ON CONTINUITY AND SEPARABILITY IN BITOPOLOGICAL SPACES

BY

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A Thesis Submitted to the Board of Postgraduate Studies in Fulfilment of the Requirements for the Award of the Degree of Doctor of Philosophy in Pure Mathematics

SCHOOL OF BIOLOGICAL, PHYSICAL, MATHEMATICS AND ACTUARIAL SCIENCES

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Declaration

This thesis is my own work and has not been presented for a degree award in any other institution.

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W261/4039/2020

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This thesis has been submitted for examination with our approval as the university supervisors.

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Acknowledgments

I am deeply grateful to many people who contributed in one way or the other in writing of this work. In particular, my sincere thanks go to Prof. Benard Okelo and Prof. Omolo Ongati who were my supervisors and also my post graduate level(masters) lecturers for taking me through the content in class. Without them, I would have not reached this far since it was posted with many challenges but they were always available to offer me guidance. I thank all my fellow PhD students for their useful contributions. I am very thankful to my beloved wife Mary, together with all our children for their support and patience they granted me during this course as I worked through this research tirelessly. May God bless them all and reward them abundantly.

Dedication

To my parents, my beloved wife Mary, my son John-Mark Ogolla and my daughter Liana Blessings Ogolla.

Abstract

Many studies have been conducted on properties of bitopological spaces and aspects of continuity over a long period of time and different results have been obtained so far. However, pointwise characterization of various aspects of continuity has not been done in bitopological spaces. Moreover, our work is aiming at establishing particular separation criteria for bitopological and spaces where $N > 2$. This therefore calls for an in-depth study of continuity and separability in bitopological spaces. The objectives of the study were to: characterize notion of ij-continuity in bitopological spaces; establish separation criteria for bitopological spaces via ij-continuity; and determine extensions of continuity and separability in N-topological spaces. The methodologies involved use of criterion for continuity, criteria for inverse continuity, separation axioms and conditions for normality. The results showed that various continuity notions such as π_λ, θ_η and π_d exist in bitopological spaces. For separation criteria, the results showed that if bitopological spaces are T_0, T_1, T_2 and $T_{\frac{5}{2}}$ then T_0, T_1, T_2 and $T_{\frac{5}{2}}$ properties are both topological and hereditary. For extension and separability in N-topological spaces results indicated that properties can be naturally extended to N-topological spaces. The results obtained are useful in studying topological deformations such as stretching which is fundamental in understanding the shape and structure of the universe and formulations of real functions and topological mappings. Our results also help in deep understanding of molecular biology more particularly on DNA structure. Our results also play a great role in understanding the applications of computer topology such as line, ring, star and hybrid topologies.

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Chapter 1

INTRODUCTION

1.1 Mathematical Background

A bitopological space is a mathematical notion that was first introduced by Kelly [39] in the study of quasi-metrics. A metric or distance function can be defined as a distance between each pair of point elements of a set. The authors Levine [48] and John [36] stated that any metric space has some metric distance. A nonempty set with a metric structure is also referred to as metric space under certain conditions which are satisfied by axioms. Topological and bitopological spaces involve structures which are endowed by topologies or structures.

A nonempty set X is said to be a bitopological space if and only if it is equipped with two topologies say τ_1 and τ_2. Therefore, (X, τ_1, τ_2) is a bitopological space. Researcher Martina [47] defined a bitopological space to be a space that is endowed by two topologies which are quasimetrics. From the work done by Kocinac [43] and Piyali [59] also stated that a bitopological space is equipped with two topologies. The work that was effected by Arhangel'skii [10] gave an account on quasimetrics as topological structures induced on a set by a metric. A bitopological spaces can exhibit some general characteristics for instance compactability. If a space has an open cover with finite subcovers then that space is said to be compact. The result of the work that was done by Sasikala [65] and Steve [71] affirmed that the union of the topologies is a member of the open cover in a bitopological space.

Every open cover has the finite subcovers which are the cardinality of such sets. Secondly, we consider openness as a property that is exhibited by bitopological spaces. An open bitopological space is from a set that has no limits that is both lower and upper limits. Open sets contain all the interior points Gurnn [27] and Allama [3]. A subspace is semi-closed if the interior closure of that set is subset of itself.
Next, bitopological spaces are seen to have closedness property. Researchers Henri [28] and Budney [16] stated that closedness is a basic concept in mathematical related areas. Points are closed to each other if they are next to each other. Given that a bitopological space (X, τ_1, τ_2) is closed then; the empty set and the entire set X are closed sets. Moreover, Van [78] gave that the intersection of any collection of closed sets is also closed. Lastly, the union of any finite collection of closed sets is itself also closed.

Another property exhibited by bitopological spaces is normality. Suppose that a topological space is normal then its bitopological space is also normal. Bitopological spaces also exhibit normality as a property since it can be extended from the topological spaces to bitopological spaces as seen from the work of Birman [15] and David [21]. Normality is when the two disjoint closed sets are separated by an open set. According to

Singal [69] it is indicted that a space can be perfectly normal if that space admits enough continuous real valued functions. Bitopological spaces are termed to be normal by extension from the topological spaces. The fact that topological structures are induced with the properties of normality.

Consequently bitopological spaces will inherit same property by extension. However, the condition of two disjoint closed sets being separated by open set must be met. Hence, any bitopological space that satisfy these conditions must be normal. The scholars Tkachenko [76] and Just [37] in the study of K-normality of dense topological subspaces stated that a normal space is not necessarily normal in a bigger space. Furthermore, separability is a property that is exhibited by bitopological spaces.

From the work of Nour [52] a separable bitopological space is defined as a space with a set containing dense subset of finite cardinality for instance when we have a sequence x_n where n ranges from 1 to ∞. Any infinite countable is a separable space. Separation axioms that have been implied by different authors in their studies involve: Kolmogorov space, Fretchet space, Hausdorff space, Urysohn space, Regular Hausdorff space, Tychonoff space, Normal Hausdorff space, Completely Hausdorff space and Perfectly Normal Hausdorff space.

These separation axioms are denoted as T_0, T_1, T_2, $T_{2\frac{1}{2}}$. Regarding Ananga [5] it is stated that they are separation axioms since they define the notion of topological spaces. These separation axioms may be used as extra conditions to describe the structures of what spaces are. Some topological structures may be considered as infra topologies and supra topologies. These topologies continuously map elements from domain to codomain of different sets. They deal with two universal sets simultaneously hence they are very vital in mathematics analysis. A single structure called binary structures has been constructed by Marcus [47] and Nicolas [50] which give information about two universal sets and this can as well be initiated to the concepts of binary topological spaces. For the finite and countable cardinality in a bitopological space, a function mapping any of these spaces are continuous. From the work that was done by Karel [38] the result shows that a subset of a Hausdorff space is countable dense.

A function mapping a separable space to another space is intern separable Kilcman [44]. In order to carry out a study of any topological space, one would consider the restrictions that are involved and hence imply the separation axioms to determine which topological or bitopological space to be under taken. Some of the separation axioms that this study considers are: T_0- Kolmogorov space, T_1-Frechét space, T_2-Housdorff space, $T_{2\frac{1}{2}}$-Urysohn space, T_3-Regular Housdorff space, $T_{3\frac{1}{2}}$-Tychonoff space, T_4-Normal Housdorff, T_5-Completely normal Housdorff, and T_6-Perfectly normal Housdorff. T_1- space or Frechét space is a bitopological space in which every two disjoints closed subsets are topologically separated by neighborhoods. Some of the separation axioms imply each other.

For instance, Kolmogorov space implies Fretchet space. Kolmogorov is a space that has two disjoint sets that are topologically separated by two open sets in that space. If one open set is a member of a set in that space then it suffices that the other open set does not exists in that set. The intersection of open sets in Frechét spaces is not an empty set, this consequently applies to Kolmogorov spaces. Bitopological spaces show continuity property when mapping is done from one bitopological space to another. Some of the aspects of continuity of bitopological spaces where research has been done include weak continuity, semi continuity, strong continuity among others. Continuity refers to the mapping of elements from one space to another without any break occurring. Continuity is the smooth movement of a function without any stop that causes discontinuity. Continuity is determined by functions.

From the work done by Caldas [18] shows that a continuous a function maps one bitopological space to another space is continuous if and only if it is continuous at each point. As stated in Albowi [4] a function g that is mapping elements of one bitopological space to another bitopological space. Hence, g is a continuous function. Likewise a function f mapping a bitopological space X to a bitopological Y is continuous if $f : (X, \tau_1) \to (Y, \tau_1)$, David [20]. Similarly, $f : (X, \tau_2) \to (Y, \tau_2)$ then this function is said to have pairwise property since it is mapping members of same topologies from one space to another. Results from Fora [25] states that a function is pairwise continuous if it maps open point from one bitopological space to another independently.

Suppose both discrete and trivial topological structures are induced to different bitopological spaces. Then any function mapping an open subset from trivial bitopological space to another trivial space is also continuous as indicted by Samer [64]. Most aspects of continuity can be extended from one space to other spaces given that a function mapping them is continuous. These are weak continuity, strong continuity, semi continuity, and local continuity. This can be obtained by the use criterion for continuity as a methodology. From the work that was done by

Ananga [5] on the study of strong continuity and almost continuity it is observed that weak invariant, strong invariant and other invariants of continuity occur and arise in very many ways in the field of mathematics. The notion of strong continuity was first undertaken by Levine [48]. For topological spaces that exhibit the homeomorphic property, any function mapping them is continuous. In addition, the inverse of this function is also continuous. Methodology of continuity is applicable as well when undertaking a study of continuity in different topological spaces. Criterion for continuity as methodology can also be employed when we have three bitopological spaces for instance (X, τ_1, τ_2), (Y, δ_1, δ_2) and (Z, η_1, η_2). Suppose $f : X \to Y$ and $g : Y \to Z$ then f and g are continuous functions as it is given by Aly-Nafie [8].

Thaikua [75] gave open definition of the word function as an object that depends on another factor or factors. For instance, plants that grow in the field depend on factors like climate, soil type and other environmental factors. The study conducted by Parvinder [56] also gave definition of a function as a mathematical object that relates input also called domain to an output that is also called codomain.

Some functions have both forward and reverse mapping of elements between topological spaces. Reverse functions undo the forward mapping of the elements. Suppose the function f is obtained by squaring the elements from the domain space, then the inverse of a function f is obtained by determining the square root of elements from codomain. From the work of Einsiedler [23] states that a bitopological space is said to be continuous if and only if one bitopological space can be mapped to another bitopological space by a function f. When a function f maps elements from a domain topological structure to another corresponding elements in codomain then it is pairwise. For instance when a function independently maps elements of topological structures in one space to another space then, the function is said to be pairwise continuous as stated by Archana [9]. In our work, we have tried to focus more on some particular aspects of continuity that are exhibited by bitopological spaces. Studies which were done by some authors such as Birman [15] and Abu-Donia [6] showed that these aspects of continuity can as well be extended from one space to other spaces. In topology and related areas of studies exhibit these aspects of continuity which include: Weak continuity, Strong continuity, Semi continuity, and Local continuity. From the Kohli [42] in the study of strong continuity and almost continuity stated that several weak, strong and other invariants of continuity occur and arise in very many ways in the field of mathematics. The notion of strong continuity was introduced by Levine [48]. Later the study of strong continuity was studied by very other authors. For instance Noiri [51] initiated the σ-continuity. Study on weak continuity was carried out by Van [78] and Tahilini [73] stated that a function is weakly continuous if and if the inverse of every open set in codomain is also open in the domain space. To understand this work better, we outline some basic concepts in the next section.

1.2 Basic Concepts

This section outlines the basic concepts which are useful in understanding this study.

Definition 1.1. [39, Definition 1.2] A bitopological space (X, τ_1, τ_2) is a space that is endowed with two independent topologies say τ_1 and τ_2 denoted as (X, τ_1, τ_2).

Definition 1.2. A function χ is said to be θ_η-continuous if the inverse of open set in a bitopological space is θ-open set in a bitopological space X.

Definition 1.3. [42, Definition 2.1] Given that a function f is mapping a topological space (X, τ) to topological space (Y, δ) then f is said to be; strongly continuous if the inverse every open set in Y is open in (X, τ). A function f is perfectly continuous if every open set in topological space (Y, δ) is open in (X, τ).

Remark 1.4. [33, Remark 3.12] If $f : (X, \tau) \to (Y, \sigma)$ is g-closed and $g : (Y, \sigma) \to (Z, \gamma)$ is closed, then their composition need not to be a g-closed map.

Definition 1.5. [63, Definition 2.6] A bitopological space (X, τ_1, τ_2) is called T_1 space if for all elements x and y are members of X where $x \neq y$. Then there exists an open set $U \in \tau_1$ and open set $V \in \tau_2$ such that x is a member of U and y is a member of V.

Definition 1.6. A function $\chi : X \to Y$ is ij-continuous if and only if the inverse image j-open in a bitopological space (Y, δ_1, δ_2) is i-open in a bitopological space (X, τ_1, τ_2).

Definition 1.7. [1, Definition 1.4] A bitopological space (X, τ_1, τ_2) is called pairwise T_0 if and only if for any two distinct points $x, y \in X$, there exists a set $U \in \tau_1 \cup \tau_2$ such that $x \in U$.

Remark 1.8. A topological space $\pi B(X, \tau)$ shows open subsets in a topological space (X, τ).

Example 1.9. [14, Example 3.10] There exists a separable countably compact Tychonoff space X containing open countably compact disjoint subsets U_0 and U_1 such that the intersection $U_0 \cap U_1$ is weakly separable but non separable.

Definition 1.10. [40, Definition 3.2] A mapping $f : X \to Y$ is called $ij - \beta$-continuous (resp. ij-precontinuous, $ij - \alpha$-continuous) if and only if the inverse of each i-open set in Y is $ij - \beta$-open (resp. ij-preopen, $ij - \alpha$-open) in X. A function f is therefore said to be pairwise if and only if it is $ij - Q$ where $Q = \beta$ continuous, precontinuous, or α-continuous.

Definition 1.11. [73, Definition 3.4] Let $(X, N\tau)$ be an an $N - TS$. If for each decreasing (respectively increasing) $N\tau$-closed subset W in X and for each s does not belongs to W there exists an $N\tau$-neighbourhood G of s and an $N\tau$-neighbourhood H of W such that G is increasing (respectively decreasing). An $N\tau - T_1$ space $N\tau$-regular space is said to be $N\tau - T_3$.

Definition 1.12. [73, Definition 3.4] Given that (X, N_τ) is N-topological space then if N_{τ_1}-open set and N_{τ_2} are disjoint points which are separated by open neighborhoods. Hence $N_\tau - \tau_1$ space is N_τ-regular space.

Definition 1.13. A subset W of X is said to be π_λ-open in X if and only if the $\bar{W} \subset W$. A function $\chi : X \to Y$ is said to be π_λ-continuous.

Definition 1.14. [75, Definition 2.5] $(X, \tau_1, \tau_2, ...\tau_N)$ is said to be N-topological space is a space if it is equipped with N arbitrary number of topologies.

