,4} &= -2U_{6,3}+5U_{7,3}-U_{8,3}-U_{7,2}\\
m=8: -2.5U_{7,4}+6U_{8,4}-2.5U_{9,4} &= -2U_{7,3}+5U_{8,3}-U_{9,3}-U_{8,2}\\
m=9: -2.5U_{8,4}+6U_{9,4}-2.5U_{10,4} &= -2U_{8,3}+5U_{9,3}-U_{10,3}-U_{9,2}\\
m=10: -2.5U_{9,4}+6U_{10,4}-2.5U_{11,4} &= -2U_{9,3}+5U_{10,3}-U_{11,3}-U_{10,2}
\end{aligned}$$

This is a system of equations obtained using the C-CNDS algorithm when $n=3$ with the initial and boundary conditions, the system of equations can be written in matrix vector form as shown:

$$\begin{bmatrix}
6 & -2.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-2.5 & 6 & -2.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -2.5 & 6 & -2.5 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -2.5 & 6 & -2.5 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -2.5 & 6 & -2.5 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -2.5 & 6 & -2.5 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -2.5 & 6 & -2.5 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -2.5 & 6 & -2.5 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.5 & 6 & -2.5\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.5 & 6
\end{bmatrix}
\begin{bmatrix}
U_{1,4}\\ U_{2,4}\\ U_{3,4}\\ U_{4,4}\\ U_{5,4}\\ U_{6,4}\\ U_{7,4}\\ U_{8,4}\\ U_{9,4}\\ U_{10,4}
\end{bmatrix}$$

$$=\begin{bmatrix}
0.333994\\ 0.13311\\ 0.08637\\ 0.02305\\ 0.03632\\ 0.00037\\ 0.01992\\ 0.00283\\ 0.01948\\ 0.02820
\end{bmatrix}$$

The solution to $u(x,t)$ obtained using MATLAB software when $m=1,2,3,...,10$ and

$n = 3$ are as follows: $U_{1,4} = 0.0215, U_{2,4} = 0.0481, U_{3,4} = 0.0534, U_{4,4} = 0.0478, U_{5,4} = 0.0412$
$U_{6,4} = 0.0323, U_{7,4} = 0.0258, U_{8,4} = 0.0180, U_{9,4} = 0.0120, U_{10,4} = 0.0050.$

n=4(t=2)

$$\begin{aligned}
m=1: -2.5U_{0,5}+6U_{1,5}-2.5U_{2,5} &= -2U_{0,4}+5U_{1,4}+U_{2,4}-U_{1,3}\\
m=2: -2.5U_{1,5}+6U_{2,5}-2.5U_{3,5} &= -2U_{1,4}+5U_{2,4}+U_{3,4}-U_{2,3}\\
m=3: -2.5U_{2,5}+6U_{3,5}-2.5U_{4,5} &= -2U_{2,4}+5U_{3,4}+U_{4,4}-U_{3,3}\\
m=4: -2.5U_{3,5}+6U_{4,5}-2.5U_{5,5} &= -2U_{3,4}+5U_{4,4}+U_{5,4}-U_{4,3}\\
m=5: -2.5U_{4,5}+6U_{5,5}-2.5U_{6,5} &= -2U_{4,4}+5U_{5,4}+U_{6,4}-U_{5,3}\\
m=6: -2.5U_{5,5}+6U_{6,5}-2.5U_{7,5} &= -2U_{5,4}+5U_{6,4}+U_{7,4}-U_{6,3}\\
m=7: -2.5U_{6,5}+6U_{7,5}-2.5U_{8,5} &= -2U_{6,4}+5U_{7,4}+U_{8,4}-U_{7,3}\\
m=8: -2.5U_{7,5}+6U_{8,5}-2.5U_{9,5} &= -2U_{7,4}+5U_{8,4}+U_{9,4}-U_{8,3}\\
m=9: -2.5U_{8,5}+6U_{9,5}-2.5U_{10,5} &= -2U_{8,4}+5U_{9,4}+U_{10,4}-U_{9,3}\\
m=10: -2.5U_{9,5}+6U_{10,5}-2.5U_{11,5} &= -2U_{9,4}+5U_{10,4}+U_{11,4}-U_{10,3}
\end{aligned}$$

This is a system of solution equations obtained using the C-CNDS algorithm when $n=4$ with the initial and boundary conditions, the system of equations can be written in matrix vector form as shown:

$$\begin{bmatrix}
6 & -2.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-2.5 & 6 & -2.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -2.5 & 6 & -2.5 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -2.5 & 6 & -2.5 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -2.5 & 6 & -2.5 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -2.5 & 6 & -2.5 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -2.5 & 6 & -2.5 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -2.5 & 6 & -2.5 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.5 & 6 & -2.5\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.5 & 6
\end{bmatrix}
\begin{bmatrix}
U_{1,5}\\ U_{2,5}\\ U_{3,5}\\ U_{4,5}\\ U_{5,5}\\ U_{6,5}\\ U_{7,5}\\ U_{8,5}\\ U_{9,5}\\ U_{10,5}
\end{bmatrix}$$

$$= \begin{bmatrix}
1.302545\\ 0.1124\\ 0.06194\\ 0.02391\\ 0.0212\\ 0.00212\\ 0.00731\\ 0.00814\\ 0.0240\\ 0.0240
\end{bmatrix}$$

The solution to $u(x,t)$ obtained using MATLAB software when $m = 1,2,3,...,10$ and

$n = 4$ are as follows: $U_{1,5} = 0.01213, U_{2,5} = 0.0351, U_{3,5} = 0.0416, U_{4,5} = 0.0375, U_{5,5} = 0.0332$
$U_{6,5} = 0.0275, U_{7,5} = 0.0208, U_{8,5} = 0.0147, U_{9,5} = 0.00928, U_{10,5} = 0.00207$

Table 4.1: Numerical Solutions at Varying Time Levels: Case One, C-CNDS

Grid Point, (m, n)	$n_1 = 0.5$	$n2 = 1.0$	$n_3 = 1.5$	$n_4 = 2.0$
$(m_1 = 0.5)$	0.0417	0.0317	0.0215	0.01213
$(m_2 = 1.0)$	0.07006	0.06106	0.0481	0.0351
$(m_3 = 1.5)$	0.07609	0.06709	0.0534	0.0416
$(m_4 = 2.0)$	0.06669	0.0569	0.0478	0.0372
$(m_5 = 2.5)$	0.05818	0.05118	0.0412	0.332
$(m_6 = 3.0)$	0.04909	0.03909	0.0323	0.0272
$(m_7 = 3.5)$	0.04154	0.03454	0.0258	0.0208
$(m_8 = 4.0)$	0.03351	0.02551	0.0180	0.0147
$(m_9 = 4.5)$	0.02351	0.0141	0.0121	0.00928
$(m_{10} = 5.0)$	0.0161	0.0101	0.005	0.00207

4.2.1 Discussion

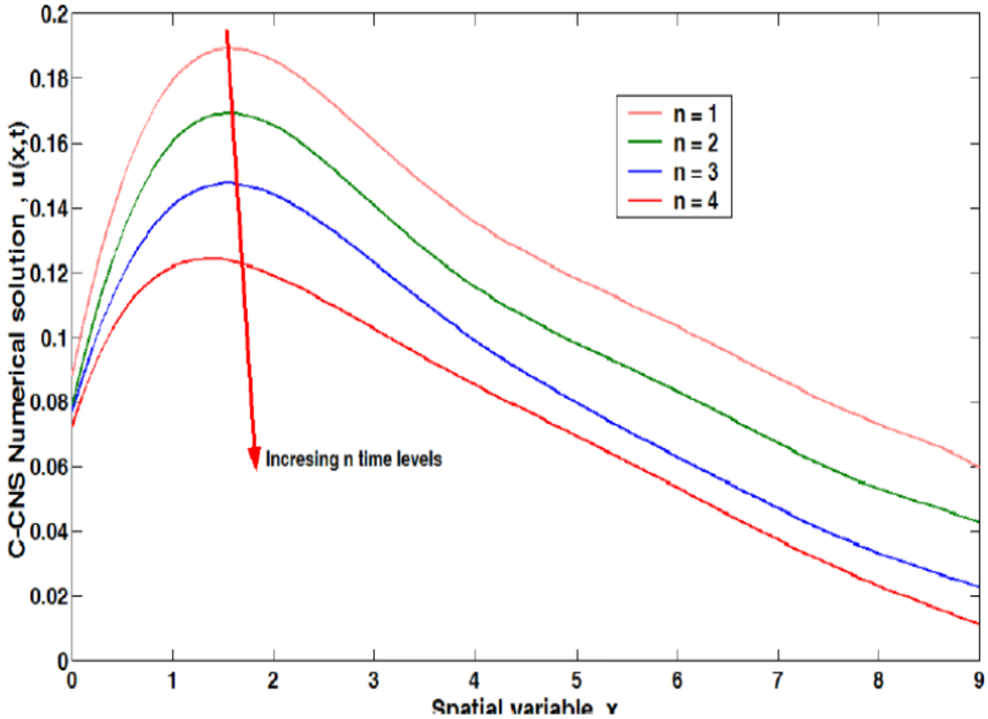
This is the first case considered when h is taken as 0.5 and came up with specific scheme for C-CNDS below:

$$\begin{aligned} &-2.5U_{m-1,n+1} + 6U_{m,n+1} - 2.5U_{m+1,n+1} \\ &= 3U_{m-1,n} + 4.5U_{m,n} - 1.5U_{m+1,n} - U_{m,n-1} \end{aligned}$$

Algebraic equations from the scheme was expressed in matrix form and analyzed using MATLAB. From the tabular and graphical presentations, table 4.1 and graph 4.1 respectively. It was observed that for a given value of x, $u(x, t)$ increases rapidly to a maximum within a short time then gradually decreases almost to zero as x tend to infinity.