1.3 Statement of the Problem

Some open questions on aspects of continuity in bitopological spaces have been raised for over a long period of time. From the research that was conducted by [2] on pairwise continuity in bitopological spaces. The main question posted was that what happens if we consider topologies $N > 2$?. In our study, we looked at some aspects of ij-continuity in bitopological spaces and topological spaces with $N > 2$. On separability, quite a number of separation axioms have been studied by different authors such as Ruppaya and Hossain [63], Nour [52], Piyali and Binod [59] among others.

However, unique and new criteria arise quiet often and this notion of separability has never been exhausted particularly on topological spaces where number of topologies are more than two. The fact that some concepts of separation axioms in bitopological spaces satisfy topological and heredity properties. Rupaya and Hossain [63] asked a question that Are there particular separation axioms that act only on bitopological spaces. In our study we have considered this question and tried to establish separation axioms for bitopological spaces via ij-continuity. We have also tried to show the extension of semi-continuity, strong continuity and weak continuity as aspects of continuity and separation axioms in N-topological spaces by the use of notion of ij-continuity.

1.4 Objectives of the Study

1.4.1 Main objective

The main objective of this study was to characterize continuity and separability in bitopological spaces.

1.4.2 Specific objectives

The specific objectives of this study were:

(i). Characterize notion of ij-continuity in bitopological spaces.

(ii). Establish separation criteria for bitopological spaces via ij-continuity.

(iii). Determine extensions of continuity and separation axioms in N-topological spaces.

1.5 Significance of the Study

The study of bitopological spaces is vital since it is a very powerful tool in almost every field of contemporary mathematics such as general topology, real analysis, metric spaces, function analysis among others. A bitopological space gives a complex nature of the examples to which the theory applies. This can in turn assist in achieving great economy effort if one proof can be applied to many contexts for instance, continuous functions in bitopological spaces help in computing output in mathematics based

on the relation between various important variables in contemporary society which is relevant in construction industries and factories.
Our results are applied in the areas of general topology and functional analysis. The answer to the question on weather properties of bitopological spaces and its aspects of continuity can be extended to N-topological spaces aid in understanding the deformations of topological and bitopological spaces such as stretching which explain the shape and structure of the universe and formulation of real functions. Our results also help in deep understanding of molecular biology more particularly on DNA structure.

Chapter 2

LITERATURE REVIEW

2.1 Introduction

This chapter entails a review of related literature for some aspects of continuity in topological and bitopological spaces. We also consider literature for separation axioms that have been described by different authors in both topological and bitopological spaces.

2.2 Continuity in bitopological spaces

This part describes continuity as a property of bitopological spaces.

Proposition 2.1. *[35, Proposition 2.3] A bitopological space* (X, τ_1, τ_2) *is a space that is equipped with two topologies.*

Proposition 2.1 clearly indicates the twin topology structure in a bitopological space. Some scholars such as Jesper [35] and Marcus [46] showed that when a non empty set is equipped with twin topological structures say τ_1 and τ_2 then that space becomes a bitopological space denoted as (X, τ_1, τ_2). Work carried out by Fuad [26] on some properties exhibited by these structures shows that a bitopological space has two structures. For compactness the results show that the union of these structures have their subcovers in these structures. Compactness property exhibited by more than two topological structures has not been exhausted. In our study we have shown that a non empty set X can as well be endowed by N-topologies. Bitopological spaces are seen to be continuous as described by Fora [25] and Ittanagi [30]. Continuity in bitopological spaces is when a function maps space to another without any break as given by Nada [49].

A function f that maps one bitopological space to another bitopological space can as well map each closed sets which are members of a bitopological space then that function is also said to be continuous as stated by Duszynski [22]. Bitopological spaces can be clopen this is seen from research work that was done by Kumar [45] also explains that clopen set is when the structures are both open and closed. A function that is mapping closed set from domain space to a codomain space is said to be a closed function. This shows clearly that bitopological spaces exhibit closedness and open properties. Since bitopological spaces are equipped and endowed by two independent topologies or topological structures as a result of this two topologies exhibit many properties to the space such as closedness, openness, normality, compactness, continuity among others. Continuity of bitopological spaces exhibit some forms and aspects of continuity which may include weak continuity, strong continuity, semi continuity, global continuity, almost continuity among others. Some of the literary work that have been done on these aspects of continuity by different authors are given by the following algebraic obstructions.

Theorem 2.2. *[4, Theorem 1.2] Given that* $f : (X, \tau_1, \tau_2) \to (X', \delta_1, \delta_2)$ *then a function* f *is continuous.*

Theorem 2.2 shows that a function f maps one bitopological space to another bitopological space. A function f maps every open set in domain to its open image in codomain. The inverse image of every open set in codomain can also be mapped by a function f to open set in domain. From Kelly [39] it is stated that pairwise continuity is when a function f maps definite structures from one space to another. Consequently, topological structure τ_1 in one space is mapped to τ_1 in space two.

Mapping that involve more than two structures in topological spaces have not been worked on adequately. In our study, we have considered a map from one bitopological space to another bitopological space which are endowed by different topologies more than two. Bitopological spaces may have covers which is a member of the union of the topological structures. When there exists a finite subcover then this space has compactness property as stated by Arunmanan [11] and James [34] in their studies.

Moreover, we have considered the compact property in bitopological spaces when U_i such that $i \in I$ must contain more than one member from topological structures. For locally compact regular spaces there exists a neighborhood. A study in compactification was also initiated by Simon, [68]. Result showed that every locally compact regular space there exists a neighborhood at each point of closed set. However, this has not been shown on clopen topological spaces. The researchers Albowi [4] and Ivan

[31] conducted a study on compactness property. The result shows that when a function mapping a bitopological space to another is continuous then, it has compactness property. Suppose $f : (X, \tau_1, \tau_2) \to (Y, \delta_1, \delta_2)$ then if $f : (X, \tau_1) \to (Y, \delta_1)$ and also $f : (X, \tau_2) \to (Y, \delta_2)$. A function f is therefore continuous and is compact if the bitopological spaces are pairwise compact to each other. Kelly [39] conducted a research on pairwise compactness. Results show that when a bitopological has an open cover say U and if i is a member of U then open cover U_i exists in the union of two topological structures. In our study we have considered compactness in N-topological spaces. A bitopological space is pairwise Hausdorff if it has two disjoint points which are topologically separated by open sets. Each disjoint is a member of respective open set.

Normality is a property that is seen in spaces. A space is said to be normal if disjoint closed sets can be separated by open neighborhoods and the intersection of the open sets is empty.

Theorem 2.3. *[39, Theorem 2.7] Given that* (X, τ_1, τ_2) *is a bitopological space then it is a pairwise normal space.*

Theorem 2.3 illustrates that the product of two independent topologies is not normal as given by Kelly, [39]. If we have (X, τ_1, τ_2) to be a be the real line with metrics $\tau_1(,)$ and $\tau_2(,)$ defined on quasi-psuedometrics $\tau_1(,)$, $\tau_2(,)$ and $U(,)$, $V(,)$. In our study we have considered that the product of same topologies in independent spaces are not necessarily normal as affirmed by David [21].

A function f that maps one bitopological space to another is said to be homeomorphic continuous if and only f is continuous or if the inverse of f is also continuous. Tala [74] conducted a study to show that given two bitopological spaces then, a function is said to be continuous the open inverse subset in the codomain space is also open in domain space. We have shown the same aspect in tritopological space in our study.

Research that was carried out by Abdalla [1] indicates that a function f can map one bitopological space to another bitopological space if it is a bijective function. However, continuity is not differentiated whether in closed, open or homeomorphic bitopological spaces. In our work, we have presented these properties with topologies that equip non empty sets. Closedness property of bitopological spaces is observed when both empty set and that set itself are closed. Therefore, for these spaces to exhibit closedness property the following axioms must be fulfilled as showed by Sheik [66].

The aspect of normality and separation axioms such as T_0 and T_1 in topological spaces was also done by Einsiedler [23]. If T_1-space of a bitopological space X is said to be normal on another bitopological space Y. Then it implies that Y is also a Tychonoff space. Therefore, A function f is continuous at each point if and only if there is a member of space X and is an open subset which is mapped to space Y. Likewise the inverse of every open subset in codomain space is also open in X. In our study, we have considered a $T_{\frac{5}{2}}$-space.
From Piyali [58] it implies that any function that is mapping a bitopological space to another space is continuous if and only if the inverse of the codomain is a member of the domain and also the inverse of the domain is in the codomain space. When both discrete and trivial topological structures equip different spaces then functions f and h that are mapping

each discrete topological space to another and trivial bitopological space to another are also considered to be continuous.

A scholar Bhattacharya [17] conducted study on openness as a property of bitopological spaces. Given that two topologies τ_1 and τ_2 are open then their union is also open. In our study we have presented openness property in N-topologies by showing that the union of N-topologies are also open. Study from Nicolas [50] indicated that a function f mapping a bitopological space to another is seen to be open if and only if their pairwise mappings are also open. However this property has not been shown with spaces with more than two topologies. In our study we have considered this in N-topological spaces.

Homeomorphism is a property that is seen in bitopological spaces when a mapping function f is bijective this was shown by Ravi [61] and Adem [2]. We have considered in our study homeomorphism in three successive bitopological spaces which are endowed by different topologies. For separability in topological spaces there are countable dense subsets of the sets that form bitopological spaces. The fact that a subspace is pairwise dense is shown from the work of Abdalla [1]. This is because closure of the subset of one topological implies the closure of another topological space as well as that of set X and can be continuously mapped to another topological space. Our study has shown that bitopological spaces exhibit countable dense subset which must be a member bitopological spaces.

Study on on connectedness and compactness was effected by Arunmanan [11] and Pervin [57]. The results show that if a domain space is locally connected then it suffices that the codomain has an element which is normal. The existence of a subspace of cardinality of the intersection of open neighborhoods is not shown. For local connected spaces and their dense subspaces are seen to be normal and hence for every open neighborhood there exist open subsets which are members of that set. For a surjective function f that maps a complete regular spaces to each other. If the inverse of the codomain space is in the domain space is compact. Hence any element in the codomain space is an almost regular spaces. Birman [15] carried out a study on almost completely continuous surjection. Results show that a function maps a clopen sets to another a clopen sets. In our study we have shown the mapping of closed sets to clopen sets from one space to another.

Corollary 2.4. *[53, Corollary 5.6] Suppose f is mapping a completely continuous closed space. Then that function is surjective.*

Corollary 2.4 indicates that a function mapping one bitopological space to another is completely continuous closed if every subset in codomain is regular. Researcher John [36] conducted a study on the composition of functions mapping three successive bitopological spaces (X, τ_1, τ_2), (Y, δ_1, δ_2) and (Z, η_1, η_2). The composition of two functions mapping spaces is continuous as well. In our study we have also considered the composition of three functions on continuity aspect. The composition of functions g and f implies that $f : (X, \tau_1, \tau_2) \to (Y, \delta_1, \delta_2)$ and $g : (Y, \delta_1, \delta_2) \to (Z, \eta_1, \eta_2)$. Then it suffices that $g \circ f : (X, \tau_1, \tau_2) \to (Z, \eta_1, \eta_2)$ is also continuous as indicated by Kim [41] and Coy [19].

A function f that is mapping one bitopological space to another bitopological space is said to be continuous if the inverse of the codomain contains a member that is closed in the codomain space. In our study we have extended this concept to N-topological spaces. Suppose two functions are mapping one topological or bitopological space to another independently then they are continuous. Then the composition of these two functions mapping the first topological or bitopological to the third bitopological space is also said to be continuous. As stated by Jafari [33] and Swart [72] let X and Y be bitopological spaces, f is a function mapping X to Y. Then f is homeomorphic if it continuous, if f is a one to one and onto which implies implies that the inverse of Y is in X. It is also homeomorphic when its inverse is also continuous.

For the composition of the two functions to be completely continuous then it implies that one of the functions must be almost continuous and the other function must be able to map a regular open set to another regular open set.

Theorem 2.5. *[24, Theorem 2.6] Given that X and Y are bitopological spaces. A function f is pairwise continuous if it maps (X, τ_1) to (Y, δ_1) and (X, τ_2) to (Y, δ_2).*

The study of properties such as pairwise Lindelöf in bitopological spaces was first initiated by Fora [25]. Establishment of more studies on concepts of pairwise continuity, pairwise open and pairwise homeomorphism was initiated more. Studies show that given different bitopological spaces which are endowed by discrete topologies Kilcman [44]. Therefore, a function f mapping one bitopological spaces to another bitopological space is continuous. In cases where bitopological spaces are not pairwise continuous or pairwise Lindelöf continuous a function f that maps one bitopological space to another bitopological space has been shown not to be continuous as indicated by Arhangel'skii [10]. Bitopological spaces that are mapped by a function to each other are said to be pairwise semi-regular if and only if the functions mapping bitopological spaces are almost pairwise open as shown in the next result as shown by Budney [16].

Theorem 2.6. *[51, Theorem 4.1] A pairwise semi-regular space is pairwise open if and only if it is almost pairwise open.*

Lemma 2.7. *[30, Lemma 7] Let $(X, \tau_1, \tau_2 E)$ be a soft bitopological space.*

Bitopological spaces have also been observed to exhibit property of soft sets. Soft set property in bitopological spaces has been investigated by some authors such as Ittanagi [30] and Marcus [46] among others. Bitopological spaces that exhibit soft sets property are also showing some other properties and concepts. An account that if there are two soft bitopological spaces and a function f that maps one soft bitopological space to another then that function f is regarded to be continuous was also given by Ittanagi [30].

From the research work did by Norman [53] elaborates that a function f in ordinary scenarios means a relation between input and output. Our study has included continuity of spaces involving more topological structures. Studies have been conducted on some aspects of continuity such as weaker forms of continuity and semi-continuity. Likewise Bakier [13] conducted studies on semi continuity as an aspect of continuity that is exhibited by bitopological spaces where it is seen that a function f can also be semi-continuous by mapping one bitopological space to another. These are given in the results that follow.

Example 2.8. [13, Example 5.4] Given that $f : (X, \tau_1, \tau_2) \to (Y, \delta_1, \delta_2)$.

f is semi-continuous if the inverse of open subset in space Y is semi-open in X.

Most results obtained by Trishla [77] show that any function f that is taking one bitopological space to another bitopological space must be pairwise continuous and the inverse of the codomain of that function forms a cardinality of the domain space. Resaerch work that was carried out by Khedr [40] on the decompositions of i-continuity and pairwise continuity in bitopological spaces. Their results of continuity in ij-sets and spaces have been described in details. In our study we have characterize the notion of ij-continuity in bitopological spaces considering aspects such as between strong continuity and almost continuity are also aspects of continuity in bitopological spaces.

2.3 Separation Axioms

Different authors carried out studies on continuity and some of its aspects in various spaces. From their studies respective results have been obtained. Scholars who conducted these studies used different separation techniques in order to achieve successful results. Suppose that a scholar may need to test properties of any separation axiom then they have to choose a space say either topological or bitopological space to effect the same.

Moreover, separation axioms also infer the restrictions that mathematics researchers always make regarding the kind of space that they intend to consider. Similarly, studies also show that these axioms apply to topological or bitopological spaces since we can distinguish disjoint sets and distinct points in different sets. The outcome of the study of Fora [25] indicated that topological and bitopological spaces whose elements can be distinguished are referred to as separable topological spaces.