The amplitude of the sound wave decreases with increasing distance from the sound source. It then show that the displacement of particles steadily decreases with increase in distance from the source.

4.1 Graphical Presentation Using C-CNDS: Case One



4.3 Forward Crank Nicholson Difference Scheme: Case One

Using the F-CNDS $h = k = 0.5, r = \xi = 0.5 and \beta = 2$ to obtain a system of solutions as shown below:

n=1(t=0.5)

$$\begin{aligned}
m = 1 : -2.5U_{0,2} + 6U_{1,2} - 2.5U_{2,2} &= -1.5U_{0,1} + 4.5U_{1,1} + 3U_{2,1} - U_{1,0} \\
m = 2 : -2.5U_{1,2} + 6U_{2,2} - 2.5U_{3,2} &= -1.5U_{1,1} + 4.5U_{2,1} + 3U_{3,1} - U_{2,0} \\
m = 3 : -2.5U_{2,2} + 6U_{3,2} - 2.5U_{4,2} &= -1.5U_{2,1} + 4.5U_{3,1} + 3U_{4,1} - U_{3,0} \\
m = 4 : -2.5U_{3,2} + 6U_{4,2} - 2.5U_{5,2} &= -1.5U_{3,1} + 4.5U_{4,1} + 3U_{5,1} - U_{4,0} \\
m = 5 : -2.5U_{4,2} + 6U_{5,2} - 2.5U_{6,2} &= -1.5U_{4,1} + 4.5U_{5,1} + 3U_{6,1} - U_{5,0} \\
m = 6 : -2.5U_{5,2} + 6U_{6,2} - 2.5U_{7,2} &= -1.5U_{5,1} + 4.5U_{6,1} + 3U_{7,1} - U_{6,0} \\
m = 7 : -2.5U_{6,2} + 6U_{7,2} - 2.5U_{8,2} &= -1.5U_{6,1} + 4.5U_{7,1} + 3U_{8,1} - U_{7,0} \\
m = 8 : -2.5U_{7,2} + 6U_{8,2} - 2.5U_{9,2} &= -1.5U_{7,1} + 4.5U_{8,1} + 3U_{9,1} - U_{8,0} \\
m = 9 : -2.5U_{8,2} + 6U_{9,2} - 2.5U_{10,2} &= -1.5U_{8,1} + 4.5U_{9,1} + 3U_{10,1} - U_{9,0} \\
m = 10 : -2.5U_{9,2} + 6U_{10,2} - 2.5U_{11,2} &= -1.5U_{9,1} + 4.5U_{10,1} + 3U_{11,1} - U_{10,0}
\end{aligned}$$

This is a system of equations obtained using the C-CNDS algorithm when $n = 1$ with the initial and boundary conditions, the system of equations can be written in matrix vector form as shown:

$$\begin{bmatrix} 6 & -2.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2.5 & 6 & -2.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2.5 & 6 & -2.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2.5 & 6 & -2.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2.5 & 6 & -2.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2.5 & 6 & -2.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2.5 & 6 & -2.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2.5 & 6 & -2.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.5 & 6 & -2.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.5 & 6 \end{bmatrix} \begin{bmatrix} U_{1,2} \\ U_{2,2} \\ U_{3,2} \\ U_{4,2} \\ U_{5,2} \\ U_{6,2} \\ U_{7,2} \\ U_{8,2} \\ U_{9,2} \\ U_{10,2} \end{bmatrix}$$

$$= \begin{bmatrix} 1.384539154 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution to $u(x, t)$ obtained using MATLAB software when $m = 1, 2, 3, ..., 10$ and $n = 1$ are as follows:
$U_{1,2} = 0.08815, U_{2,2} = 0.18010, U_{3,2} = 0.1834, U_{4,2} = 0.1648, U_{5,2} = 0.1312$
$U_{6,2} = 0.12113, U_{7,2} = 0.010228, U_{8,2} = 0.08740, U_{9,2} = 0.07331, U_{10,2} = 0.05978$

n=2(t=1)

$$\begin{aligned}
m=1: -2.5U_{0,3}+6U_{1,3}-2.5U_{2,3} &= -1.5U_{0,2}+4.5U_{1,2}+3U_{2,2}-U_{1,1}\\
m=2: -2.5U_{1,3}+6U_{2,3}-2.5U_{3,3} &= -1.5U_{1,2}+4.5U_{2,2}+3U_{3,2}-U_{2,1}\\
m=3: -2.5U_{2,3}+6U_{3,3}-2.5U_{4,3} &= -1.5U_{2,2}+4.5U_{3,2}+3U_{4,2}-U_{3,1}\\
m=4: -2.5U_{3,3}+6U_{4,3}-2.5U_{5,3} &= -1.5U_{3,2}+4.5U_{4,2}+3U_{5,2}-U_{4,1}\\
m=5: -2.5U_{4,3}+6U_{5,3}-2.5U_{6,3} &= -1.5U_{4,2}+4.5U_{5,2}+3U_{6,2}-U_{5,1}\\
m=6: -2.5U_{5,3}+6U_{6,3}-2.5U_{7,3} &= -1.5U_{5,2}+4.5U_{6,2}+3U_{7,2}-U_{6,1}\\
m=7: -2.5U_{6,3}+6U_{7,3}-2.5U_{8,3} &= -1.5U_{6,2}+4.5U_{7,2}+3U_{8,2}-U_{7,1}\\
m=8: -2.5U_{7,3}+6U_{8,3}-2.5U_{9,3} &= -1.5U_{7,2}+4.5U_{8,2}+3U_{9,2}-U_{8,1}\\
m=9: -2.5U_{8,3}+6U_{9,3}-2.5U_{10,3} &= -1.5U_{8,2}+4.5U_{9,2}+3U_{10,2}-U_{9,1}\\
m=10: -2.5U_{9,3}+6U_{10,3}-2.5U_{11,3} &= -1.5U_{9,2}+4.5U_{10,2}+3U_{11,2}-U_{10,1}
\end{aligned}$$

This is a system of equations obtained using the C-CNDS algorithm when $n = 2$ with the initial and boundary conditions, the system of equations can be written in matrix vector form as shown:

$$\begin{bmatrix}
6 & -2.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-2.5 & 6 & -2.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -2.5 & 6 & -2.5 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -2.5 & 6 & -2.5 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -2.5 & 6 & -2.5 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -2.5 & 6 & -2.5 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -2.5 & 6 & -2.5 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -2.5 & 6 & -2.5 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.5 & 6 & -2.5\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.5 & 6
\end{bmatrix}
\begin{bmatrix}
U_{1,3}\\ U_{2,3}\\ U_{3,3}\\ U_{4,3}\\ U_{5,3}\\ U_{6,3}\\ U_{7,2}\\ U_{8,3}\\ U_{9,3}\\ U_{10,3}
\end{bmatrix}$$

$$= \begin{bmatrix}
2.1712059893\\ 1.228425\\ 1.04955\\ 1.8601\\ 0.70659\\ 0.378969\\ 0.126531\\ 0.597888\\ 0.198795\\ 0.10997
\end{bmatrix}$$

The solution to $u(x,t)$ obtained using MATLAB software when $m = 1, 2, 3, ..., 10$ and

$n = 2$ are as follows:
$U_{1,3} = 0.7915, U_{2,3} = 0.16110, U_{3,3} = 0.16340, U_{4,3} = 0.14480, U_{5,3} = 0.11230$
$U_{6,3} = 0.010113, U_{7,3} = 0.08728, U_{8,3} = 0.06740, U_{9,3} = 0.05331, U_{10,3} = 0.04278$

n=3(t=1.5)

$$\begin{aligned}
m = 1 : -2.5U_{0,4} + 6U_{1,4} - 2.5U_{2,4} &= -1.5U_{0,3} + 4.5U_{1,3} + 3U_{2,3} - U_{1,2}\\
m = 2 : -2.5U_{1,4} + 6U_{2,4} - 2.5U_{3,4} &= -1.5U_{1,3} + 4.5U_{2,3} + 3U_{3,3} - U_{2,2}\\
m = 3 : -2.5U_{2,4} + 6U_{3,4} - 2.5U_{4,4} &= -1.5U_{2,3} + 4.5U_{3,3} + 3U_{4,3} - U_{3,2}\\
m = 4 : -2.5U_{3,4} + 6U_{4,4} - 2.5U_{5,4} &= -1.5U_{3,3} + 4.5U_{4,3} + 3U_{5,3} - U_{4,2}\\
m = 5 : -2.5U_{4,4} + 6U_{5,4} - 2.5U_{6,4} &= -1.5U_{4,3} + 4.5U_{5,3} + 3U_{6,3} - U_{5,2}\\
m = 6 : -2.5U_{5,4} + 6U_{6,4} - 2.5U_{7,4} &= -1.5U_{5,3} + 4.5U_{6,3} + 3U_{7,3} - U_{6,2}\\
m = 7 : -2.5U_{6,4} + 6U_{7,4} - 2.5U_{8,4} &= -1.5U_{6,3} + 4.5U_{7,3} + 3U_{8,3} - U_{7,2}\\
m = 8 : -2.5U_{7,4} + 6U_{8,4} - 2.5U_{9,4} &= -1.5U_{7,3} + 4.5U_{8,3} + 3U_{9,3} - U_{8,2}\\
m = 9 : -2.5U_{8,4} + 6U_{9,4} - 2.5U_{10,4} &= -1.5U_{8,3} + 4.5U_{9,3} + 3U_{10,3} - U_{9,2}\\
m = 10 : -2.5U_{9,4} + 6U_{10,4} - 2.5U_{11,4} &= -1.5U_{9,3} + 4.5U_{10,3} + 3U_{11,3} - U_{10,2}
\end{aligned}$$

This is a system of equations obtained using the F-CNDS algorithm when $n = 3$ with the initial and boundary conditions, the system of equations can be written in matrix vector form as shown:

$$\begin{bmatrix} 6 & -2.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2.5 & 6 & -2.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2.5 & 6 & -2.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2.5 & 6 & -2.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2.5 & 6 & -2.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2.5 & 6 & -2.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2.5 & 6 & -2.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2.5 & 6 & -2.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.5 & 6 & -2.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.5 & 6 \end{bmatrix} \begin{bmatrix} U_{1,4} \\ U_{2,4} \\ U_{3,4} \\ U_{4,4} \\ U_{5,4} \\ U_{6,4} \\ U_{7,4} \\ U_{8,4} \\ U_{9,4} \\ U_{10,4} \end{bmatrix}$$

$$= \begin{bmatrix} 4.733901087 \\ -0.06025 \\ 0.74795 \\ 0.56000 \\ 0.42674 \\ 0.417275 \\ 0.322135 \\ 1.761452 \\ 2.21045 \\ -0.87296 \end{bmatrix}$$

The solution to $u(x, t)$ obtained using MATLAB software when $m = 1, 2, 3, ..., 10$ and $n = 3$ are as follows:
$U_{1,4} = 0.7715, U_{2,4} = 0.14110, U_{3,4} = 0.1434, U_{4,4} = 0.1248, U_{5,4} = 0.09712$
$U_{6,4} = 0.0813, U_{7,4} = 0.06228, U_{8,4} = 0.04740, U_{9,4} = 0.03331, U_{10,4} = 0.02278.$

n=4(t=2)

$$
\begin{aligned}
m = 1: -2.5U_{0,5} + 6U_{1,5} - 2.5U_{2,5} &= -1.5U_{0,4} + 4.5U_{1,4} + 3U_{2,4} - U_{1,3}\\
m = 2: -2.5U_{1,5} + 6U_{2,5} - 2.5U_{3,5} &= -1.5U_{1,4} + 4.5U_{2,4} + 3U_{3,4} - U_{2,3}\\
m = 3: -2.5U_{2,5} + 6U_{3,5} - 2.5U_{4,5} &= -1.5U_{2,4} + 4.5U_{3,4} + 3U_{4,4} - U_{3,3}\\
m = 4: -2.5U_{3,5} + 6U_{4,5} - 2.5U_{5,5} &= -1.5U_{3,4} + 4.5U_{4,4} + 3U_{5,4} - U_{4,3}\\
m = 5: -2.5U_{4,5} + 6U_{5,5} - 2.5U_{6,5} &= -1.5U_{4,4} + 4.5U_{5,4} + 3U_{6,4} - U_{5,3}\\
m = 6: -2.5U_{5,5} + 6U_{6,5} - 2.5U_{7,5} &= -1.5U_{5,4} + 4.5U_{6,4} + 3U_{7,4} - U_{6,3}\\
m = 7: -2.5U_{6,5} + 6U_{7,5} - 2.5U_{8,5} &= -1.5U_{6,4} + 4.5U_{7,4} + 3U_{8,4} - U_{7,3}\\
m = 8: -2.5U_{7,5} + 6U_{8,5} - 2.5U_{9,5} &= -1.5U_{7,4} + 4.5U_{8,4} + 3U_{9,4} - U_{8,3}\\
m = 9: -2.5U_{8,5} + 6U_{9,5} - 2.5U_{10,5} &= -1.5U_{8,4} + 4.5U_{9,4} + 3U_{10,4} - U_{9,3}\\
m = 10: -2.5U_{9,5} + 6U_{10,5} - 2.5U_{11,5} &= -1.5U_{9,4} + 4.5U_{10,4} + 3U_{11,4} - U_{10,3}
\end{aligned}
$$

This is a system of equations obtained using the F-CNDS algorithm when $n = 4$ with the initial and boundary conditions, the system of equations can be written in matrix vector form as shown:

$$
\begin{bmatrix}
6 & -2.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-2.5 & 6 & -2.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -2.5 & 6 & -2.5 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -2.5 & 6 & -2.5 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -2.5 & 6 & -2.5 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -2.5 & 6 & -2.5 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -2.5 & 6 & -2.5 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -2.5 & 6 & -2.5 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.5 & 6 & -2.5\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.5 & 6
\end{bmatrix}
\begin{bmatrix}
U_{1,5}\\ U_{2,5}\\ U_{3,5}\\ U_{4,5}\\ U_{5,5}\\ U_{6,5}\\ U_{7,5}\\ U_{8,5}\\ U_{9,5}\\ U_{10,5}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
0.82355922\\ 0.157925\\ 0.64695\\ 0.47446\\ 0.34894\\ 0.29471\\ 0.19938\\ 0.13253\\ 0.011395\\ -0.58307
\end{bmatrix}
$$

The solution to $u(x, t)$ obtained using MATLAB software when $m = 1, 2, 3, ..., 10$ and

$n = 4$ are as follows:
$U_{1,5} = 0.07293, U_{2,5} = 0.120880, U_{3,5} = 0.122126, U_{4,5} = 0.09932, U_{5,5} = 0.09672$
$U_{6,5} = 0.070012, U_{7,5} = 0.05298, U_{8,5} = 0.03727, U_{9,5} = 0.02348, U_{10,5}0.011207$

Table 4.2: Plot of $u_{x,t}$ against $x$for varied n attained using F-CNS: Case One

Grid Point, (i, j, n)	$n_1 = 0.5$	$n_2 = 1.0$	$n_3 = 1.5$	$n_4 = 2.0$
$(m_1 = 0.5)$	0.08815	0.07915	0.07715	0.07293
$(m_2 = 1.0)$	0.18010	0.16110	0.14110	0.12080
$(m_3 = 1.5)$	0.1834	0.1634	0.1434	0.122126
$(m_4 = 2.0)$	0.1648	0.14480	0.1248	0.09932
$(m_5 = 2.5)$	0.1312	0.1123	0.09712	0.08672
$(m_6 = 3.0)$	0.121130	0.010113	0.0813	0.070012
$(m_7 = 5.5)$	0.010228	0.08728	0.06228	0.05298
$(m_8 = 4.0)$	0.08740	0.06740	0.04740	0.03727
$(m_9 = 4.5)$	0.07331	0.05331	0.03331	0.02348
$(m_{10} = 5.0)$	0.05978	0.04278	0.02278	0.011207