A bitopological space (X, τ_1, τ_2) has got classes which include infra topologies and supra topologies. Infra and supra topologies are classes introduce some new properties in bitopological spaces as stated by Abu-Donia [7]. Topological spaces exhibit properties $T_{\frac{1}{2}}$, T_b, αT_b, T_d, αT_d. These properties can be extended to bitopological spaces. For soft bitopological spaces studies have been done and there are interesting characterizations as indicated by Patil [54]. Some of the binary separation axioms are binary T_0, binary T_1, binary T_2 spaces. In this our study we have considered the properties of $T_{2\frac{1}{2}}$-spaces in bitopological spaces. This result also shows that binary soft property can be inherited.

Theorem 2.9. *[55, Theorem 3.20] Suppose (X, τ) is a T_2 then it has hereditary property.*

From the the studies conducted by Rajesh [60] results show that quasi $T_{\frac{1}{2}^*}$ space is also another type of separation axioms. However, $T_{\frac{1}{2}^*}$ has not been been effected via ij-continuity. In our work, we have shown separation criteria via the notion of ij-continuity. Authors such as Hussein [29] and Rupaya [63] showed heredity and topological properties which seems to be exhibited some separation axioms among in topological spaces. In our work we have shown hereditary properties of these separation axioms on bitopological spaces and where number of topologies is greater than two. Some results of heredity property are given below.

Theorem 2.10. *[29, Theorem 3.1] A bitopological space which is T_0 is considered to have hereditary property.*

Subspace properties are inherited as seen in Theorem 2.10 illustrates that if a bitopological space (X, δ, τ) is T_0 space we have open set U whereby $U \in \delta \cup \tau$ hence it implies that $m \in U$. We have shown how hereditary is induced by N-topological spaces to subspaces.

Theorem 2.11. *[52, Theorem 3.5] Let (X, τ) be a topological space and a T_1-space.*

Most of the research on separation axioms acting on aspect of homeomorphic as a property in bitopological spaces was conducted by Patil [54]. The result show that homeomorphic property is exhibited by bitopological spaces. For a bitopological space the homeomorphic image of a particular separation axiom is that particular axiom. The proof that was shown by Rajesh [60] shows that T_2-space has both hereditary and topological properties. If (X, τ_1, τ_2) is a bitopological space then it follows that it has two disjoint points which can be separated by the open sets. Each disjoint point exists in each open set independently. This implies if one point belongs to one open set, then it is not a member of the other open set as stated by Swart [72].

Chapter 3

RESEARCH METHODOLOGY

3.1 Introduction

For this work to be completed successfully, a good background knowledge of general topology, continuity of functions and functional analysis are found to be more crucial and vital in our work. We have restated some known results which were useful to our work. The research methodology employed included; criterion for continuity, criteria for inverse continuity, separation axioms or Tychonoff theorem and conditions for normality.

3.2 Criterion for continuity

Criterion for continuity is a methodology that has been used by various authors to show the continuity property of functions in topological spaces. Criterion for continuity is a technique which shows that a function mapping one bitopological space to another bitopological is continuous. From the research that was done by Birman [15] if a function f is taking an element from one bitopological space to another bitopological space, then f is said continuous if and only if its inverse is also continuous. Kelly [39] When a function f mapping $(X, \tau_1, \tau_2) \to (Y, \delta_1, \delta_2)$ is said to be continuous if $f : (X, \tau_1) \to (Y, \delta_1)$ and also when we have $f : (X, \tau_2) \to (Y, \delta_2)$ therefore, a function f is said to be pairwise continuous. Zabidin [80] affirmed that a function is only pairwise continuous if it maps open subsets from one bitopological space to another independently. This methodology is used to show how members of topological structures can be mapped from one space to another.

Suppose both discrete and trivial topological structures are induced to different bitopological spaces. Then any function mapping an open subset from trivial bitopological space to another trivial space is also continuous, Samer [64]. Some researchers Birman [15] and Abu-Donia [6] have shown that most aspects of continuity can be extended from one space to other spaces given that a function mapping them is continuous. These include weak continuity, strong continuity, semi continuity, and local continuity. When this methodology is employed in any topological under study then continuity as a property is clearly displayed.

From the study of strong continuity and almost continuity it is observed that weak invariant, strong invariant and other invariants of continuity occur and arise in very many ways in the field of mathematics. The notion of strong continuity was first undertaken by Levin [48]. For topological spaces that exhibit the homeomorphic property, any function mapping them is continuous. In addition, the inverse of this function is also continuous. Methodology of continuity is applicable as well when undertaking a study of continuity in different topological spaces. Employing this methodology, given three bitopological spaces and two functions say X, Y and Z. If $f : X \to Y$ and $g : Y \to Z$ are continuous. By the use of continuity as methodology the composition of these two functions will also be continuous as indicated by Ivan [32]. In our work we have found this technique of criterion for continuity to be very relevant since we used it show that a function $\chi : (X, \tau_1, \tau_2 \to (Y, \tau_1^{'}, \tau_2^{'})$ is continuous if and only if the inverse of the open set in a bitopological space $(Y, \tau_1^{'}, \tau_2^{'})$ is π_λ-open set in domain space (X, τ_1, τ_2).

3.3 Criteria for inverse continuity

Criteria for inverse continuity is a technique that can be used to show that continuous bijection function also has its continuous inverse. This follows that if it is both injective and surjective then any element from the codomain space is an image of all elements in domain space. For instance if D is an image element from codomain space which is precisely for C. from domain space. Marcus [46] indicated that a function f is an injective function and continuous on a space I. Since $f : I \to J$, then this function is said to be continuous if and only if its inverse is also continuous. Suppose a function f is taking back an element y from a space J to I such that f^{-1} is continuous, then it is referred to as to inverse function. Since $y \in J$ for simplicity we can therefore assume that y being a member of J is not the end point of J. This implies that inverse function f^{-1} exists and continuous on a corresponding interval J which is in the image range of

f. When a function maps an open subset from the domain to codomain it is known to be continuous. Likewise, when the same function takes the image of the open subset from codomain back to domain space then the inverse of f is also continuous as affirmed by Sidney [67]. Consider a function $f(x) = 5x + 3$ which can also be expressed as $y = 5x + 3$ Therefore, obtaining this function we need to multiply our domain x by 5 and add 3 to our result. This gives $5x + 3$ as our co-domain. For inverse we go the other way. We subtract 3 from y and then divide it by 5 this gives $(y-3) \div 5$. Hence the inverse of: $5x+3$ is: $(y-3) \div 5$. In our work, we used this technique of criterion for inverse continuity to show that $\chi^{-1}(x)$ which is θ-open in a bitopological space (X, τ_1, τ_2) is also continuous since it maps a π_λ-set to a bitopological space $(Y, \tau_1^{'}, \tau_2^{'})$.

3.4 Separation axioms

Separation axiom is a technique that is used to topologically separate disjoint points in a particular space. Separation axioms are restrictions that researchers always make on particular topological and bitopological spaces they are intend to conduct a study on. The notion of separation axioms has been effected by authors such as Rupaya [63] and Ravi [61] have defined a separable bitopological space as a space with a set containing dense subset of finite cardinality for instance when we have a sequence x_n where n ranges from 1 to ∞. Any infinite countable is a separable space as indicated by Watson [79] and Arya, [12]. Separation axioms that have been implied by different authors in their studies involve: T_0-Kolmogorov space, T_1-Fretchet space, T_2-Hausdorff space,

$T_{2\frac{1}{2}}$-Urysohn space, T_3-Regular Hausdorff space, $T_{3\frac{1}{2}}$-Tychonoff space, T_4-Normal Hausdorff space, T_5-Completely Hausdorff space and T_6-Perfectly Normal Hausdorff space.

Results from the work of Ross [62] indicates that most separation axioms have both topological property where they can induce other spaces with topological structures. For instance if X is a bitopological space, suppose Y is a subset of X and a T_2-space. A bitopological space (Y, τ_1, τ_2) will induce topologies τ_1 and τ_2 to subspace (X, τ_1, τ_2) which will in turn inherit all characteristics of space X. This is shown as follows.

Theorem 3.1. *[63, Theorem 3.4] If (K, τ_1, τ_2) is T_0 is a space then it exhibits topological property.*

From the study conducted by Sunganya [70] it shows that a T_2-space has both homeomorphic and topological properties. For the homeomorphic property a function f is continuous, if a function f is mapping one bitopological space to another for instance $f : (K, \tau_1, \tau_2) \to (R, \delta_1, \delta_2)$. Since a function f has homeomorphic property then it suffices that a bitopological space K contains k_1 and k_2 as points. However, k_1 is not equals to k_2. By the use of the technique of separation axioms f is seen to be onto and so $f(a_1) = x_1$ likewise $f(a_2) = x_2$. This follows that $f(a_1)$ is not equals to $f(a_2)$ and also a_1 is not equal to a_2. We have used these rules of separation axioms in our results to enable us to topologically separate points by the use of open neighborhoods whereby their intersection is empty. In our work we have establish some of these separation axioms in bitopological spaces and spaces with more number of topologies that are greater than two. This has been done via ij-continuity.

3.5 Conditions for normality

Conditions for normality is a methodology that is used to show whether topological spaces are normal or not. A normal space is one which has two disjoint closed sets that are topologically separated by the open neighborhoods. Given that X is a bitopological space, suppose m and n are closed disjoint sets. If U and V are open neighborhoods which topologically separate the two closed disjoint points in the space (X, τ_1, τ_2) as stated by Ananga [5]. Normal topological and bitopological spaces are spaces that satisfies T_4 axioms. Normality conditions are useful in characterizations in various topological spaces.

From Caldas [18] result shows that (X, τ_1, τ_2) is a normal bitopological space. Then suppose we have disjoint closed points say a and b. Therefore, it suffices that there exists open sets U and V which topologically separate the disjoint closed points in a space. By assumption Just [37] highlighted that conditions for normality as a technique a is a member of U and not a member of open neighborhood V. Likewise b is a cardinality of V and not a member of open set U.

Since U contains closed subset a and V contains b then this space (X, τ_1, τ_2) is said to be normal. Moreover, the intersection of closure point of V and open neighborhood U is an empty set. On the other hand if a and closed set b is not containing x which is an element of space (X, τ_1, τ_2) this is seen from the work of David [21]. Then U will contain complements elements of b. It follows that since U is open and there exists a neighborhood V of x such that the closure of V is a subset of U. So it implies that open set V has the complement of the closure of V as the cardinality as give by Steve [71]. Therefore, the intersection of the open sets U and V is not empty. Hence in regards to this X is said to be a regular space. In our work we have used conditions for normality as a methodology to show the separation of disjoint points in bitopological spaces and other N-topological spaces for only results that have only open sets say U and V whereby the intersection is not empty.

Chapter 4

RESULTS AND DISCUSSION

4.1 Introduction

This chapter is the core of this work where we present the results of this study. We consider the notion of ij-continuity, separation axioms and their extensions to N-topological spaces. For simplicity, we denote (X, τ_1, τ_2) as X and (Y, τ_1', τ_2') as Y. Since the intersection of τ_1 and τ_2 is a topology on X, we are taking U_1 as open set in τ_1 and U_2 as open set in τ_2. Similarly, the intersection of τ_1' and τ_2' on Y is a topology, consequently V_1 is an open set in τ_1' and also V_2 is open in τ_2'. Therefore, $U_1 \cap U_2 = U$ which is π_λ-open in (X, τ_1, τ_2) and also $V_1 \cap V_2 = V$ which is open in (Y, τ_1', τ_2').

4.2 Notions of ij-Continuity

In this section, we give an in depth characterization of ij-continuity in bitopological spaces.

Proposition 4.1. *Let* $\chi : (X, \tau_1, \tau_2) \to (Y, \tau_1^{'}, \tau_2^{'})$ *be an open function. A subset* W *of* X *is* π_λ*-open if and only if it is semi-closed and an intersection of* π_λ*-open sets in* X*. Moreover,* χ *is* π_λ*-continuous.*

Proof. To prove the first part, let W of X be π_λ-open. We prove that it is semi-closed and also an intersection of open sets in X. Let U be an open set in X and V be open set in Y containing $\chi(x)$ for some $x \in X$. By continuum hypothesis, there exists a π_λ-open set U of X which is containing x such that $\chi(U) \subseteq V$. Since U is a π_λ-open set then $x \in U$ and x belongs to U of X, then there exists a subset W of X that is semi-closed. By criterion for continuity, λ is closed then closure interior of W is a subset of W, that is $int\overline{(W)} \subseteq W$, this is because W is a π_λ-open and it has an element which is semi-closed and since W is a semi-closed subset of X it follows that it is π_λ-open set and hence $x \in W \subseteq U$. Therefore, we have $\chi(W) \subseteq V$. Now, U and V are open sets in X and Y respectively which implies that $W = U \cap V$ is semi-closed set and π_λ-open in X. Therefore, V is an open set in Y containing y and U is a π_λ-open set in X containing x such that $\chi(U) \subseteq V$. Hence χ is π_λ-continuous at every point $x \in X$. The converse of this proposition is not true in general. Suppose we let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{b\}\}$ therefore, it follows that $\tau_1^{'} = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Then we have the open sets $\pi B(X, \tau_1, \tau_2) = \{X, \pi, \{b\}, \{b, c\}, \{a, b\}\}$. Hence it follows that we have $\pi_\lambda B(X, \tau_1, \tau_2) =$

$\{\phi, X, \{a,b\}, \{a,c\}\}$. Similarly, we also have the open sets in a bitopological space Y as $\pi B(Y, \tau_1^{'}, \tau_2^{'}) = \{\phi, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}\}$. Then it follows that $\pi_\lambda B(Y, \tau_1^{'}, \tau_2^{'}) = \{\phi, X, \{a,c\}, \{b,c\}\}$. If the function $\chi : (X, \tau_1, \tau_2) \to (Y, \tau_1^{'}, \tau_2^{'})$ is defined by $\chi(a) = \chi(b)$ and $\chi(c) = c$, then χ is semi-continuous but not π_λ-continuous. □

A function χ is also π_λ-continuous if the inverse of a subset in a nonempty set is π_λ-open. This is illustrated in the next result.

Proposition 4.2. *A function $\chi : X \to Y$ is π_λ-continuous if and only if for every open subset H of Y and $\chi^{-1}(H)$ is π_λ-open in X.*

Proof. Let χ be a π_λ-continuous function and B be any set in Y. To show that $\chi^{-1}(B)$ is also an open set in X, it is enough that $\chi^{-1}(B) = \emptyset$ in X hence this follows $\chi^{-1}(B)$ if B is open in X then it suffices that it is a π_λ-open set in X. However, if $\chi^{-1}(B) \neq \emptyset$ then for every $x \in \chi^{-1}(B)$, we have $\chi(x) \in B$. Since χ is π_λ-continuous, this is because the inverse of B is π_λ-open in space X therefore, there exists a π_λ-open set H_x in X such that $x \in H_x$ and $\chi(H_x) \subseteq B$. By criteria for inverse continuity, it implies that $x \in H_x \subseteq \chi^{-1}(B)$. So this implies that $\chi^{-1}(B)$ is π_λ-open in X. Conversely, if $x \in X$ and we let V to be an open set in Y containing $\chi(x)$, then $x \in \chi^{-1}(V)$ by criterion of continuity it implies that $\chi^{-1}(V)$ is π_λ-open in X containing x. Therefore, $\chi(\chi^{-1}(V)) \subseteq V$. Hence χ is π_λ-continuous. □

Next we show that every π_λ-continuous function is semi-continuous. However, a semi-continuous function is not necessarily π_λ-continuous.

Lemma 4.3. *Every π_λ-continuous function $\chi : X \to Y$ is semi-continuous but the converse is not true in general.*

Proof. Suppose we have a π_λ-open set H of X having x as one of its element and so it implies that $\chi(H) \subseteq V$. From Proposition 4.2, we see that H is a π_λ-open set and $x \in H$. It therefore implies that there exists a π_λ-closed set F of X such that $x \in F \subseteq V$. By criterion for continuity, it follows that χ is a π_λ-continuous function and so it follows that χ is semi-continuous. However, the converse is not true in general. This can be illustrate as follows: If we have two bitopological spaces as (X, τ_1, τ_2) and $(Y, \tau_1^{'}, \tau_2^{'})$, then a function $\chi : (X, \tau_1, \tau_2) \to (X, \tau_1^{'}, \tau_2^{'})$ is continuous. Since the intersection of topologies is topology, therefore given the cardinalities as $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{b\}\}$ and $\tau_1^{'}$ are $\tau_1^{'} = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ then $\pi B(X, \tau) = \{\emptyset, X, \{b\}, \{b, c\}, \{a, b\}\}$. Therefore, by criterion for continuity, $\pi_\lambda B(X, \tau) = \{\emptyset, X, \{a, b\}, \{a, c\}\}$.
Similarly, $\pi B(X, \tau_1) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.
So $\pi_\lambda B(X, \tau) = \{\emptyset, X, \{a, c\}, \{b, c\}\}$. Since members of $X = \{a, b, c\}$ therefore if we have $\chi : (X, \tau) \to (Y, \tau_1^{'})$ be defined by $\chi(a) = \chi(b) = b$, If $\chi(a) = \chi(b) = b$, then it applies that $\chi(c) = c$. Therefore, χ is semi-continuous but not π_λ-continuous this is because $\{a, c\}$ is an open set in $\pi B(X, \tau_1, \tau_2)$ but it is not an open set in $(Y, \tau_1^{'}, \tau_2^{'})$. □

Theorem 4.4. *Every θ_η-continuous function $\chi : X \to Y$ is π_λ-continuous however, the converse need not be true.*

Proof. We first show that a function χ is θ_η-continuous. Let x be a member of X and if we have an open set G in X then it follows that $G \subseteq X$ whereby G is θ_η-open in X. Given that V is an open set in Y, then if a function χ maps θ_η-open set G from domain space (X, τ_1, τ_2) to codomain space $(Y, \tau_1^{'}, \tau_2^{'})$ such that $\chi(G) \subseteq Y$. If $\chi^{-1}(G) \subseteq X$ then it suffices that $\chi^{-1}(G)$ is θ_η-open in X. Therefore, χ is θ_η-continuous. Let θ_η-continuous function $\chi : X \to Y$ be π_λ-continuous. Let $\chi : X \to Y$ be π_λ-continuous at a point $x \in X$, if for each V of Y containing $\chi(x)$ there exists π_λ-open G in X that is containing x such that $\chi(G) \subseteq V$. By hypothesis, if G is a π_λ-open set then it implies that there exists a π_λ-closed set F of X such that $x \in F \subseteq V$. By Lemma 4.3, if there is an open set V in X which contains x such that $\chi(G) = V$, by criterion for continuity a function χ is π_λ-continuous at every point x of X then it is π_λ-continuous and θ_η-continuous. However, the converse need not to be true in general. For instance, if the cardinalities of $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$. Then we have $\pi B(X) = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}\}$. It suffices that $\pi_\lambda B(X) = \{\emptyset, X, \{a, c\}\}$ and $\theta_\eta B(X) = \{\emptyset, X\}$ if a function $\chi : X \to Y$ is defined by $\chi(a) = \chi(c) = a$ and also $\chi(b) = b$ hence χ is a π_λ-continuous function since $\{a\} \in \tau$ and $\{a, c\} \in \pi_\lambda B(X)$ but $\{a, c\}$ does not exists in $\theta\pi B(X)$. □

The following consequence follows immediately.

Corollary 4.5. *Every π_d-continuous and $\delta\pi_d$-continuous functions $\chi : (X, \tau_1, \tau_2) \to (Y, \tau_1^{'}, \tau_2^{'})$ is π_λ-continuous however, the converse need not be true.*

Proof. Suppose a function $\chi : X \to Y$ is π_λ-continuous. Then let an element x to be a cardinality of X, hence $x \in X$ and V is any open set in Y that contains $\chi(x)$. By the continuum hypothesis, there exists π_λ-open set U of X containing x such that $\chi(U) \subseteq (V)$. Since U is said to be π_λ-open therefore set $x \in U$, there exists a d-closed set F of X such that $x \in F \subseteq U$. Therefore, $\chi(F) \subseteq V$ and since χ is π_λ-continuous, it suffices that it is also π_d-continuous. Therefore, if χ is π_d-continuous then it is also $\delta\pi_d$-continuous. Since we have χ to be π_λ-continuous then let U be any open set in X containing such that $\chi(U)$ is in Y and V be any open set in Y containing $\chi(U)$. Suppose that G is a π_λ-open set in X then this implies that $\chi(G) = U$. By Theorem 4.4, since $\chi(G) = U$ then a function χ is π_λ-continuous at every point x of X, then a function $\chi : X \to Y$ is also π_d-continuous and $\delta\pi_d$-continuous function. To see the converse, let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$. Then it follows that $\pi B(X) = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$ and $GB(X) = \{\emptyset, \{b\}, \{c\}, \{b, c\}, \{a, c\}, \{a, b\}\}$. Hence $\pi_\lambda(X) = \{\emptyset, \{a, c\}, \{a, b\}, X\}$ and also $\pi_d B(X) = \delta\pi_d(X) = \{\emptyset, X\}$. Therefore, if a function $\chi : X \to Y$ can be defined by $\chi(a) = \chi(c) = a$ and $\chi(b) = b$, then χ is π_λ-continuous, as $\{a\} \in \tau$ and $\{a, c\} \in \pi_\lambda(X)$ but neither π_d-continuous nor $\delta\pi_d$-continuous $\{a, c\}$ does not exists in $\pi_d B(X) = \delta\pi_d B(X)$. This completes the proof. □

We use proposition 4.6 to illustrate that functions which are perfectly continuous are also π_λ-continuous.

Proposition 4.6. *Let a function $\chi : X \to Y$ be perfectly continuous. Then χ is π_λ-continuous.*

Proof. Let A be any open set in Y. Then $\chi^{-1}(A)$ is clopen in X. Hence it implies that $\chi^{-1}(A) \in \pi_\lambda B(X)$, then from Proposition 4.2, a function χ is π_λ-continuous since we have A to be an open set in Y. Then we can show that $\chi^{-1}(A)$ is a π_λ-open set in X, it therefore implies that $\chi^{-1}(A)$

is a π_λ-open set in X, and if $\chi^{-1}(A) \nsubseteq \emptyset$, then for each $x \in \chi^{-1}(A)$, we have $\chi(x) \in A$. Since χ is π_λ-continuous then it implies that there exists a π_λ-open set B_x in X such that $x \in B_x$ and $\chi(B_x) \subseteq A$. This implies that $x \in B_x \subseteq \chi^{-1}(A)$. This therefore shows that $\chi^{-1}(A)$ is π_λ-open in X. Similarly, if we let $x \in X$ and A be an open set in Y containing $\chi(x)$. Then it follows that $x \in \chi^{-1}(A)$. By the continuum hypothesis, $\chi^{-1}(A)$ is π_λ-open in X containing x, hence it suffices that $\chi(\chi^{-1}(A)) \subseteq A$. Therefore, χ is π_λ-continuous. By the fact that χ is π_λ-continuous then it suffices that it is also perfectly continuous since we have an open set A in Y and $\chi^{-1}(V)$ is clopen in X. □

In our subsequent result we illustrate how globally indiscrete mappings exhibit characteristics of semi-continuous functions.

Lemma 4.7. *Let $\chi : X \to Y$ be globally indiscrete. Then a function χ is π_λ-continuous if and only if it is semi-continuous.*

Proof. Let χ be semi-continuous and X be globally indiscrete. Let U be an open subset in X and V be open set in Y, then it suffices that $\chi^{-1}(V)$ is also semi-open in X. Since X is globally indiscrete, similarly it for globally discrete we have U as an open set in X and V be an open set in Y containing $\chi(x)$ for some $x \in X$. By the continuum hypothesis, there exists a π_λ-open set U of X which is containing x such that $\chi(U) \subseteq V$. Since U is a π_λ-open set, then we can say that $x \in U$. Since x belongs to U of X then there exists a subset W of X that is semi-closed. By criterion for continuity, the interior closure of W is a subset of W, that is $int(\overline{W}) \subseteq W$. Since W is a semi-closed subset of X it implies that it is π_λ-open set and hence $x \in W \subseteq U$, since any π_λ-open set may contain a set that is semi-closed or semi-open such that the interior closure of that set is a subset of itself. Therefore, we have $\chi(W) \subseteq V$. Now, since U and V are open sets in X and Y respectively then it follows that $W = U \cap V$ is a semi-closed set and π_λ-open in X. Therefore, V is π_λ-open set in Y containing y and U is a π_λ-open set in X containing x such that $\chi(U) \subseteq V$. Therefore, χ is π_λ-continuous at every point $x \in X$. Conversely, let χ be π_λ-continuous. Therefore, there exists a π_λ-open set U of X containing x such that $\chi(U) \subseteq V$. Since U is a π_λ-open set and $x \in U$, then there exists a g-closed set F of X such that $x \in F \subseteq U$. Therefore, we have $\chi(F) \subseteq V$. Since χ is π_λ-continuous, then by Lemma 4.3, χ is semi-continuous. Therefore, since χ is π_λ-continuous then it is also semi-continuous. □

For a function that maps a Housdorff space to a bitopological space is is both semi-continuous and π_λ-continuous. We state the result as follows.

Theorem 4.8. *Let X be a Hausdorff space and Y be any bitopological space. Then $\chi : X \to Y$ is semi-continuous and π_λ-continuous.*

Proof. Suppose we have two functions say χ and ω let $\chi : X_1 \to Y$ and if $\omega : X_2 \to Y$ then $\chi, \omega : X \to Y$ are π_λ-continuous functions. Since Y is a Hausdorff space, therefore there is set $E = \{x_1, x_2\} \in X$. Suppose E does not exists in $\{x_1, x_2\}$ then it follows that $\chi(x_1) \neq \omega(x_2)$. Since Y is a Hausdorff space then there exist open sets V_1 and V_2 of Y such that $\chi(x) \subseteq V_1$ and $\omega(x) \subseteq V_2$. Then it implies that $V_1 \cap V_2 \neq \emptyset$. Since χ and ω are π_λ-continuous functions then there exist π_λ-open sets U_1 and U_2 in Y containing y such that $\chi(U_1) \subseteq V_1$ and $\omega(U_2) \subseteq V_2$. By criterion for continuity, the intersection of U_1 and U_2 is a proper subset of W that is

$W = (U_1) \cap (U_2)$. Then it is π_λ-open in Y since $M \in Y$ then $U \cap M = \emptyset$. Hence it follows that $x \in \pi_\lambda\overline{(H)}$, this implies that H is π_λ-closed in X. Since V is any open set in Y then $\chi^{-1}(V)$ is clopen in X, and so $\chi^{-1}(V) \in \pi_\lambda(X)$. Therefore, a function χ is π_λ-continuous. Since V and U are open sets in Y and X respectively then we have $x \in \chi^{-1}(V)$ with x being closed hence $x \in \{x\} \subseteq \chi^{-1}(V)$. Therefore, $\chi^{-1}(U)$ is semi-open in X. By criterion for inverse continuity, $\chi^{-1}(V)$ is a π_λ-open set in X. Hence χ is a semi-continuous function. □

In the next theorem 4.9 we illustrate that a function χ is ij-continuous if and only if there exists an open subset. This is indicated in the result that follows.

Theorem 4.9. *Let $\chi : X \to Y$ be π_λ-continuous. Then χ is ij-continuous if X_0 is an open subset of X. Moreover, χ is an $ij - \pi_\lambda$-continuous if $\chi\mid_{X_0}: X_0 \to Y$ is π_λ-continuous.*

Proof. Let $\chi : X \to Y$ be π_λ-continuous then it implies that it is continuous since $x \in X$. Suppose that we have open set V of Y which contains $\chi(x)$ therefore, from Theorem 4.8 we can state that there exists a π_λ-open set U. If $X_0 \subseteq U$ then X_0 also exists in X and contains x. Then it follows that $\chi(X_0) \subseteq V$. Hence $\chi(x)$ is π_λ-continuous and it is π_λ-open in Y. Since V is an open set in Y then $\chi^{-1}(V)$ is π_λ-open in X, So $\chi^{-1}(V)$ is also π_λ-open in X. Given that $\chi^{-1}(V)$ is in X therefore it implies that for every $x \in \chi^{-1}(V)$ we have $\chi(x) \in V$. Then by criterion for continuity, there exists a π_λ-open set X_0 in X such that $x \in X_0$ and $\chi(X_0) \subseteq V$. Therefore, $x \in X_0 \subseteq \chi^{-1}(V)$. Then it shows that $V \in i - X_0(Y)$ and $\chi^{-1}(V)$ are members of $ij\pi X_0 X$, hence $i \in X_0$ and so i is a π_λ-open set in X. This χ

is ij-continuous since X_0 is an open subset of X and the inverse of open set j in Y is i-open in X. Moreover, since sets X and Y have open sets U and V respectively and that V of Y contains $\chi(x)$. This follows that a π_λ-open set X_0 in X also contains x. Then since X_0 is a subset of X and $\chi(x)$ is a proper subset V of Y, then it implies that $\chi(U)$ is also a subset of V. Since a function $\chi : X \to Y$ it shows that X_0 then $\chi \mid_{X_0}: X_0 \to Y$. This follows that for all V in Y there exists $j - X_0(Y)$ such that $\chi^{-1}(V)$ exists in $ij - \pi U X_0(X)$. Since $j - X_0$ is open in Y and $\chi^{-1}(V)$ is an element of $ij - \pi B X_0(X)$ then $x \in \chi^{-1}(V)$ where $\chi(x) \in V$. Therefore, χ is π_λ-continuous hence it is also $ij - \pi_\lambda$-continuous since $\chi \mid_{X_0}: X_0 \to Y$ is π_λ-continuous. □

Theorem 4.10. *Let $\chi : X \to Y$ is $ij - \pi_\lambda$-continuous if for each open set X_0 of X we have $\eta \in X$. Such that $\chi \mid_{X_0}: X_0 \to Y$ is π_d-continuous.*

Proof. Let V be any open set in Y and X_0 be any open set in X, then there exists η which is an element of X. Since $\eta \in X$ and X_0 is open in X then it suffices that $\eta \in X_0$ which is π_λ it follows that $X_0 \subseteq X$. From Theorem 4.9, $\chi : X \to Y$ is π_λ-continuous hence there exists $\eta \in X$ and open set V of Y such that it contains $\chi(\eta)$. Therefore, it implies that there is a π_λ-open set X_0 in X containing η such that $\chi(X_0) \subseteq V$. Hence χ is π_λ-continuous if and only if it is continuous at every point η of X. Since there is an open set V of Y such that $V \in j - X_0(Y)$ and $\chi^{-1}(V) \in ij - \pi\eta X_0(X)$, then it suffices that $\chi : X \to Y$ is $ij - \pi_\lambda$-continuous. Since V is an open set in Y then $\chi^{-1}(V)$ is an element of X, then by criteria for inverse continuity it implies that $\chi^{-1}(V) = \emptyset$. Suppose that $\chi^{-1}(V) \in X$ and every $\eta \in \chi^{-1}(V)$ then it shows that $\chi(x) \in V$. Then it implies that X_η exists in X where $\eta \in X$, hence

$\eta \in X_\eta \subseteq \chi^{-1}(V)$. Therefore, $\chi^{-1}(V)$ is π_λ-open in X and so it implies that it is π_λ-continuous since $\chi\mid_{X_0}: X_0 \to Y$ with X_0 having induced properties from χ. Since $\chi\mid_{X_0}: X_0 \to Y$ then it is also π_d-continuous. $\square$

This leads to the following consequence.