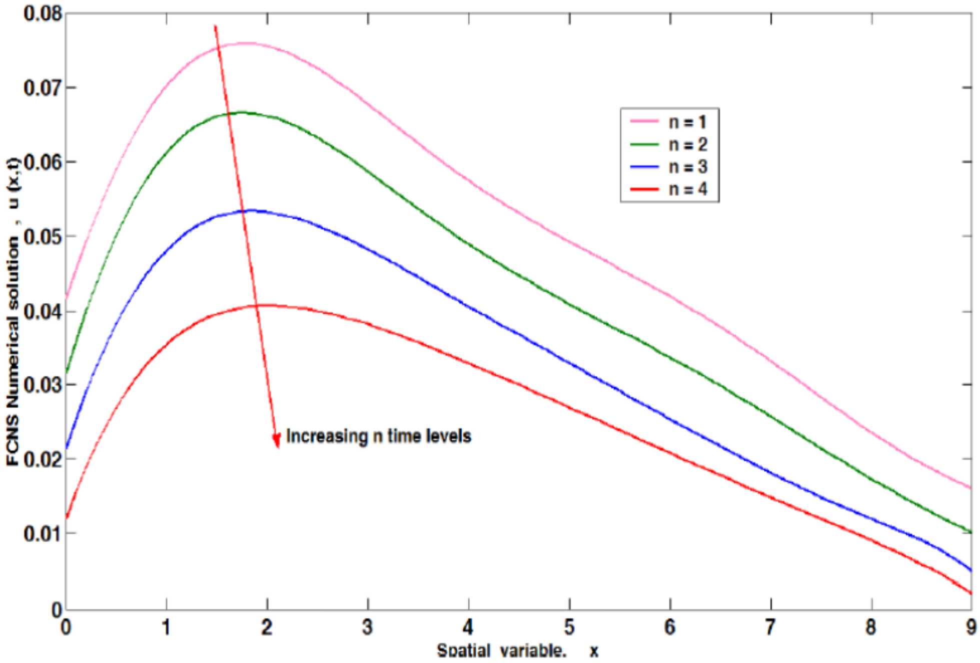
4.3.1 Discussion

In this case the value of $h = k = 0.5$ was maintained. But the numerical scheme was changed from C-CNDS to F-CNDS. The specific scheme developed is:

$$\begin{aligned} &-2.5U_{m-1,n+1} + 6U_{m,n+1} - 2.5U_{m+1,n+1} \\ &= 3U_{m-1,n} + 4.5U_{m,n} - 1.5U_{m+1,n} - U_{m,n-1} \end{aligned}$$

Algebraic equations from the scheme was expressed in matrix form and analyzed using MATLAB. The graphical and tabular presentations, graph 4.2 and table 4.2 respectively show that there was no significant difference when numerical scheme is changed from C-CNDS and F-CNDS.
the result show a similarity in the trend, that the value of $u(x, t)$ gradually decreases as x tend to infinity. Therefore, the amplitude of the wave decreases with increase in distance from the source.

4.2 Graphical Presentation Using F-CNDS: Case One



Case Two

We found the solution of the Third Order Advection-Viscous Wave Equation at two levels namely $n = 1, 2, 3, 4$. Difference Schemes and MATLAB software were used. An approximate temporal mesh size considered for the calculation was taken for different values of $h, k, r, \beta, \xi and dt$ we get the results as follows for given values thus:

$$h = \frac{1}{5}, k = \frac{1}{5}, r = \frac{1}{2}, \beta = 5 and \xi = \frac{1}{5}$$

The schemes become:

C-CNDS

$$\begin{aligned} -(r+\beta)U_{m-1,n+1} + (1+2r+2\beta)U_{m,n+1} - (r+\beta)U_{m+1,n+1} \\ = (r-\beta+\xi)U_{m-1,n} + (2-2r+2\beta)U_{m,n} + (r-\beta+\xi)U_{m+1,n} - U_{m,n-1} \end{aligned}$$

$$\begin{aligned} -5.5U_{m-1,n+1} + 12U_{m,n+1} - 5.5U_{m+1,n+1} \\ = -4.7U_{m-1,n} + 11U_{m,n} - 4.3U_{m+1,n} - U_{m,n-1} \end{aligned}$$

F-CNDS

$$\begin{aligned} -(r+\beta)U_{m-1,n+1} + (1+2r+2\beta)U_{m,n+1} - (r+\beta)U_{m+1,n+1} \\ = (r-\beta)U_{m-1,n} + (2-2r+2\beta-\psi)U_{m,n} + (r+\beta+\psi)U_{m+1,n} - U_{m,n-1} \end{aligned}$$

$$\begin{aligned} -5.5U_{m-1,n+1} + 12U_{m,n+1} - 5.5U_{m+1,n+1} \\ = 4.5U_{m-1,n} + 10.8U_{m,n} + 5.7U_{m+1,n} - U_{m,n-1} \end{aligned}$$

The solution at $t = 0$ and $t = 1$ are known and referred as the zero and the first time level solutions respectively. We fix n(from $n = 1$) and vary m iteratively as follows:

4.4 Central Crank Nicholson Difference Scheme: Case Two

Using the C-CNDS $h = k = 0.2, r = \xi = 0.2 and \beta = 2$ to obtain a system of solutions as shown below:

n=1(t=0.2)

$$\begin{aligned}
m=1: -5.5U_{0,2}+12U_{1,2}-5.5U_{2,2} &= -4.7U_{0,1}+11U_{1,1}-4.3U_{2,1}-U_{1,0}\\
m=2: -5.5U_{1,2}+12U_{2,2}-5.5U_{3,2} &= -4.7U_{1,1}+11U_{2,1}-4.3U_{3,1}-U_{2,0}\\
m=3: -5.5U_{2,2}+12U_{3,2}-5.5U_{4,2} &= -4.7U_{2,1}+11U_{3,1}-4.3U_{4,1}-U_{3,0}\\
m=4: -5.5U_{3,2}+12U_{4,2}-5.5U_{5,2} &= -4.7U_{3,1}+11U_{4,1}-4.3U_{5,1}-U_{4,0}\\
m=5: -5.5U_{4,2}+12U_{5,2}-5.5U_{6,2} &= -4.7U_{4,1}+11U_{5,1}-4.3U_{6,1}-U_{5,0}\\
m=6: -5.5U_{5,2}+12U_{6,2}-5.5U_{7,2} &= -4.7U_{5,1}+11U_{6,1}-4.3U_{7,1}-U_{6,0}\\
m=7: -5.5U_{6,2}+12U_{7,2}-5.5U_{8,2} &= -4.7U_{6,1}+11U_{7,1}-4.3U_{8,1}-U_{7,0}\\
m=8: -5.5U_{7,2}+12U_{8,2}-5.5U_{9,2} &= -4.7U_{7,1}+11U_{8,1}-4.3U_{9,1}-U_{8,0}\\
m=9: -5.5U_{8,2}+12U_{9,2}-5.5U_{10,2} &= -4.7U_{8,1}+11U_{9,1}-4.3U_{10,1}-U_{9,0}\\
m=10: -5.5U_{9,2}+12U_{10,2}-5.5U_{11,2} &= -4.7U_{9,1}+11U_{10,1}-4.3U_{11,1}-U_{10,0}
\end{aligned}$$

This is a system of equations obtained using the C-CNDS algorithm when $n = 1$ with the initial and boundary conditions, the system of equations can be written in matrix vector form as shown:

$$\begin{bmatrix}
12 & -5.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-5.5 & 12 & -5.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -5.5 & 12 & -5.5 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -5.5 & 12 & -5.5 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -5.5 & 12 & -5.5 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -5.5 & 12 & -5.5 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -5.5 & 12 & -5.5 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -5.5 & 12 & -5.5 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -5.5 & 12 & -5.5\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5.5 & 12
\end{bmatrix}
\begin{bmatrix}
U_{1,2}\\ U_{2,2}\\ U_{3,2}\\ U_{4,2}\\ U_{5,2}\\ U_{6,2}\\ U_{7,2}\\ U_{8,2}\\ U_{9,2}\\ U_{10,2}
\end{bmatrix}$$

$$= \begin{bmatrix}
1.208055028\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0
\end{bmatrix}$$

The solution to $u(x, t)$ obtained using MATLAB software when $m = 1, 2, 3, ..., 10$ and $n = 1$ are as follows:

$U_{1,2} = 0.63622, U_{2,2} = 0.0835006, U_{3,2} = 0.081609, U_{4,2} = 0.0708669, U_{5,2} = 0.0574818$
$U_{6,2} = 0.049309, U_{7,2} = 0.0376154, U_{8,2} = 0.0277615464, U_{9,2} = 0.01937331, U_{10,2} = 0.009193351$

n=2(t=0.4)

$$\begin{aligned}
m=1: -5.5U_{0,3}+12U_{1,3}-5.5U_{2,3} &= -4.7U_{0,2}+11U_{1,2}-4.3U_{2,2}-U_{1,1}\\
m=2: -5.5U_{1,3}+12U_{2,3}-5.5U_{3,3} &= -4.7U_{1,2}+11U_{2,2}-4.3U_{3,2}-U_{2,1}\\
m=3: -5.5U_{2,3}+12U_{3,3}-5.5U_{4,3} &= -4.7U_{2,2}+11U_{3,2}-4.3U_{4,2}-U_{3,1}\\
m=4: -5.5U_{3,3}+12U_{4,3}-5.5U_{5,3} &= -4.7U_{3,2}+11U_{4,2}-4.3U_{5,2}-U_{4,1}\\
m=5: -5.5U_{4,3}+12U_{5,3}-5.5U_{6,3} &= -4.7U_{4,2}+11U_{5,2}4.3U_{6,2}-U_{5,1}\\
m=6: -5.5U_{5,3}+12U_{6,3}-5.5U_{7,3} &= -4.7U_{5,2}+11U_{6,2}-4.3U_{7,2}-U_{6,1}\\
m=7: -5.5U_{6,3}+12U_{7,3}-5.5U_{8,3} &= -4.7U_{6,2}+11U_{7,2}-4.3U_{8,2}-U_{7,1}\\
m=8: -5.5U_{7,3}+12U_{8,3}-5.5U_{9,3} &= -4.7U_{7,2}+11U_{8,2}-4.3U_{9,2}-U_{8,1}\\
m=9: -5.5U_{8,3}+12U_{9,3}-5.5U_{10,3} &= -4.7U_{8,2}+11U_{9,2}-4.3U_{10,2}-U_{9,1}\\
m=10: -5.5U_{9,3}+12U_{10,3}-5.5U_{11,3} &= -4.7U_{9,2}+11U_{10,2}-4.3U_{11,2}-U_{10,1}
\end{aligned}$$