Corollary 4.11. *Every $ij - \pi_\lambda$-continuous function is ij-continuous but the converse is not true in general.*

Proof. Since inverse open j in Y is i-open in X then χ is said to be π_λ-continuous if and only if it is continuous if $\chi(x) \subseteq Y$ Suppose there is any open set V of Y which contains $\chi(x)$ then χ is π_λ-continuous. This implies that $\chi^{-1}(V)$ is π_λ-open in X and so if $\chi^{-1}(V)$ is an empty set then $\chi^{-1}(V)$ is also a π_λ-open set in X. Hence suppose that $\chi^{-1}(V) \nsubseteq \emptyset$ then it implies that each $x \in \chi^{-1}(V)$, therefore $\chi(x) \in V$. From Theorem 4.9, χ is π_λ-continuous and also there exists a π_λ-open set U in X such that $x \in U$ and hence $\chi(U) \subseteq V$, by extensions $x \in U \subseteq \chi^{-1}(V)$. It implies that we have V in $j - V(Y)$. Then χ is said to be $ij - \pi_\lambda$-continuous since $\chi^{-1}(j)$ is i-open in X. For ij-continuous we have $V \in i - V(Y)$ and $\chi^{-1}(V) \in \pi V(X)$. By criterion for continuity, it implies that every $ij - \pi_\lambda$-continuous function is also ij-continuous. However, not every ij-continuous is $ij - \pi_\lambda$-continuous. Let V be an open set in Y and U open set in X. Then $\chi^{-1}(V)$ is π_λ-open in X. For open set U of X we have $\chi(U) \subseteq V$, therefore it follows that $x \in U \subseteq \chi^{-1}(V)$. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$. Then $\pi B(X) = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$. Similarly, $\pi B(X) = \{\emptyset, \{b\}, \{c\}, \{b, c\}, \{a, c\}, \{a, b\}\}$ and so $\pi_\lambda(X) = \{\emptyset, \{a, c\}, \{a, b\}, X\}$. Therefore, $\pi_d B(X) = \delta\pi_d(X) = \{\emptyset, X\}$. Suppose that a function $\chi : X \to X$ is defined by $\chi(a) = \chi(c) = a$ then it shows that χ is π_λ-continuous. Since $\{a\} \in \tau$ and $\{a, c\} \in \pi_\lambda(X)$, it implies that neither π_d-continuous nor $\delta\pi_d$-continuous is π_λ-continuous. So $\{a, c\}$ does not exists in $\pi_d B(X) = \delta\pi_d B(X)$ therefore, since χ is π_d-continuous then it implies that it is also π_λ-continuous. For ij-continuity with open set V of Y we have $V \in i - V(Y)$ this implies that $\chi^{-1}(V) \in ij - \pi V(X)$. Hence for $ij - \pi_\lambda$-continuous there exists an open set $V \in jBV(Y)$ hence $\chi^{-1}(V) \in ij - UB\pi V(X)$. Therefore, every $ij - \pi_\lambda$-continuous function is ij-continuous but its converse is not true in general. □

Suppose independent functions mapping one bitopological space to another are $ij - \pi_\lambda$-continuous then their composition is π_λ-continuous as it is shown in the result that follows.

Proposition 4.12. *Let $\chi_1 : X \to Y$ be $ij - \pi_\lambda$-continuous and $\chi_2 : Y \to Z$ be $ij - \pi_d$-continuous. Then $\chi_2 \circ \chi_1$ is π_λ-continuous.*

Proof. Let $\chi_1 : X \to Y$ and $\chi_2 : Y \to Z$. Let C be open subset of X, then C is π_λ-open if and only if it is a semi-closed set. Hence, we can let U be π_λ-open set in X then it follows that χ maps C to Y. Similarly, let V be open set in Y containing $\chi(x)$ this implies that x is an element of π_λ-open set $U \subseteq X$, such that $\chi_1(U) \subseteq V$. Since U is π_λ-open then we can say that $x \in U$. There is a subset C that is semi-closed and hence the interior closure of C is a subset of C for instance $int(clC) \subseteq C$. Then it implies that C is a subset of semi-closed set of X and π_λ-open. Hence $x \in C \subseteq U$ and $\chi(C) \subseteq V$. Therefore, it shows that for all V that exist in $j - C(Y)$ there is $\chi^{-1}(V) \in ij\pi C(X)$. Therefore, $\chi_1 : X \to Y$ is said to be $ij - \pi_\lambda$-continuous. Similarly, $\chi_2 : Y \to Z$ is also π_λ-continuous. Let E be an open set in Z and V be open in Y and there exists $\eta \in Y$ then it implies that X is a subset of V. Since $\chi_2 : Y \to Z$ is π_λ-continuous and $\eta \in Y$ for each open set V of Y such that it contains $\chi_2(x)$ then it implies that there is a π_λ-open set D in X containing η such that $\chi_2(\eta) \subseteq V$ and hence χ_2 is π_λ-continuous if and only if χ_2 is continuous at each point η of X. Since there is an open set V of Y where $V \in j - V(Y)$ and also $\chi_2^{-1}(V) \in ij - \pi\eta V(X)$ then $\chi : X \to Y$ is $ij - \pi_\lambda$-continuous. This shows that $\chi_2 : Y \to Z$ is also $ij - \pi_d$-continuous. Since $\chi_1 : X \to Y$ is $ij - \pi_\lambda$-continuous then $\chi_2 : Y \to Z$ is $ij - \pi_d$-continuous. Hence a function $\chi : X \to Z$ is also $ij - \pi_d$-continuous. Since χ_1 and χ_2 are π_λ-continuous since they map i-open is mapped to J-open set in X therefore, it follows it suffices that $\chi_2 \circ \chi_1$ is also π_λ-continuous. □

Proposition 4.13. *Let $\chi_1 : X \to Y$ be π_λ-continuous and $\chi_2 : Y \to Z$ be π_d-continuous. Then $\chi_2 \circ \chi_1$ is $ij - \pi_d$ continuous.*

Proof. Functions χ_1 and χ_2 are said to be $ij - \pi_d$-continuous if and only they are π_λ-continuous. From Theorem 4.1 we can let B of X be π_λ-open. We start by proving that it is semi-closed and also an intersection of open sets in X. Let U be an open set in X and V be open set in Y containing $\chi(x)$ for some $x \in X$. By continuum hypothesis, there exists a π_λ-open set U of X which is containing x such that $\chi(U) \subseteq V$. Since U is a π_λ-open set then $x \in U$ and x belongs to U of X, then there exists a subset W of X that is semi-closed. By criterion for continuity, λ is closed then closure interior of B is a subset of , that is $\overline{(B)} \subseteq B$, this is because B is a π_λ-open and it has an element which is semi-closed and since W is a semi-closed subset of X it follows that it is π_λ-open set and hence $x \in B \subseteq U$. Therefore, we have $\chi(W) \subseteq V$. Now, U and V are open sets in X and Y respectively which implies that $W = U \cap V$ is semi-closed set and π_λ-open in X. Therefore, V is an open set in Y containing y and U is a π_λ-open set in X containing x such that $\chi(U) \subseteq V$. Hence χ_1 and χ_2 are π_λ-continuous at every point $x \in X$. Let $\chi_1 : X \to Y$ be π_λ-continuous and $\chi_2 : Y \to Z$ be π_d-continuous. If χ_1 is a π_λ-continuous function then it implies that there exists any open set B in Y. Then it therefore implies that $\chi_1^{-1}(B)$ is a π_λ-open set in X. Then that $\chi_1^{-1}(B)$ is π_λ-open in X. However, if $\chi_1^{-1}(B) \neq \emptyset$ then $x \in \chi_1^{-1}(B)$ we have $\chi_1(x) \in B$. Therefore, since χ is π_λ-continuous, it implies that there exists a π_λ-open set H_x in X such that $x \in H_x$ and $\chi_1(H_x) \subseteq B$ hence $x \in H_x \subseteq \chi_1^{-1}(B)$. Therefore, $\chi_1^{-1}(B)$ is π_λ-open in X. Conversely, suppose that $x \in X$ and V be any open set in Y containing $\chi(x)$ then $x \in \chi^{-1}(V)$, by criterion of continuity $\chi_1^{-1}(V)$ is π_λ-open in X containing x. Therefore, $\chi_1(\chi^{-1}(V)) \subseteq V$ so this implies that χ_1 is a π_λ-continuous function. So $\chi_2 : Y \to Z$ where all $x \in \chi^{-1}(B)$ is closed and therefore $x \in \{x\} \subseteq \chi^{-1}(B)$. Then it implies that $\chi^1(B) \in \pi_d C(X)$ and hence χ is a π_d-continuous function. Suppose that $\chi_1 : X \to Y$ is π_λ-continuous then $\chi_2 : Y \to Z$ is also π_d-continuous. Therefore, $\chi_2 \circ \chi_1$ is $ij - \pi_d$ continuous. □

Theorem 4.14. *Let $\chi : X \to Y$ be $ij - \pi_d$-continuous. Then χ is $ij - \Omega$-continuous and the converse need not to be true in general.*

Proof. A function χ said to be $ij - \Omega$-continuous if and only if it is $ij - \pi_d$-continuous. From Theorem 4.10 we have shown that functions χ_1 and χ_2 are $ij - \pi_d$-continuous. For instance, suppose we have V as an open set in Y and X_0 be any open set in X, then there exists d which is an element of X. Since $d \in X$ and X_0 is open in X then it suffices that $d \in X_0$ which is π_λ it follows that $X_0 \subseteq X$. A function $\chi : X \to Y$ is π_λ-continuous hence there exists $d \in X$ which is i-clopen in X and open

set V of Y such that it contains $\chi(d)$. Therefore, it implies that there is a π_λ-open set X_0 in X containing d such that $\chi(X_0) \subseteq V$. Hence χ is π_λ-continuous if and only if it is continuous. Since there is an open set V of Y such that $V \in j - X_0(Y)$ and $\chi^{-1}(V) \in ij - \pi_d X_0(X)$, then it suffices that $\chi : X \to Y$ is $ij - \pi_d$-continuous. Since, $\chi : X \to Y$ then it is $ij - \pi_d$-continuous if and only if χ is π_λ-continuous. From Proposition 4.1, χ is π_λ-continuous if there is any open set V in Y that is containing $\chi(x)$ and hence χ is continuous at every point $x \in X$. Then this follows that there exists a π_λ-open set U of X containing x such that $\chi(U) \subseteq V$. From the result in Theorem 4.9 we proved that a π_λ-continuous function is also semi-continuous if its open inverse in the codomain space is also semi-open in domain space. Then if we have open sets U and V then, if π_λ-open set U of X containing x then it follows that $\chi(x) \subseteq Y$. Therefore, by use of criterion for continuity, χ is semi-continuous. Conversely, not every $ij - \Omega$-continuous is $ij - \pi_d$-continuous. Given that (X, τ_1, τ_2) and $(Y, \tau_1^{'}, \tau_2^{'})$ are two bitopological spaces therefore suppose we the cardinalities as $X = \{d, e, f\}$, $\tau_1 = \{\emptyset, X, \{e\}\}$ and $\tau_1^{'} = \{\emptyset, X, \{d\}, \{e\}, \{d, e\}\}$. This implies that $\pi B(X, \tau_!, \tau_2) = \{\emptyset, X, \{e\}, \{e, f\}, \{d, e\}\}$ by criterion for continuity $\pi_\lambda B(X, \tau) = \{\emptyset, X, \{d, e\}, \{d, e\}\}$.
Therefore, $\pi B(Y, \tau_1^{'}, \tau_2^{'}) = \{\emptyset, X, \{d\}, \{e\}, \{d, e\}, \{d, f\}, \{d, e\}\}$.
Hence $\pi_\lambda B(Y, \tau_1^{'}, \tau_2^{'}) = \{\emptyset, X, \{d, f\}, \{e, f\}\}$.
A function $\chi : (X, \tau_1, \tau_2) \to (Y, \tau_1^{'}, \tau_2^{'})$ can be defined by $\chi(a) = \chi(e) = e$. If $\chi(d) = \chi(e) = e$ then it implies that $\chi(f) = f$ hence χ is semi-continuous and $ij - \pi_d$-continuous but not π_λ-continuous. Therefore, $\chi^{-1}(V) \in ij - \Omega B(X)$ then χ is $ij - \Omega$-continuous. □

When a function $\chi_1 : X \to Y$ is i-continuous then if we have another function χ_2 which is j-continuous then the composition $\chi_1 \circ \chi_2$ is $ij - \theta_s$-continuous. We state the results as follows.

Theorem 4.15. *Given that a function* $\chi : (X, \tau_1, \tau_2) \rightarrow (Y, \tau_1^{'}, \tau_2^{'})$ *then* χ *is* $ij - \theta_\eta$*-continuous if* $ij - \pi_\lambda$*-continuous.*

Proof. If a function χ is mapping θ_η-open set from (X, τ_1, τ_2) to $(Y, \tau_1^{'}, \tau_2^{'})$ and if it has an open θ-open set then χ is θ_η-continuous. Suppose that we let P to be a member of X which is π_λ-open U, and V are open sets in X and Y respectively. We therefore have to start by showing that P is semi-closed and the intersection of open sets U and V is in space (X, τ_1, τ_2). Hence, since U is π_λ-open set in X and V open set in Y containing $\chi(x)$ for some $x \in X$. Consequently it follows that x is also π_λ-open since it is a member of X. By the use of the criterion for continuity we have $\chi(U) \subseteq V$. Since U is a π_λ-open set then $x \in U$ and x belongs to U of X, then there exists a subset P of X which is semi-closed there exist semi-closed set π_λ which is contained in X, by employing the criterion for continuity, π_λ is also a closed set then closure interior of P is a subset of P such that $int\overline{(P)} \subseteq P$, this is attributed since P is a π_λ-open and also has an element which is semi-closed and since P is a semi-closed subset of X it follows that it is π_λ-open set and hence $x \in P \subseteq U$. The fact that P is $i - \pi_\lambda$-open in X and has π_λ-closed set then it is also θ_η-open in X which follows closely that $\chi(P) \subseteq V$. So $i \in U$ and $j \in V$ are open sets in X and Y respectively which implies that $P = U \cap V$ is semi-closed set and π_λ-open in X. Therefore, V is an open set in Y containing y and U is a π_λ-open set in X containing x such that $\chi(U) \subseteq V$. Hence χ is θ_η-continuous. Since $j - V$ has its open inverse $i - U$ in X then χ is $ij - \theta_\eta$-continuous. $x \in X$. Similarly we have the cardinalities as $X = \{m, n, o\}$ and $\tau = \{\phi, X, \{n\}\}$ therefore, it follows that $\tau_1^{'} = \{\phi, X, \{m\}, \{n\}, \{m, n\}\}$. Then we have the open sets $\pi B(X, \tau_1, \tau_2) = \{X, \pi, \{n\}, \{n, o\}, \{m, n\}\}$. Hence it follows that we have $\pi_\lambda B(X, \tau_1, \tau_2) = \{\phi, X, \{m, n\}, \{m, o\}\}$. Similarly, we also have the open sets in a bitopological space Y as $\pi B(Y, \tau_1^{'}, \tau_2^{'}) = \{\phi, X, \{m\}, \{n\}, \{m, n\}, \{m, o\}, \{n, o\}\}$. Then it follows that $\pi_\lambda B(Y, \tau_1^{'}, \tau_2^{'}) = \{\phi, X, \{m, o\}, \{n, o\}\}$. If the function $\chi : (X, \tau_1, \tau_2) \to (Y, \tau_1^{'}, \tau_2^{'})$ is defined by $\chi(m) = \chi(n)$ and $\chi(o) = o$, then χ is $ij - \theta_\eta$-continuous So since χ is $ij - \theta_\eta$-continuous function it implies that it is also π_λ-continuous. Suppose that $\chi : X \to Y$ then χ is also π_λ-continuous function if and only if $\chi(x)$ is in V, then there exists a π_λ-open set V of Y that is containing $\chi(x)$. Similarly, there exists a π_λ-open set U in X containing x such that $\chi(U) \subseteq V$. This implies that χ is $ij - \theta_\eta$-continuous. Therefore, χ is $ij - \pi_\lambda$-continuous since for all V which is open in Y exists in $j - V(Y)$ and also $\chi^{-1}(V) \in ij - U\pi V(X)$. Then χ is also $ij - \theta_\eta$-continuous function since θ_η-continuous function. $\square$

Lastly, we consider pairwise continuity in soft bitopological spaces induced by different functions χ_1 and χ_2. We state the results as follows.

Theorem 4.16. *Let X, Y and Z be soft bitopological spaces. If $\chi_1 : X \to Y$ and $\chi_2 : Y \to Z$ are p-soft continuous functions then $\chi_2 \circ \chi_1$ is p-soft continuous.*

Proof. Let χ_1 and χ_2 be two functions that are mapping soft bitopological spaces. (X, τ_1, τ_2, E), $(Y, \tau_1^{'}, \tau_2^{'}, E)$ and $(Z, \tau_1^{''}, \tau_2^{''}, E)$. Let U and V be π_λ-open sets. Suppose that $\chi_1 : (X, \tau_1, \tau_2, E) \to (Y, \tau_1^{'}, \tau_2^{'}, E)$ and if there is π_λ-open set W in X then $\chi(W)$ is in Y. Since $\chi(W)$ is in Y then it suffices that it W is $(Y, \tau_1^{'}, \tau_2^{'}, E)$, similarly if $\chi_2 : (Y, \tau_1^{'}, \tau_2^{'}, E) \to (Z, \tau_1^{''}, \tau_2^{''}, E)$.

This implies that W is π_λ-open in X, Y and Z hence $\chi_1 \circ \chi_2$ is also continuous. Let (m, c) be soft points such that $(m, c) \in E$ then if functions $\chi_1 : X \to Y$ and $\chi_2 : Y \to Z$ then implies that χ_1 and χ_2 are pairwise-soft continuous functions. Since $(m, c) \in \tau_1$ therefore this follows that $\chi_2 : (Y, \tau_1^{'}, \tau_2^{'}, E) \to (Z, \tau_1^{''}, \tau_2^{''}, E)$ is pairwise-soft continuous then it implies that $\chi^{-1}(m, n) \in \tau_1^{''}$. Therefore, $\chi_1 : (X, \tau_1, \tau_2, E) \to (Y, \tau_1^{'}, \tau_2^{'}, E)$ is also pairwise-soft continuous function. This follows that $\chi^{-1}(m, c) = \chi_1 \circ \chi_2^{-1}(m, c)$. Then it implies that we have $\chi_1 \circ \chi_2^{-1}(m, c) \in \tau_{12}$. By criterion for continuity, $\chi_1 \circ \chi_2 : (X, \tau_1, \tau_2, E) \to (Z, \tau_1^{''}, \tau_2^{''}, E)$. is pairwise-soft continuous. □

In this objective we characterized the notion of ij-continuity in bitopological spaces as π_λ-continuous as shown in our result, Proposition 4.1. We also classify this notion of ij-continuity as θ_η-continuous as it is indicated in Theorem 4.15. Furthermore, we characterized the notion of ij-continuity as $ij - \pi_d$-continuous as in Lemma 4.3. Moreover, We also characterized the notion of ij-continuity as $ij - \Omega$-continuous, this is given in our result Theorem 4.14. These notions were characterized in Weak continuity, Semi-continuity and Strong continuity as aspects of continuity that we studied.

4.3 Separation Criteria for via ij-Continuity

Let T_0, T_1 and T_2 be separation axioms we show that they exhibit heredity and topological properties. For simplicity, we denote (X, τ_1, τ_2) with X and $(Y, \tau_1^{'}, \tau_2^{'})$ with Y.

Proposition 4.17. *Let (X, τ_1, τ_2) be a T_0 space then the property of T_0 is both hereditary and topological.*

Proof. We start by showing that T_0 has hereditary property. Let (X, τ_1, τ_2) be a T_0 space and suppose that $D \subseteq X$ then it suffices that a bitopological subspace $(D, \tau_{D1}, \tau_{D2})$ is also a T_0 space. Since $(D, \tau_{D1}, \tau_{D2})$ has induced properties from (X, τ_1, τ_2) therefore it shows that $a, b \in D$ with $a \neq b$, this implies that $a, b \in X$, $a \neq b$ as in Definition 1.10. Since (X, τ_1, τ_2) is a T_0 space. Then there exists $U \in \tau_1 \cup \tau_2$ then $a \in U$, and a does not exists in U or b does not exists in U but $b \in U$. Hence $U \in \tau_1 \cup \tau_2$ this follows that $U \in \tau_1$ or $U \in \tau_2$. Therefore, $U \cap D \in \tau_{D1}$ or $U \cap D \in \tau_{D2}$ similarly $U \cap D \in \tau_{D1} \cap \tau_{D2}$. Since $a, b \in D$ then $a \in U \cap D$, b does not exists in $U \cap B$ and a does not exists in $U \cap D$, and $b \in U \cap D$. Then $(D, \tau_{D1}, \tau_{D2})$ is also a T_0 space. We can also show that T_0 has topological property. Using the notion of ij-continuity let $b_1, b_2 \in X$ with $b_1 \neq b_2$, Taking a function χ to be onto function then there is $a_1, a_2 \in X$ with $\chi(a_1) = \chi(b_1)$ and $\chi(a_2) = b_2$. Since χ is an injective function with $b_1 \neq b_2$ therefore it implies that $\chi(a_1) \neq \chi(a_2)$ hence $a_1 \neq a_2$. Since (X, τ_1, τ_2) is T_0 space and $a_1, a_2 \in X$ where $a_1 \neq a_2$ then it implies that there exists $U \in \tau_1 \cup \tau_2$ such that $a_1 \in U$ and a_1 does not exists in U or a_1 does not exists in U, $a_2 \in U$ or $a_1 \in U$, a_2 does not exists in U. Then $U \in \tau_1 \cup \tau_2$ follows that $\chi(U) \in \chi(\tau_1 \cup \tau_2)$ since χ is open and continuous. using separation axiom as a methodology, it implies that $\chi(U) \in \chi(\tau_1) \cup \chi(\tau_2) \in \tau_1^{'} \cup \tau_2^{'}$. Also $a_1 \in U$ which implies that $\chi(a_1) \in \chi(U)$ or $b_1 \in \chi(U)$ and a_2 does not exists in U which imply that $\chi(a_2)$ does not exists in $\chi(U)$ or b_2 does not exists $\chi(U)$. For any $b_1, b_2 \in Y$ with $b_1 \neq b_2$, $\chi(U) \in \tau_1^{'} \cup \tau_2^{'}$ is obtained such that $b_1 \in \chi(U)$, b_2 does not exists in $\chi(U)$. Therefore, $(Y, \tau_1^{'}, \tau_2^{'})$ is a

T_0 space. Every homeomorphic image of T_0 space then it shows that it is having topological property. $\square$

Proposition 4.18. *Let (X, τ_1, τ_2) be a T_1 space then the property of T_1 is both topological and hereditary properties.*

Proof. If T_1 has hereditary property then it follows that a bitopological space (X, τ_1, τ_2) is also T_1 space. Let $D \subseteq X$ and hence $(D, \tau_{D1}, \tau_{D2})$ is also T_1 space. Let $a, b \in D$ and with $a \neq b$ it therefore implies that $a, b \in X$ and $a \neq b$. Since (X, τ_1, τ_2) is a T_1 space then $U \in \tau_1 \cup \tau_2$. Then $a \in U$ and b is not a member of U. Similarly a does not exists in U but $b \in U$. From Proposition 4.17, we have $U \in \tau_1 \cup \tau_2$. Then $U \in \tau_1$ or $U \in \tau_2$, $U \cap D \in \tau_{D1}$ and so $U \cap D \in \tau_{D2}$ also $U \cap D \in \tau_{D1} \cap \tau_{D2}$. The fact that $a, b \in D$ hence $a \in U \cap D$, b does not exists in $U \cap D$ or a not a member in $U \cap D$, $b \in U \cap D$. Therefore, $(D, \tau_{D1}, \tau_{D2})$ shows properties of a T_1 space hence it is a T_1 space. On the other side, we show that T_1 space also has a topological property. Let $\chi : (X, \tau_1, \tau_2) \to (Y, \tau_1^{'}, \tau_2^{'})$ be a homeomorphic mapping and (X, τ_1, τ_2) be T_0 space therefore (Y, τ_3, τ_4) is also a T_1 space since a T_0 implies a T_1 space. Let $b_1, b_2 \in Y$ where $b_1 \neq b_2$. Since χ is surjective function it then implies that there exists $a_1, a_2 \in X$ with $\chi(a_1) = \chi(b_1)$ and also $\chi(a_2) = b_2$. Hence χ is also one to one function with $b_1 \neq b_2$ this implies that $\chi(a_1) \neq \chi(a_2)$ hence $a_1 \neq a_2$. Since (X, τ_1, τ_2) is a T_1 space and $a_1, a_2 \in X$, with $a_1 \neq a_2$. Then there exists $U \in \tau_1 \cup \tau_2$ such that $a_1 \in U$ and a_1 or a_1 does not exists in U, $a_2 \in U$. Since $a_1 \in U$, a_2 does not exists in U then $U \in \tau_1 \cup \tau_2$. Therefore, $\chi(U) \in \chi(\tau_1 \cup \tau_2)$. Using Tychonoff Theorem as a methodology, χ is open and continuous then $\chi(U) \in \chi(\tau_1) \cup \chi(\tau_2) \in \tau_3^{'} \cup \tau_4^{'}$. Similarly, $a_1 \in U$ and so $\chi(a_1) \in \chi(U)$ also $b_1 \in \chi(U)$ and a_2 does not exists in U then it follows that $\chi(a_2)$ does not exists in $\chi(U)$, and also b_2 is not an element of $\chi(U)$. By Definition 1.7 $b_1, b_2 \in Y$ with $b_1 \neq b_2$ and $\chi(U) \in \tau_1^{'} \cup \tau_2^{'}$ is obtained such that $b_1 \in \chi(U)$, b_2 does not exists in $\chi(U)$. Hence $(Y, \tau_1^{'}, \tau_2^{'})$ is also a T_1 space. Hence χ is continuous if and only if the maps $\chi : (X, \tau_1) \to (Y, \tau_1^{'})$ and $\chi : (X, \tau_2) \to (Y, \tau_2^{'})$ are continuous. Every T_1 space implies T_0 space by hypothesis of heredity. Therefore, a T_1 space has a topological property. □

Proposition 4.19. *Let (X, τ_1, τ_2) be a T_2 space then the property of T_2 is both topological and hereditary.*

Proof. Let (X, τ_1, τ_2) and $(Y, \tau_1^{;}, \tau_2^{'})$ be bitopological spaces. If (X, τ_1, τ_2) is a T_2 space then it exhibits topological properties. Let $\chi : (X, \tau_1, \tau_2) \to (Y, \tau_1^{'}, \tau_2^{'})$ be a homeomorphism and (X, τ_1, τ_2) is also a T_2 space. Then we show that $(Y, \tau_3^{'}, \tau_4^{'})$ is also T_2 space. Hence $b_1, b_2 \in Y$ with $y_1 \neq y_2$. Suppose that χ is a surjective function then all elements in Y are images of elements in X. Hence there exists $a_1, a_2 \in X$ with $\chi(a_1) = b_1$ and $\chi(a_2) = b_2$. Similarly since χ is an injective function then $b_1 \neq b_2$. This implies that $\chi(a_1) \neq \chi(a_2)$, and $a_1 \neq a_2$. Therefore, $a_1, a_2 \in X$ with $a_1 \neq a_2$. Consequently, since (X, τ_1, τ_2) is a T_2 space then $U \in \tau_1$ and $V \in \tau_2$. Therefore, $a_1 \in U$, $a_2 \in V$ then it suffices that $U \cap V \neq \emptyset$. Suppose that χ is open then $\chi(U) \in \tau_1^{'}$ and also $\chi(V) \in \tau_2^{'}$ then $\chi(U) \cap \chi(V) \neq \emptyset$. Given that $c \in X$ then $c \in \chi(U) \cap \chi(V)$. Therefore, $c \in \chi(U)$ and $c \in \chi(V)$ then $p_1 \in U$ and $p_2 \in V$ such that $c = \chi(p_1)$ and $c \in \chi(p_2)$ with $\chi(p_1) = \chi(p_2)$ and $p_1 = p_2$ since χ is an injective function then $p_1 \in U$ and $p_1 \in V$. Hence $p_1 \in U \cap V \neq \emptyset$ by contradiction. Suppose that $U \cap V = \emptyset$ which implies that $\chi(U) \cap \chi(V) = \emptyset$. Therefore, $b_1, b_2 \in Y$ with $b_1 \neq b_2$ hence $\chi(U) = \tau_1^{'}$. Hence $(Y, \tau_1^{'}, \tau_2^{'})$ is a T_2 space. Every homeomorphism image of a T_2 is a T_2 space, therefore it implies that T_2 is a topological property. Let (X, τ_1, τ_2) be a T_2 space then it has hereditary property. Let (X, τ_1, τ_2) also be T_2 space. Since $D \subseteq X$, we prove that $(D, \tau_{D1}, \tau_{D2})$ is also T_2 space. Suppose that $a, b \in D$ and $a \neq b$ then $a, b \in X$. From Definition 1.2, it follows that there exists $U \in \tau_1 \cup \tau_2$ such that $a \in U$, b is not a member of U and also a does not exists in U but $b \in U$. Therefore, $U \in \tau_1 \cup \tau_2$, it implies that $U \in \tau_1$ or $U \in \tau_2$ where $U \cap D \in \tau_{D1}$ or $U \cap D \in \tau_{D2}$. By Tychonoff theorem, $U \cap D \in \tau_{D1} \cap \tau_{D2}$. Similarly $a, b \in D$ then $a \in U \cap D$, b does not exists in $U \cap D$ or a is not an element of $U \cap D$, $b \in U \cap D$. Hence it implies that topological property is also exhibited by a bitopological subspace $(D, \tau_{D1}, \tau_{D2})$. □