This is a systems of equations obtained using the C-CNDS algorithm when $n=2$ with the initial and boundary conditions, the system of equations can be written in matrix vector form as shown:

$$\begin{bmatrix}
12 & -5.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-5.5 & 12 & -5.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -5.5 & 12 & -5.5 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -5.5 & 12 & -5.5 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -5.5 & 12 & -5.5 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -5.5 & 12 & -5.5 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -5.5 & 12 & -5.5 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -5.5 & 12 & -5.5 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -5.5 & 12 & -5.5\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5.5 & 12
\end{bmatrix}
\begin{bmatrix}
U_{1,3}\\ U_{2,3}\\ U_{3,3}\\ U_{4,3}\\ U_{5,3}\\ U_{6,3}\\ U_{7,3}\\ U_{8,3}\\ U_{9,3}\\ U_{10,3}
\end{bmatrix}$$

$$= \begin{bmatrix}
-2.631831\\ -2.4226461\\ 0.20051851\\ 0.14880186\\ 0.08719607\\ 0.11048832\\ 0.062642435\\ 0.045279437\\ 0.082627125\\ -0.091054557
\end{bmatrix}$$

The solution to $u(x,t)$ obtained using MATLAB software when $m=1,2,3,...,10$ and $n=2$ are as follows:

$U_{1,3} = 0.051317, U_{2,3} = 0.71311, U_{3,3} = 0.065571, U_{4,3} = 0.063579, U_{5,3} = 0.050412, U_{6,3} = 0.044391, U_{7,3} = 0.033645, U_{8,3} = 0.021648, U_{9,3} = 0.153551, U_{10,3} = 0.007541$

n=3(t=0.6)

$$\begin{aligned}
m=1: -5.5U_{0,4}+12U_{1,4}-5.5U_{2,4} &= -4.7U_{0,3}+11U_{1,3}-4.3U_{2,3}-U_{1,2}\\
m=2: -5.5U_{1,4}+12U_{2,4}-5.5U_{3,4} &= -4.7U_{1,3}+11U_{2,3}-4.3U_{3,3}-U_{2,2}\\
m=3: -5.5U_{2,4}+12U_{3,4}-5.5U_{4,4} &= -4.7U_{2,3}+11U_{3,3}-4.3U_{4,3}-U_{3,2}\\
m=4: -5.5U_{3,4}+12U_{4,4}-5.5U_{5,4} &= -4.7U_{3,3}+11U_{4,3}-4.3U_{5,3}-U_{4,2}\\
m=5: -5.5U_{4,4}+12U_{5,4}-5.5U_{6,4} &= -4.7U_{4,3}+11U_{5,3}-4.3U_{6,3}-U_{5,2}\\
m=6: -5.5U_{5,4}+12U_{6,4}-5.5U_{7,4} &= -4.7U_{5,3}+11U_{6,3}-4.3U_{7,3}-U_{6,2}\\
m=7: -5.5U_{6,4}+12U_{7,4}-5.5U_{8,4} &= -4.7U_{6,3}+11U_{7,3}-4.3U_{8,3}-U_{7,2}\\
m=8: -5.5U_{7,4}+12U_{8,4}-5.5U_{9,4} &= -4.7U_{7,3}+11U_{8,3}-4.3U_{9,3}-U_{8,2}\\
m=9: -5.5U_{8,4}+12U_{9,4}-5.5U_{10,4} &= -4.7U_{8,3}+11U_{9,3}-4.3U_{10,3}-U_{9,2}\\
m=10: -5.5U_{9,4}+12U_{10,4}-5.5U_{11,4} &= -4.7U_{9,3}+11U_{10,3}-4.3U_{11,3}-U_{10,2}
\end{aligned}$$

This is a system of equations obtained using the C-CNDS algorithm when $n=3$ with the initial and boundary conditions, the system of equations can be written in matrix vector form as shown:

$$\begin{bmatrix}
12 & -5.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-5.5 & 12 & -5.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -5.5 & 12 & -5.5 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -5.5 & 12 & -5.5 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -5.5 & 12 & -5.5 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -5.5 & 12 & -5.5 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -5.5 & 12 & -5.5 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -5.5 & 12 & -5.5 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -5.5 & 12 & -5.5\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5.5 & 12
\end{bmatrix}
\begin{bmatrix}
U_{1,4}\\ U_{2,4}\\ U_{3,4}\\ U_{4,4}\\ U_{5,4}\\ U_{6,4}\\ U_{7,4}\\ U_{8,4}\\ U_{9,4}\\ U_{10,4}
\end{bmatrix}$$

$$= \begin{bmatrix}
0.913268575\\ -0.37495\\ 0.02923\\ 0.02806\\ -0.00604\\ 0.049207\\ 0.019069\\ -0.61790\\ 1.587318\\ -0.72169
\end{bmatrix}$$

The solution to $u(x,t)$ obtained using MATLAB software when $m=1,2,3,...,10$ and $n=3$ are as follows:
$U_{1,4}=0.22915, U_{2,4}=0.04851, U_{3,4}=0.055934, U_{4,4}=0.046974, U_{5,4}=0.043412,$

$U_{6,4} = 0.037423, U_{7,4} = 0.030258, U_{8,4} = 0.021676, U_{9,4} = 0.01398, U_{10,4} = 0.050312.$

n=4(t=0.8)

$$
\begin{aligned}
m = 1 : -5.5U_{0,5} + 12U_{1,5} - 5.5U_{2,5} &= -4.7U_{0,4} + 11U_{1,4} - 4.3U_{2,4} - U_{1,3} \\
m = 2 : -5.5U_{1,5} + 12U_{2,5} - 5.5U_{3,5} &= -4.7U_{1,4} + 11U_{2,4} - 4.3U_{3,4} - U_{2,3} \\
m = 3 : -5.5U_{2,5} + 12U_{3,5} - 5.5U_{4,5} &= -4.7U_{2,4}11U_{3,4} - 4.3U_{4,4} - U_{3,3} \\
m = 4 : -5.5U_{3,5} + 12U_{4,5} - 5.5U_{5,5} &= -4.7U_{3,4} + 11U_{4,4} - 4.3U_{5,4} - U_{4,3} \\
m = 5 : -5.5U_{4,5} + 12U_{5,5} - 5.5U_{6,5} &= -4.7U_{4,4} + 11U_{5,4} - 4.3U_{6,4} - U_{5,3} \\
m = 6 : -5.5U_{5,5} + 12U_{6,5} - 5.5U_{7,5} &= -4.7U_{5,4} + 11U_{6,4} - 4.3U_{7,4} - U_{6,3} \\
m = 7 : -5.5U_{6,5} + 12U_{7,5} - 5.5U_{8,5} &= -4.7U_{6,4} + 11U_{7,4} - 4.3U_{8,4} - U_{7,3} \\
m = 8 : -5.5U_{7,5} + 12U_{8,5} - 5.5U_{9,5} &= -4.7U_{7,4} + 11U_{8,4} - 4.3U_{9,4} - U_{8,3} \\
m = 9 : -5.5U_{8,5} + 12U_{9,5} - 5.5U_{10,5} &= -4.7U_{8,4} + 11U_{9,4} - 4.3U_{10,4} - U_{9,3} \\
m = 10 : -5.5U_{9,5} + 12U_{10,5} - 5.5U_{11,5} &= -4.7U_{9,4} + 11U_{10,4} - 4.3U_{11,4} - U_{10,3}
\end{aligned}
$$

This is a system of equations obtained using the C-CNDS algorithm when $n = 4$ with the initial and boundary conditions, the system of equations can be written in matrix vector form as shown:

$$
\begin{bmatrix}
12 & -5.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-5.5 & 12 & -5.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -5.5 & 12 & -5.5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -5.5 & 12 & -5.5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -5.5 & 12 & -5.5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -5.5 & 12 & -5.5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -5.5 & 12 & -5.5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -5.5 & 12 & -5.5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -5.5 & 12 & -5.5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5.5 & 12
\end{bmatrix}
\begin{bmatrix}
U_{1,5} \\ U_{2,5} \\ U_{3,5} \\ U_{4,5} \\ U_{5,5} \\ U_{6,5} \\ U_{7,5} \\ U_{8,5} \\ U_{9,5} \\ U_{10,5}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
3.56344209 \\ -0.83523 \\ 0.114391 \\ 0.000526 \\ 0.032708 \\ 0.027095 \\ 0.019353 \\ 0.002463 \\ 0.051903 \\ -0.06517
\end{bmatrix}
$$