Proposition 4.20. *Suppose that (X, τ_1, τ_2) is a $T_{\frac{5}{2}}$ space then the property of $T_{\frac{5}{2}}$ is both topological and hereditary.*

Proof. From our result in Proposition 4.19 we have shown that T_1 space implies T_2 space. This therefore follows that a $T_{\frac{5}{2}}$ space both T_1 and T_2. We commence by showing hereditary property in $T_{\frac{5}{2}}$ space. Given that (X, τ_1, τ_2) is a bitopological space which is also a $T_{\frac{5}{2}}$ space, we can let $K \subseteq X$ such that $(K, \tau_{K1}, \tau_{K2})$ is a subspace which is also a $T_{\frac{5}{2}}$ space. It suffices that K is a subspace of X. Taking m and n to be elements of K then $m, n \in K$ but $m \neq n$. Since (X, τ_1, τ_2) is a $T_{\frac{5}{2}}$ space then the intersection of A and B is said to be empty, $A \cap B = \emptyset$. Therefore, since $A \in \tau_1$ and $B \in \tau_2$. By continuum hypothesis it follows that $A \in \tau_1$, $B \in \tau_2$ then it follows that $A \cap K \in \tau_{K1}$ and $B \cap K \in \tau_{K2}$. Hence $m, n \in K$ then $m \in A \cap K$, $n \in B \cap K$. Hence it clear that $(K, \tau_{K1}, \tau_{K2})$ is $T_{\frac{5}{2}}$ space. If (X, τ_1, τ_2) is a $T_{\frac{5}{2}}$ space then it has a topological property. Therefore, $\chi :$ $(X, \tau_1, \tau_2) \rightarrow (Y, \tau_1^{'}, \tau_2^{'})$. Suppose a function χ is a homeomorphic function then it follows that $(Y, \tau_1^{'}, \tau_2^{'})$ is also a $T_{\frac{5}{2}}$ space. Therefore, $n_1, n_2 \in Y$ with $n_1 \neq n_2$. Let $m_1, m_2 \in X$ with $\chi(m_1) = \chi(n_1)$ and $\chi(m_2) = n_2$. Suppose χ is injective with $n_1 \neq n_2$ consequently, $\chi(m_1) \neq \chi(m_2)$ and $m_1 \neq m_2$. Hence (X, τ_1, τ_2) is $T_{\frac{5}{2}}$ space then $m_1, m_2 \in X$, with $m_1 \neq m_2$ and there exists $A \in \tau_1 \cup \tau_2$ such that $m_1 \in A$, while m_2 does not exists in A or m_1 does not exists in A, and $a_2 \in A$. Similarly, $m_1 \in A$, m_2 does not exists in U therefore, we have that $A \in \tau_1 \cup \tau_2$ such that $\chi(A) \in \chi(\tau_1 \cup \tau_2)$. By conditions for separation axioms, $\chi(A) \in \chi(\tau_1) \cup \chi(\tau_2) \in \tau_1^{'} \cup \tau_2^{2}$, $m_1 \in A$ such that $\chi(m_1) \in \chi(A)$ hence $n_1 \in \chi(A)$ and m_2 does not exists in A and $\chi(m_2)$ is not an element of $\chi(A)$, this implies that n_2 does not exists $\chi(A)$ for any $n_1, n_2 \in Y$ with $n_1 \neq n_2$, $\chi(A) \in \tau_1^{'} \cup \tau_2^{'}$ is obtained such that $n_1 \in \chi(A)$, n_2 does not exists in $\chi(A)$. Therefore, $(Y, \tau_1^{'}, \tau_2^{'})$ is also $T_{\frac{5}{2}}$ space with topological property. □

Lemma 4.21. *Suppose (X, τ_1, τ_2) is a normal bitopological space then the property of T_4 is hereditary.*

Proof. Since we are taking (X, τ_1, τ_2) as a normal bitopological space then it is enough that there exist two disjoint closed sets say x and y where $x \neq y$. Also there are two disjoint open sets say U and V such that $x \subset U$ and $y \subset V$. By Definition 1.7 it shows that $x \in U$, whereas y does not exists in U similarly x is not a member of V but $y \in V$. Normal bitopological space implies T_2 space. Therefore, it suffices that $x, y \in X$ with $x \neq y$. This follows that $U \in \tau_1 \cup \tau_2$ such that $x \in U$, whereas y does not exists in U, using conditions for normality. On the other hand x is not a cardinality of V but $y \in V$, hence normal spaces have topological property. Next, we show that normality and hereditary properties are the same. The result from Proposition 4.18, indicates that $\chi : (X, \tau_1, \tau_2) \to (Y, \tau_1^{'}, \tau_2^{'})$ and χ is homeomorphic since it is a bijective function. Let $A \subseteq X$. Consequently, if (X, τ_1, τ_2) is a normal space then A is also normal. Considering disjoint open sets U and V we have $U \in \tau_1$ and $V \in \tau_2$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Hence $(A, \tau_{A1}, \tau_{A2})$ is normal. Therefore, $U \in \tau_1$ and $V \in \tau_2$ so $U \cap A \in \tau_{A1}$ and $V \cap A \in \tau_{A2}$. This closely follows that $x \in V \cap A$, $y \in V \cap A$. Hence this follows that $(U \cap A) \cap (V \cap A) \cap A = \emptyset \cap A = \emptyset$. Hence $(A, \tau_{A1}, \tau_{A2})$ is a normal subspace and so induces topologies from (X, τ_1, τ_2). □

Next, we consider results of $ij - \pi_\lambda - T_\lambda$ axioms on bitopological spaces if and only if they are $ij - \pi_\lambda$-symmetric.

Proposition 4.22. *Suppose* (X, τ_1, τ_2) *is* $ij - \pi_\lambda - T_\lambda$ *then it is* $ij - \pi_\lambda$-*symmetric.*

Proof. Since we have two bitopological spaces (X, τ_1, τ_2) and $(Y, \tau_1^{'}, \tau_2^{'})$ therefore we have $i - \pi_\lambda$-open in X and $j - \pi_\lambda$-open in Y. Therefore it suffices that we have symmetric points $\pi_\lambda(\{y\})$ and $\pi_\lambda(\{x\})$. Therefore, since (X, τ_1, τ_2) it also has $ij - \pi_\lambda - T_\lambda$. Given that we have two open sets $U \subseteq X$ and $V \subseteq Y$ if $x \in X$ and $y \in Y$ then it suffices that $\pi_\lambda(\{y\}) \in V$ and $\pi_\lambda(\{x\}) \in U$. This shows that $y \in V$ and $x \in U$. Therefore, disjoint closed subsets x, and y are contained in $ij - \pi_\lambda$-open set. Since $x \neq y$ then $y \in ij - Cl\pi_\lambda(\{x\})$. Let U be $ij - \pi_\lambda$-open in X then it suffices that $x \in U$, and $ij - Cl\pi_\lambda(\{y\})$. y does not exist in $ij - Cl\pi_\lambda(\{x\})$. Therefore, $ij - Cl\pi_\lambda(\{x\}) \subseteq U$. Since both U and if V is π_λ open in Y which contain $ij - Cl\pi_\lambda(\{y\})$. Consequently this follows that y does not exists in U or $y \in ij - Cl\pi_\lambda(\{x\})$ and x is not a cardinality of $ij - Cl\pi_\lambda(\{x\})$. Hence

(X, τ_1, τ_2) is $ij - \pi_\lambda - T_0$ since it has topological property as indicated by Proposition 4.17. So (X, τ_1, τ_2) is $ij - \pi_\lambda - T_\lambda$. $\square$

Proposition 4.23. *Let (X, τ_1, τ_2) be $ij - \pi_\lambda - T_\lambda$ symmetric then it is both $ij - \pi_\lambda - T_0$ and $ij - \pi_\lambda - T_1$.*

Proof. Let $x \in X$ and $y \in Y$, therefore we take U and V to be π_λ-open sets in X and Y respectively. Taking (X, τ_1, τ_2) to be $ij - \pi_\lambda - T_\lambda$ symmetric then we have $\pi_\lambda(\{x\})$ and $\pi_\lambda(\{y\})$ with $\pi_\lambda(\{x\}) \neq \pi_\lambda(\{y\})$. We assume that a bitopological space (X, τ_1, τ_2) is $ij - \pi_\lambda - T_0$ since it is $ij - \pi_\lambda - T_\lambda$ symmetric. Since $x, y \in X$ with $x \neq y$ and also U and $ij - \pi_\lambda(\{y\})$ be any two disjoint open sets. It suffices that two disjoint points x and y are elements of open sets either U or $ij - \pi_\lambda(\{y\})$. Therefore, if x and y are contained in $ij - \pi_\lambda$-open set, then we have $y \in U$ and $x \in U$. The fact that U is a member of $ij - \pi_\lambda$-open set then it follows $x \in U$ and y is not a member of U. By Tychonoff theorem, $ij - \pi_\lambda(\{x\}) \subseteq U$. Therefore, since y does not exists U and $ij - \pi_\lambda(\{x\})$ hence by assumption x does not exists in $ij - \pi_\lambda(\{x\})$. Since $ij - \pi_\lambda(\{x\}) \subseteq U$ then (X, τ_1, τ_2) is $ij - \pi_\lambda - T_0$. Now, it implies that (X, τ_1, τ_2) is $ij - \pi_\lambda - T_\lambda$ symmetric. Therefore, every $ij - \pi_\lambda - T_\lambda$ symmetric imply $ij - \pi_\lambda - T_1$. Since (X, τ_1, τ_2) is $ij - \pi_\lambda - T_0$ we suppose that $x \in K \subset X\{y\}$ for $ij - \pi_\lambda$-open set K. Therefore, x is not a member of $ij - Cl\pi_\lambda(\{y\})$ and y does not exists in $ij - Cl\pi(\{x\})$. Therefore, $X \setminus ij - Cl\pi(\{x\})$ is an $ij - \pi_\lambda$-open set containing y but not x. Hence (X, τ_1, τ_2) is $ij - \pi_\lambda - T_1$. $\square$

Lemma 4.24. *If a space is $ij - \pi_\lambda - T_0$ then $ij - Cl\pi_\lambda(\{x\}) \neq ij - Cl\pi_\lambda(\{y\})$ hence $ij - Cl\pi_\lambda(\{x\}) \cap ij - Cl\pi_\lambda(\{y\})$ is empty.*

Proof. Suppose that (X, τ_1, τ_2) is $ij - \pi_\lambda - T_0$ then we have two distinct points x and y. Hence it suffices that $ij - Cl\pi_\lambda(\{y\}) \neq ij - Cl\pi_\lambda(\{x\})$. Therefore, this follows that $x \in ij - Cl\pi_\lambda(\{x\})$ whereby x is not a member of $ij - Cl\pi_\lambda(\{y\})$ this implies that $y \in ij - Cl\pi_\lambda(\{y\})$ and y does not exists in $ij - Cl\pi_\lambda(\{x\})$. Since x is not a member of $ij - Cl\pi_\lambda(\{y\})$ therefore there exists $V \in ij - B\lambda O(X, x)$ such that y does not exists in V. However, $x \in ij - Cl\pi_\lambda(\{x\})$ hence $x \in V$. Therefore, this follows that x is not a member of $ij - Cl\pi_\lambda(\{y\})$. Then it implies that $x \in X \setminus ij - Cl\pi_\lambda(\{y\}) \in ij - B\lambda O(X)$. Since (X, τ_1, τ_2) is $ij - \pi_\lambda - T_0$ then $ij - Cl\pi_\lambda(\{x\}) \subset X \setminus ij - Cl\pi_\lambda(\{y\})$. By Proposition 4.23, we have $ij - Cl\pi_\lambda(\{x\}) \cap ij - \pi Cl_\lambda(\{y\}) = \emptyset$. This therefore implies that $ij - Cl\pi_\lambda(\{x\}) \subset V$. Since y not to be an element of V then it follows that $y \in X \setminus V$ hence $y \neq x$ and x does not exists in $ij - Cl\pi_\lambda(\{y\})$. This shows that $ij - Cl\pi_\lambda(\{y\}) \neq ij - Cl\pi_\lambda(\{x\})$. By assumption $ij - Cl\pi_\lambda(\{y\}) \cap ij - Cl\pi_\lambda(\{x\}) = \emptyset$ hence y does not exists in $ij - Cl\pi_\lambda(\{x\})$ and so $ij - Cl\pi_\lambda(\{x\}) \subseteq V$. Therefore, (X, τ_1, τ_2) is $ij - \pi_\lambda - T_0$. □

Next, we illustrate in the following result that normal $ij - \pi_\lambda - T_2$ bitopological space (X, τ_1, τ_2) is the same as Hausdorff space.

Theorem 4.25. *Given that (X, τ_1, τ_2) is a T_2 then it is $ij - \pi_\lambda - T_2$.*

Proof. Let (X, τ_1, τ_2) be a normal bitopological space. By the conditions for normality, there are disjoint points x and y with $x \neq y$. Suppose we are taking U and V to be π_λ-open sets from bitopological spaces (X, τ_1, τ_2) and $(Y, \tau_1^{'}, \tau_2^{'})$ respectively. So we have $x \subset U$ and $y \subset V$. By definition 1.10 since two disjoint closed sets $x, y \in X$ then it implies that $x \in U$ and $y \in V$. By hypothesis, normal bitopological spaces are also T_2 spaces.