The solution to $u(x, t)$ obtained using MATLAB software when $m = 1, 2, 3, ..., 10$ and

$n = 4$ are as follows: $U_{1,5} = 0.01876213, U_{2,5} = 0.0311051, U_{3,5} = 0.049416, U_{4,5} = 0.0304572, U_{5,5} = 0.364432$
$U_{6,5} = 0.029172, U_{7,5} = 0.025008, U_{8,5} = 0.013727, U_{9,5} = 0.0058147, U_{10,5} = 0.0020698$

Table 4.3: Plot of $u_{x,t}$ against $x$for varied n attained using C-CNS: Case Two

Grid Point, (m, n)	$n_1 = 0.2$	$n_2 = 0.4$	$n_3 = 0.6$	$n_4 = 0.8$
$(m_1 = 0.2)$	0.0636217	0.051317	0.022915	0.01876213
$(m_2 = 0.4)$	0.0835006	0.0713106	0.04851	0.0311051
$(m_3 = 0.6)$	0.081609	0.0655709	0.055934	0.049416
$(m_4 = 0.8)$	0.0708669	0.063579	0.046878	0.0304572
$(m_5 = 1.0)$	0.0574818	0.0504118	0.043412	0.364432
$(m_6 = 1.2)$	0.049309	0.0443909	0.037423	0.029172
$(m_7 = 1.4)$	0.0376154	0.0336454	0.030258	0.025008
$(m_8 = 1.6)$	0.0277615464	0.0216474554	0.021675894	0.013723
$(m_9 = 1.8)$	0.01937331	0.0153551	0.013980	0.0058147
$(m_{10} = 2.0)$	0.009193351	0.0071541	0.0503121	0.0020698

4.4.1 Discussion on tables and graphical output

In the second case, we further reduced the mesh size $h = k = \frac{1}{5}$ and came up with a specific C-CNDS scheme:

$$\begin{aligned} &-5.5U_{m-1,n+1} + 12U_{m,n+1} - 5.5U_{m+1,n+1} \\ &= -4.7U_{m-1,n} + 11U_{m,n} - 4.3U_{m+1,n} - U_{m,n-1} \end{aligned}$$

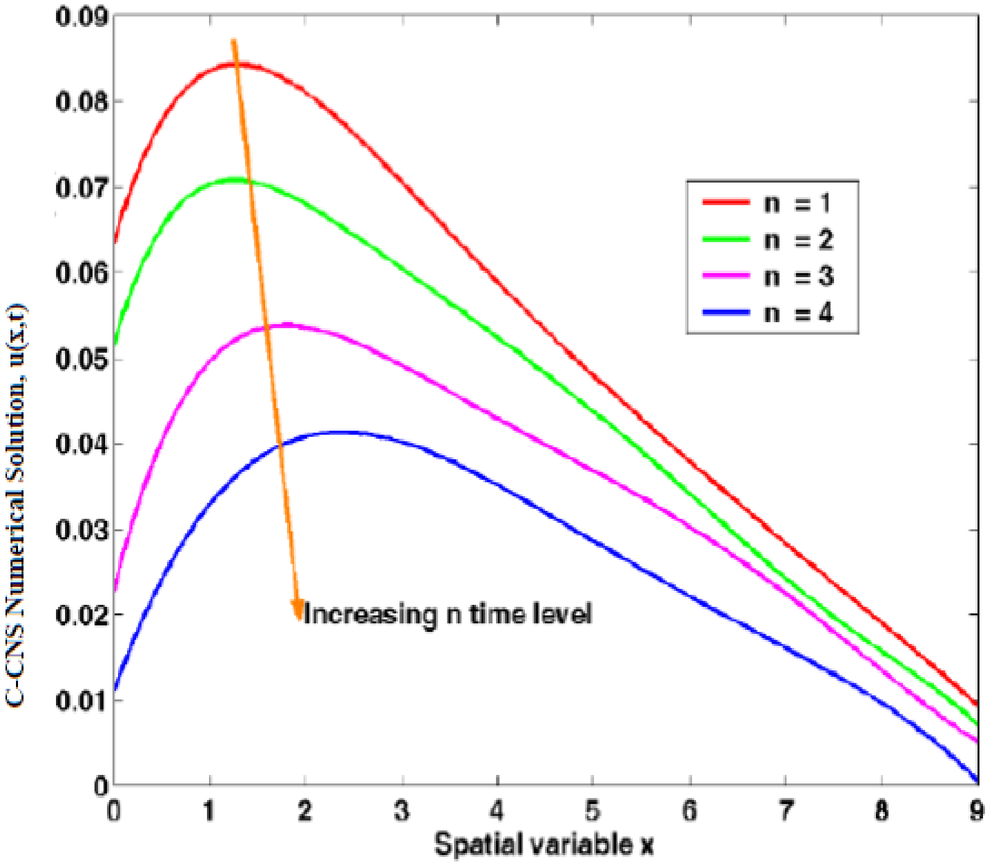
The algebraic equations and matrices derived from the scheme were presented graphically and in tabular form, graph 4.3 and table 4.4 respectively. It was also noted that the reduction in mesh size has no significant effect on the result obtained since a similar trend in sudden increase in amplitude and gradual decline was achieved. Hence amplitude of sound decreases with increase in distance from the source.

4.5 Central-Crank Nicolson Difference Scheme: Case Three

Using the C-CNDS $h = k = 0.1, r = 0.5, \xi = 0.1 and \beta = 10$ to obtain a system of equations as shown below:

n=1(t=0.1)

4.3 Graphical Presentation Using C-CNDS: Case Two



$$\begin{aligned}
m=1: -10.5U_{0,2}+22U_{1,2}-10.5U_{2,2} &= -9.6U_{0,1}+21U_{1,1}-9.4U_{2,1}-U_{1,0}\\
m=2: -10.5U_{1,2}+22U_{2,2}-10.5U_{3,2} &= -9.6U_{1,1}+21U_{2,1}-9.4U_{3,1}-U_{2,0}\\
m=3: -10.5U_{2,2}+22U_{3,2}-10.5U_{4,2} &= -9.6U_{2,1}+21U_{3,1}-9.4U_{4,1}-U_{3,0}\\
m=4: -10.5U_{3,2}+22U_{4,2}-10.5U_{5,2} &= -9.6U_{3,1}+21U_{4,1}-9.4U_{5,1}-U_{4,0}\\
m=5: -10.5U_{4,2}+22U_{5,2}-10.5U_{6,2} &= -9.6U_{4,1}+21U_{5,1}-9.4U_{6,1}-U_{5,0}\\
m=6: -10.5U_{5,2}+22U_{6,2}-10.5U_{7,2} &= -9.6U_{5,1}+21U_{6,1}-9.4U_{7,1}-U_{6,0}\\
m=7: -10.5U_{6,2}+22U_{7,2}-10.5U_{8,2} &= -9.6U_{6,1}+21U_{7,1}-9.4U_{8,1}-U_{7,0}\\
m=8: -10.5U_{7,2}+22U_{8,2}-10.5U_{9,2} &= -9.6U_{7,1}+21U_{8,1}-9.4U_{9,1}-U_{8,0}\\
m=9: -10.5U_{8,2}+22U_{9,2}-10.5U_{10,2} &= -9.6U_{8,1}+21U_{9,1}-9.4U_{10,1}-U_{9,0}\\
m=10: -10.5U_{9,2}+22U_{10,2}-10.5U_{11,2} &= -9.6U_{9,1}+21U_{10,1}-9.4U_{11,1}-U_{10,0}
\end{aligned}$$

This is a system of equations obtained using the C-CNDS algorithm when $n=1$ with the initial and boundary conditions, the system of equations can be written in matrix vector form as shown:

$$\begin{bmatrix}
22 & -10.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-10.5 & 22 & -10.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -10.5 & 22 & -10.5 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -10.5 & 22 & -10.5 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -10.5 & 22 & -10.5 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -10.5 & 22 & -10.5 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -10.5 & 22 & -10.5 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -10.5 & 22 & -10.5 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -10.5 & 22 & -10.5\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -10.5 & 22
\end{bmatrix}
\begin{bmatrix} U_{1,2}\\ U_{2,2}\\ U_{3,2}\\ U_{4,2}\\ U_{5,2}\\ U_{6,2}\\ U_{7,2}\\ U_{8,2}\\ U_{9,2}\\ U_{10,2}\end{bmatrix}$$

$$= \begin{bmatrix} 1.12762717354\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\end{bmatrix}$$

The solution to $u(x,t)$ obtained using MATLAB software when $m=1,2,3,...,10$ and $n=1$ are as follows:

$U_{1,2}=0.8362171, U_{2,2}=0.10352006, U_{3,2}=0.1016093, U_{4,2}=0.09086692, U_{5,2}=0.07748187$,
$U_{6,2}=0.0693709, U_{7,2}=0.057615464, U_{8,2}=0.04776154642, U_{9,2}=0.03937339, U_{10,2}=$

0.11193351

n=2(t=0.2)

$$
\begin{aligned}
m=1: -10.5U_{0,3}+22U_{1,3}-10.5U_{2,3} &= -9.6U_{0,2}+21U_{1,2}-9.4U_{2,2}-U_{1,1}\\
m=2: -10.5U_{1,3}+22U_{2,3}-10.5U_{3,3} &= -9.6U_{1,2}+21U_{2,2}-9.4U_{3,2}-U_{2,1}\\
m=3: -10.5U_{2,3}+22U_{3,3}-10.5U_{4,3} &= -9.6U_{2,2}+21U_{3,2}-9.4U_{4,2}-U_{3,1}\\
m=4: -10.5U_{3,3}+22U_{4,3}-10.5U_{5,3} &= -9.6U_{3,2}+21U_{4,2}-9.4U_{5,2}-U_{4,1}\\
m=5: -10.5U_{4,3}+22U_{5,3}-10.5U_{6,3} &= -9.6U_{4,2}+21U_{5,2}-9.4U_{6,2}-U_{5,1}\\
m=6: -10.5U_{5,3}+22U_{6,3}-10.5U_{7,3} &= -9.6U_{5,2}+21U_{6,2}-9.4U_{7,2}-U_{6,1}\\
m=7: -10.5U_{6,3}+22U_{7,3}-10.5U_{8,3} &= -9.6U_{6,2}+21U_{7,2}-9.4U_{8,2}-U_{7,1}\\
m=8: -10.5U_{7,3}+22U_{8,3}-10.5U_{9,3} &= -9.6U_{7,2}+21U_{8,2}-9.4U_{9,2}-U_{8,1}\\
m=9: -10.5U_{8,3}+22U_{9,3}-10.5U_{10,3} &= -9.6U_{8,2}+21U_{9,2}-9.4U_{10,2}-U_{9,1}\\
m=10: -10.5U_{9,3}+22U_{10,3}-10.5U_{11,3} &= -9.6U_{9,2}+21U_{10,2}-9.4U_{11,2}-U_{10,1}
\end{aligned}
$$

This is a system of equations obtained using the C-CNDS algorithm when $n=2$ with the initial and boundary conditions, the system of equations can be written in matrix vector form as shown:

$$
\begin{bmatrix}
22 & -10.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-10.5 & 22 & -10.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -10.5 & 22 & -10.5 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -10.5 & 22 & -10.5 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -10.5 & 22 & -10.5 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -10.5 & 22 & -10.5 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -10.5 & 22 & -10.5 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -10.5 & 22 & -10.5 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -10.5 & 22 & -10.5\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -10.5 & 22
\end{bmatrix}
\begin{bmatrix}
U_{1,3}\\ U_{2,3}\\ U_{3,3}\\ U_{4,3}\\ U_{5,3}\\ U_{6,3}\\ U_{7,3}\\ U_{8,3}\\ U_{9,3}\\ U_{10,3}
\end{bmatrix}
$$

$$
=\begin{bmatrix}
2.883683714\\ 0.416025136\\ 0.285853676\\ 0.204426462\\ 0.097388192\\ 0.183267576\\ 0.089570144\\ 0.08014536\\ 0.262283555\\ 0.142545069
\end{bmatrix}
$$

The solution to $u(x,t)$ obtained using MATLAB software when $m=1,2,3,...,10$ and $n=2$ are as follows:
$U_{1,3}=0.0713175, U_{2,3}=0.09136106, U_{3,3}=0.08557094, U_{4,3}=0.0835796, U_{5,3}=0.0741182$

$U_{6,3} = 0.064394093, U_{7,3} = 0.0536454986, U_{8,3} = 0.0416647546, U_{9,3} = 0.0353551265, U_{10,3} = 0.00915414.$

n=3(t=0.3)

$$
\begin{aligned}
m = 1: -10.5U_{0,4} + 22U_{1,4} - 10.5U_{2,4} &= -9.6U_{0,3} + 21U_{1,3} - 9.4U_{2,3} - U_{1,2}\\
m = 2: -10.5U_{1,4} + 22U_{2,4} - 10.5U_{3,4} &= -9.6U_{1,3} + 21U_{2,3} - 9.4U_{3,3} - U_{2,2}\\
m = 3: -10.5U_{2,4} + 22U_{3,4} - 10.5U_{4,4} &= -9.6U_{2,3} + 21U_{3,3} - 9.4U_{4,3} - U_{3,2}\\
m = 4: -10.5U_{3,4} + 22U_{4,4} - 10.5U_{5,4} &= -9.6U_{3,3} + 21U_{4,3} - 9.4U_{5,3} - U_{4,2}\\
m = 5: -10.5U_{4,4} + 22U_{5,4} - 10.5U_{6,4} &= -9.6U_{4,3} + 21U_{5,3} - 9.4U_{6,3} - U_{5,2}\\
m = 6: -10.5U_{5,4} + 22U_{6,4} - 10.5U_{7,4} &= -9.6U_{5,3} + 21U_{6,3} - 9.4U_{7,3} - U_{6,2}\\
m = 7: -10.5U_{6,4} + 22U_{7,4} - 10.5U_{8,4} &= -9.6U_{6,3} + 21U_{7,3} - 9.4U_{8,3} - U_{7,2}\\
m = 8: -10.5U_{7,4} + 22U_{8,4} - 10.5U_{9,4} &= -9.6U_{7,3} + 21U_{8,3} - 9.4U_{9,3} - U_{8,2}\\
m = 9: -10.5U_{8,4} + 22U_{9,4} - 10.5U_{10,4} &= -9.6U_{8,3} + 21U_{9,3} - 9.4U_{10,3} - U_{9,2}\\
m = 10: -10.5U_{9,4} + 22U_{10,3} - 10.5U_{11,4} &= -9.6U_{9,3} + 21U_{10,3} - 9.4U_{11,3} - U_{10,2}
\end{aligned}
$$

This is a system of equations obtained using the C-CNDS algorithm when $n = 3$ with the initial and boundary conditions, the system of equations can be written in matrix vector form as shown:

$$
\begin{bmatrix}
22 & -10.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-10.5 & 22 & -10.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -10.5 & 22 & -10.5 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -10.5 & 22 & -10.5 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -10.5 & 22 & -10.5 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -10.5 & 22 & -10.5 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -10.5 & 22 & -10.5 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -10.5 & 22 & -10.5 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -10.5 & 22 & -10.5\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -10.5 & 22
\end{bmatrix}
\begin{bmatrix}
U_{1,4}\\ U_{2,4}\\ U_{3,4}\\ U_{4,4}\\ U_{5,4}\\ U_{6,4}\\ U_{7,4}\\ U_{8,4}\\ U_{9,4}\\ U_{10,4}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
1.807150040629\\ 0.345422\\ 0.030994\\ 0.170211\\ 0.01989\\ 0.094573\\ 0.046949\\ -0.03035\\ 0.208831\\ -0.18651
\end{bmatrix}
$$

The solution to $u(x,t)$ obtained using MATLAB software when $m = 1, 2, 3, ..., 10$ and $n = 3$ are as follows:
$U_{1,4} = 0.0522915, U_{2,4} = 0.068541, U_{3,4} = 0.0759346, U_{4,4} = 0.0668786, U_{5,4} = 0.0634125$
$U_{6,4} = 0.05742353, U_{7,4} = 0.0502588, U_{8,4} = 0.041675895, U_{9,4} = 0.0339807, U_{10,4} = 0.070312134$

n=4(t=0.4)

$$\begin{aligned}
m=1: -10.5U_{0,5}+22U_{1,5}-10.5U_{2,5} &= -9.6U_{0,4}+21U_{1,4}-9.4U_{2,4}-U_{1,3}\\
m=2: -10.5U_{1,5}+22U_{2,5}-10.5U_{3,5} &= -9.6U_{1,4}+21U_{2,4}-9.4U_{3,4}-U_{2,3}\\
m=3: -10.5U_{2,5}+22U_{3,5}-10.5U_{4,5} &= -9.6U_{2,4}+21U_{3,4}-9.4U_{4,4}-U_{3,3}\\
m=4: -10.5U_{3,5}+22U_{4,5}-10.5U_{5,5} &= -9.6U_{3,4}+21U_{4,4}-9.4U_{5,4}-U_{4,3}\\
m=5: -10.5U_{4,5}+22U_{5,5}-10.5U_{6,5} &= -9.6U_{4,4}+21U_{5,4}-9.4U_{6,4}-U_{5,3}\\
m=6: -10.5U_{5,5}+22U_{6,5}-10.5U_{7,5} &= -9.6U_{5,4}+21U_{6,4}-9.4U_{7,4}-U_{6,3}\\
m=7: -10.5U_{6,5}+22U_{7,5}-10.5U_{8,5} &= -9.6U_{6,4}+21U_{7,4}-9.4U_{8,4}-U_{7,3}\\
m=8: -10.5U_{7,5}+22U_{8,5}-10.5U_{9,5} &= -9.6U_{7,4}+21U_{8,4}-9.4U_{9,4}-U_{8,3}\\
m=9: -10.5U_{8,5}+22U_{9,5}-10.5U_{10,5} &= -9.6U_{8,4}+21U_{9,4}-9.4U_{10,4}-U_{9,3}\\
m=10: -10.5U_{9,5}+22U_{10,5}-10.5U_{11,5} &= -9.6U_{9,4}+21U_{10,4}-9.4U_{11,4}-U_{10,3}
\end{aligned}$$

This is a system of equations obtained using the C-CNDS algorithm when $n = 4$ with the initial and boundary conditions, the system of equations can be written in matrix vector form as shown:

$$\begin{bmatrix}
22 & -10.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-10.5 & 22 & -10.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -10.5 & 22 & -10.5 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -10.5 & 22 & -10.5 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -10.5 & 22 & -10.5 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -10.5 & 22 & -10.5 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -10.5 & 22 & -10.5 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -10.5 & 22 & -10.5 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -10.5 & 22 & -10.5\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -10.5 & 22
\end{bmatrix}
\begin{bmatrix} U_{1,4}\\ U_{2,4}\\ U_{3,4}\\ U_{4,4}\\ U_{5,4}\\ U_{6,4}\\ U_{7,4}\\ U_{8,4}\\ U_{9,4}\\ U_{10,4} \end{bmatrix}$$

$$= \begin{bmatrix} 1.678070669\\ 0.15226\\ 0.216638\\ -0.00617\\ 0.066267\\ 0.054289\\ 0.048022\\ 0.019645\\ -0.38908\\ 1.114984 \end{bmatrix}$$

The solution to $u(x,t)$ obtained using MATLAB software when $m = 1,2,3,...,10$ and $n = 4$ are as follows:

$U_{1,5} = 0.038476213, U_{2,5} = 0.05110519, U_{3,5} = 0.0694169, U_{4,5} = 0.05045728, U_{5,5} = 0.05644327$
$U_{6,5} = 0.05917213, U_{7,5} = 0.05500865, U_{8,5} = 0.041675, U_{9,5} = 0.007814745, U_{10,5} = 0.00469285$

Table 4.4: Numerical Solutions at Varying Time Levels: Case Three, C-CNDS

Grid Point, (m, n)	$n_1 = 0.1$	$n_2 = 0.2$	$n_3 = 0.3$	$n_4 = 0.4$
$(m_1 = 0.1)$	0.08362174	0.0713175	0.0522915	0.038476213
$(m_2 = 0.2)$	0.10352006	0.09136106	0.068541	0.05110519
$(m_3 = 0.3)$	0.1016093	0.08557094	0.0759346	0.0694169
$(m_4 = 0.4)$	0.09086692	0.0835796	0.0668786	0.05045728
$(m_5 = 0.5)$	0.07748187	0.07041182	0.0634125	0.5644327
$(m_6 = 0.6)$	0.0693709	0.064394093	0.05742353	0.05917213
$(m_7 = 0.7)$	0.057615464	0.0536454986	0.0502588	0.05500865
$(m_8 = 0.8)$	0.04776154642	0.041647546	0.041675895	0.041675
$(m_9 = 0.9)$	0.03937339	0.0353551265	0.0339807	0.007814745
$(m_{10} = 1.0)$	0.011193351	0.00915414	0.070312134	0.004069285

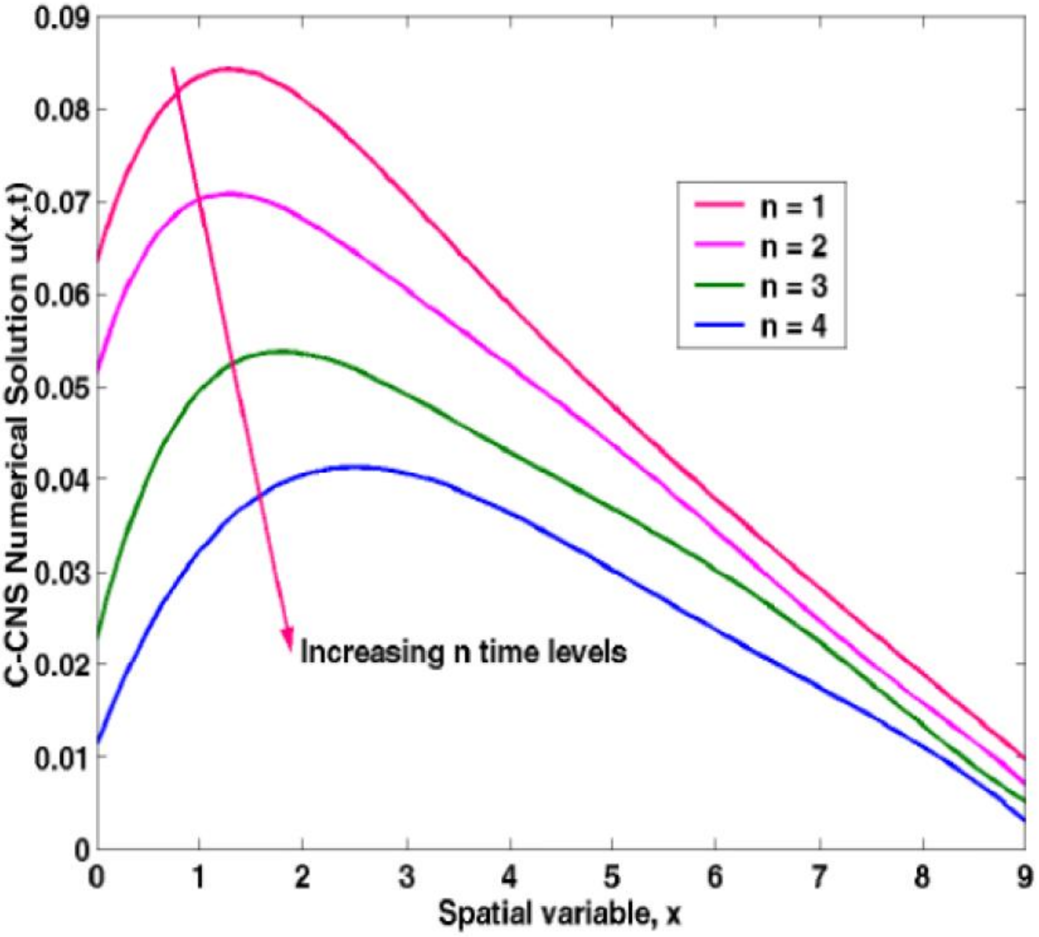
4.5.1 Discussion

In case three, we refined the mesh size to $h = k = \frac{1}{10}$ and developed a specific scheme:

$$\begin{aligned} &-10.5U_{m-1,n+1} + 22U_{m,n+1} - 10.5U_{m+1,n+1} \\ &= -9.6U_{m-1,n} + 21U_{m,n} - 9.6U_{m+1,n} - U_{m,n-1} \end{aligned}$$

The system of algebraic equations obtained expressed in matrix form was solved and presented in tabular and graphically, table 4.4 and graph 4.4. It was noted that the solution is similar to that of case one and case two. It also show that that reduction of mesh size does not cause a remarkable change in the in the result realized.

4.4 Graphical Presentation Using C-CNDS: Case Three



Chapter 5

CONCLUSION AND RECOMMENDATIONS

This chapter contains conclusion and recommendation for further studies which focuses on third order advection- viscous wave equation which is a Partial Differential Equation, with the aim of analyzing it guided by three objectives.

5.1 Conclusion

Third order advection-viscous wave equation was discretized and central Crank Nicolson scheme and forward Crank Nicolson scheme were developed equation 3.22 and 3.31. Matrix method was used to ascertain stability of both schemes.

We noted that the two schemes are unconditionally stable. To prove consistency each term of the equation was recovered from their approximations. We concluded that since the equation is stable and consistent it follows that it converges to a unique solution.

We managed to obtain matrix solution of the two developed schemes at various time levels using MATLAB software. It was seen that the schemes produce similar results. The findings were presented in tables and graphs which were then interpreted. Hence the general and specific objectives were achieved.

5.2 Recommendations

From the study we wish to recommend that there are areas that requires further analysis and development. These areas may include:

(i) To develop schemes for Backward-Centered and Centered-Centered analogue , solve the equation and compare the results.

(ii) Use Vonn Naumann method for stability and consistency analysis for both Central Crank Nicolson Difference Scheme and Forward Crank Nicolson Difference Scheme.

(iii) Find the numerical solution to this equation by varying the viscosity and advection coefficients μ and α.

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