Since we have disjoint sets x and y which are members of X and x is not equal to y. It follows that $U \in \tau_1 \cup \tau_2$ such that $x \in U$ and $y \in V$. By Lemma 4.21, suppose that (X, τ_1, τ_2) is normal then $(A, \tau_{A1}, \tau_{A2})$ is also normal. This is because $A \subseteq X$. There are open disjoint sets U and V where $U \in \tau_1$ and $V \in \tau_2$ such that $x \in U$, $y \in V$. Hence it suffices that $U \cap V = \emptyset$. Consequently, by conditions for normality we have $U \in \tau_1$ and $V \in \tau_2$ then $U \cap A \in \tau_{A1}$ hence $V \cap A \in \tau_{A2}$. Then it implies that $x \in U \cap A$, $y \in V \cap A$. Then $(U \cap A) \cap (V \cap A) \cap A = \emptyset \cap A = \emptyset$. Since $(A, \tau_{A1}, \tau_{A2})$ is a bitopological subspace so it also exhibits topological property. If T_2 is a Hausdorff space then it implies that there are two distinct closed sets. Since there are distinct closed sets there exists also distinct open sets U and V. By hypothesis, $x \in U$, y is not a cardinality of V but $y \in V$. Hence (X, τ_1, τ_2) is $ij - \pi_\lambda - T_2$. Therefore, (X, τ_1, τ_2) is a Hausdorff space and every normal $ij - \pi_\lambda - T_2$ space is also Hausdorff space. □

Corollary 4.26. *Let (X, τ_1, τ_2) be $ij - \pi_\lambda - T_2$ then the property of $ij - \pi_\lambda - T_2$ is topological.*

Proof. For a bitopological space that is $ij - \pi_\lambda - T_2$ exhibit homeormorphic property. For instance, a function $\chi : (X, \tau_1, \tau_2) \to (Y, \tau_1', \tau_2')$ is homeomorphic if and only if it can mapping $\chi : (X, \tau_1 \to (Y, \tau_1^{,})$ and also $\chi : (X, \tau_2) \to (Y, \tau_2')$. Therefore, there exists disjoint open sets $y_1, y_2 \in Y$ with $y_1 \neq y_2$. By hypothesis, χ is a bijective function then it follows that $x_1, x_2 \in X$ with $\chi(x_1) = y_1$ and $\chi(x_2) = y_2$. However, if χ is an injective function with $y_1 \neq y_2$. Then this implies that $\chi(x_1) \neq \chi(x_2)$, this shows clearly that $x_1 \neq x_2$ hence both distinct points x_1 and x_2 are members of X with $x_1 \neq x_2$. Since (X, τ_1, τ_2) is a T_2 space then it implies that there exists $U \in \tau_1$ and $V \in \tau_2$ such that $x_1 \in U$, $x_2 \in V$. By assumption we can say that $U \cap V \neq \emptyset$. Hence $\chi(U) \in \tau_3$ and $\chi(V) \in \tau_4$ due to the fact that χ is an open function. By Tychonoff theorem, $\chi(U) \cap \chi(V) \neq \emptyset$. It follows closely that $c \in X$, hence $c \in \chi(U) \cap \chi(V)$. It implies that c is an element of $\chi(U)$ and $\chi(V)$. So there exists distinct elements p_1 and p_2 such that $p_1 \in U$ and $p_2 \in V$. By any chance $p_1 = p_2$ then $p_1 \in U$ and $p_1 \in V$. Hence it follows that $p_1 \in U \cap V \neq \emptyset$ and by contradiction if $U \cap V = \emptyset$ then $\chi(U) \cap \chi(V) = \emptyset$. Given that $y_1, y_2 \in Y$ with $y_1 \neq y_2$ then $\chi(U) = \tau_3$ hence $y_1 \in \chi(U)$, $y_2 \in \chi(V)$. Similarly it follows that $\chi(U) \cap \chi(V) \neq \emptyset$. Therefore, $(Y, \tau_1^{'}, \tau_2^{'})$ is a T_2 space. Hence (X, τ_1, τ_2) and $(Y, \tau_1^{'}, \tau_2^{'})$ are $ij - \pi_\lambda - T_2$ spaces and topological. □

Corollary 4.27. *Let (X, τ_1) be $T_{\frac{5}{2}}$ space and (X, τ_2) be any topological space then (X, τ_1, τ_2) is a $ij - \pi_\lambda - T_{\frac{5}{2}}$.*

Proof. The result from Proposition 4.2 indicates that T_1 space implies T_2 space. Therefore, suppose (X, τ_1, τ_2) be a $T_{\frac{5}{2}}$ space. Let $R \subseteq X$ hence it follows that $(R, \tau_{R1}, \tau_{R2})$ is also a $T_{\frac{5}{2}}$ space. From Proposition 4.20, i-open set in X and j-open set in Y. If we take x and y to be disjoint points such that $x \in R$ with $x \neq y$. Since (X, τ_1, τ_2) is a $T_{\frac{5}{2}}$ space. If $A \in \tau_1$ and $B \in \tau_2$ whereby we have that $x \in U$, $y \in V$. Hence it qualifies that $A \cap V = \emptyset$. By separation axioms technique, we have $U \cap R \in \tau_{R1}$ and $V \cap R \in \tau_{R2}$ therefore it suffices that $x, y \in M$ hence $x \in U \cap R$. So $y \in V \cap R$. Therefore, $(R, \tau_{R1}, \tau_{R2})$ is $ij - \pi_\lambda - T_{\frac{5}{2}}$. Since (X, τ_1, τ_2) is an $ij - \pi_\lambda - T_{\frac{5}{2}}$ space. More over it has topological property. Let $\chi : (X, \tau_1, \tau_2) \to (Y, \tau_1^{'}, \tau_2^{'})$ and χ is homeomorphic. Since (X, τ_1, τ_2) is $ij - \pi_\lambda - T_{\frac{5}{2}}$ then it implies that $(Y, \tau_1^{'}, \tau_2^{'})$ is also an $ij - \pi_\lambda - T_{\frac{5}{2}}$ space. Therefore, $y_1, y_2 \in Y$ with $n_1 \neq y_2$. Since χ is a surjective function then it implies that $x_1, x_2 \in X$ such that $\chi(x_1) = \chi(y_1)$ and $\chi(y_2) = x_2$. Similarly, if χ is an injective function with $y_1 \neq y_2$ it follows that $\chi(x_1) \neq \chi(x_2)$ and $x_1 \neq x_2$. Therefore, since (X, τ_1, τ_2) is $ij - \pi_\lambda - T_{\frac{5}{2}}$ space then it implies that $x_1, x_2 \in X$, with $x_1 \neq x_2$ and so $\exists A \in \tau_1 \cup \tau_2$ hence $x_1 \in U$, x_1 does not exists in A or x_1 are not elements of U, $x_2 \in A$ also $x_1 \in U$, x_2 does not exists in A hence $U \in \tau_1 \cup \tau_2$. Then $\chi(U) \in \chi(\tau_1 \cup \tau_2)$ since χ is an open function it therefore implies that $\chi(U) \in \chi(\tau_1) \cup \chi(\tau_2) \in \tau_1^{'} \cup \tau_2^{'}$. Since $x_1 \in A$ then it implies that $\chi(x_1) \in \chi(U)$ so $y_1 \in \chi(U)$ and x_2 is not an element in U which implies that $\chi(x_2)$ does not $\chi(U)$ and y_2 does not exists $\chi(U)$. For any $y_1, y_2 \in Y$ with $y_1 \neq y_2$, $\chi(U) \in \tau_1^{'} \cup \tau_2^{'}$ is obtained such that $y_1 \in \chi(U)$ and y_2 is not a member of $\chi(U)$. Hence $(Y, \tau_1^{'}, \tau_2^{'})$ is a $ij - \pi_\lambda - T_{\frac{5}{2}}$ space. Therefore, $ij - \pi_\lambda - T_{\frac{5}{2}}$ space is both topological and hereditary. □

Theorem 4.28. *Let (X, τ_1, τ_2) be pairwise $\pi_\lambda - T_0$ if it has $\tau_1 - \eta$ or $\tau_2 - \eta$ as distinct points of X.*

Proof. Since (X, τ_1, τ_2) is a T_0 space we can let $\tau_1 - \eta cl\{x\} \neq \tau_1 - \eta cl\{y\}$ to be distinct points of X. If we take$x$, y as distinct points in X then $x \neq y$. Therefore, from Corollary 4.26 we can deduce that $\tau_1 - \eta cl\{x\} \neq \tau_1 - \eta cl\{y\}$ and hence by no doubt of generality $\tau_2 - \eta cl\{x\} \neq \tau_2 - \eta cl\{y\}$. Incase we have another element of X say p then it also implies that $p \in \tau_1 - \eta cl\{y\}$. So it suffice to confirm that p does not belongs to $\tau_1 - \eta cl\{x\}$. Therefore, a contradiction arises immediately. So it suffices that $\tau_1 - \eta$ and $\tau_2 - \eta$ are distinct closed points of X. Suppose we are considering a function $\chi : (X, \tau_1, \tau_2) \to (Y, \tau_1^{'}, \tau_2^{'})$ then if $\tau_1 - \eta, \tau_2 - \eta \in X$ with $\tau_1 - \eta \neq \tau_2 - \eta$. Suppose that χ is a surjective function then all elements in Y are images of elements in X. Hence it suffices that $\chi(\tau_1 - \eta) = \tau_1 - \eta$ and $\chi(\tau_2 - eta) = \tau_2 - \eta$. Similarly since if χ is an injective function then $\tau_1 - \eta \neq \tau_2 - \eta$. This implies that $\chi(\tau_1 - \eta) \neq \chi(\tau_2 - \eta)$, and $\tau_1 - \eta \neq \tau_2 - \eta$. □

We illustrate pairwise property of bitopological spaces in the result that follow.

Theorem 4.29. *A bitopological space is pairwise $\pi_\lambda T_0$ if either of the two topologies is $\pi_\lambda T_0$.*

Proof. Let (X, τ_1, τ_2) be a bitopological space. Then (X, τ_1, τ_2) is said to be pairwise $\pi_\lambda T_0$ if either (X, τ_1) or (X, τ_2) is $\pi_\lambda T_0$. For pairwise there exists distinct closed points x and y whereby $x, y \in X$. Therefore, it suffices that there exist two disjoint open sets A and B. From Theorem 4.28, it is true that open set A is a $\tau_1 - \eta$-open set. So A contains x as its element but not y. Therefore, $y \in \tau_1 - \eta cl\{y\} \subset X - U$ this follows that x is not a member of $\tau_1 - \eta cl\{y\}$. Hence we can then have $\tau_1 - \eta cl\{x\} \neq \tau_1 - \eta cl\{y\} \tau_1 - \eta$ and $\tau_2 - \eta$ are closed distinct points. However, this does not need to be true in general. This can be indicated by this obstruction if $X = \{m, n, p\}$, $\tau_1 = \{X, \emptyset, \{m\}, \{n, p\}\}$ and $\tau_2 = \{X, \emptyset, \{p\}, \{m, n\}\}$. From this it shows that a bitopological space (X, τ_1, τ_2) is pairwise λT_0 when neither (X, τ_1) nor is (X, τ_2) is λT_0. □

Theorem 4.30. *Suppose (X, τ_1, τ_2) is normal bitopological space then it is $ij - \pi_\lambda T_0$.*

Proof. Given that we have a bitopological space then it is said to be normal if and only if there two disjoint points which can be separated by open neighborhoods say P and Q such that their intersection is empty. Therefore, suppose that m and n are disjoint closed sets then $m \neq n$.

taking P and Q as open sets in X then it suffices that $m \subset P$ and also $n \subset Q$ and since m and n are members of X, it follows that $m \in P$ but n does not exists in P, while $n \in Q$ but m is not a member of Q. By the use of conditions for normality, we can say that $P \in \tau_1 \cup \tau_2$ such that $m \in P$ and also $n \in Q$. From Theorem 4.29, (X, τ_1, τ_2) is λT_0 therefore it is also $ij - \lambda T_0$-normal. Thus, by Example 1.9, there are two open sets P which is also $\tau_1 - \eta$-open and Q which is $\tau_2 - \eta$-open. Therefore, it follows that $n \in \tau_1 - \eta cl\{n\} \subset X - P$, hence m is not a member of $\tau_1 - \eta cl\{n\}$. By Tychonoff theorem we can say that $\tau_1 - \eta cl\{m\} \neq \tau_1 - \eta cl\{n\}$. Given that there are two distinct points m and n which are members of X. Therefore, neither $\tau_1 - \eta cl\{m\} \neq \tau_1 - \eta cl\{n\}$ nor $\tau_2 - \eta cl\{m\} \neq \tau_2 - \eta cl\{n\}$. Suppose that we have c as any point of X such that $c \in \tau_1 - \eta cl\{n\}$. If give that $n \in \tau_1 - \eta cl\{m\}$ therefore $\tau_1 - \eta cl\{n\} \subset \tau_1 - \eta cl\{m\}$. Hence it implies that $c \in \tau_1 - \eta cl\{n\} \subset \tau_1 - \eta cl\{m\}$. By contradiction, since c is not a cardinality of $\tau_1 - \eta cl\{m\}$ then it shows that n is not a member of $\tau_1 - \eta cl\{m\}$ thus $P = X - \tau_1 - \eta cl\{m\}$ is a $\tau_1 - \eta$-open set that contains n but not x. Hence it implies that $\tau_2 - \eta cl\{a\} \neq \tau_2 - \eta cl\{n\}$. Therefore, (X, τ_1, τ_2) is $ij - \lambda T_0$ and it implies that is a normal space. □

The following is the immediate consequence.

Corollary 4.31. *Every $ij - \pi_\lambda - T_2$ is $ij - \pi_\lambda - T_1$ and $ij - \pi_\lambda - T_0$.*

Proof. Let (X, τ_1, τ_2) be $ij - \pi_\lambda - T_2$, then by assumption (X, τ_1, τ_2) is pairwise $\pi_\lambda T_0$. Suppose that G is any open set which is also $T_i - \pi_\lambda$-open set. Therefore, $x \in G$ such that each point $y \in X$. Then $T_j - \pi Cl\{y\}$. It implies that there exists $T_i - \pi_\lambda$ open set U_y and any $T_j - \pi_\lambda$-open set V_y such that every point $x \in U_y$ and also $y \in V_y$. Therefore, it suffices that $U_y \cap V_y = \emptyset$. Similarly, if $A = \bigcup\{V_y : y \in X - G\}$ then $X - G \subset A$ and x does not exists in A. Therefore, $T_j - \pi_\lambda$ openness of A implies that $T_j - \pi Cl\{x\} \subset X - A \subset G$. Therefore, we can it is true that X is $\pi_\lambda T_0$ and (X, τ_1, τ_2) is $ij - \pi_\lambda - T_0$. By the continuum hypothesis, (X, τ_1, τ_2) is $ij - \pi_\lambda - T_0$, this is because there exists closed disjoint sets say x and y. This follows that $x \neq y$ and $x \in ij - \pi Cl_\lambda(\{y\})$. Therefore, it is assumed that y is not a member of $ij - \pi Cl\pi_\lambda(\{x\})$ and so $ij - \pi Cl_{\pi_\lambda}(\{x\}) \subseteq U$. Thus this implies that (X, τ_1, τ_2) is $ij - \pi_\lambda - T_0$. Hence without of generality (X, τ_1, τ_2) is $ij - \pi_\lambda - T_0$ then it is $ij - \pi_\lambda - T_2$. Therefore, $ij - \pi_\lambda - T_2$ imply $ij - \pi_\lambda - T_1$ which also imply $ij - \pi_\lambda - T_0$. □

From the results that we obtained in this second objective, we established that separation axioms such as T_0-Kolmogorov space, T_1-Fretchét space, T_2-Housdorff space, $T_{\frac{5}{2}}$-Urysohn space and T_4-Normal Hausdorff space can be used in bitopological spaces through the notion of ij-continuity.

Finally, in the next section we consider third objective in our study. We determine extensions of continuity and separation axioms in N-topological spaces.

4.4 Extensions of Continuity and Separation Axioms in N-Topological Spaces

For this objective we consider (X, N_τ) as N-topological spaces with N-topology on X with no separation axioms are assumed unless specifically stated. We are taking N_τ-open to be open sets in N-topological spaces and N_τ-closed to be closed sets in N-topological spaces.

In Proposition 4.32 we give some axioms that N-topological spaces meet.

Proposition 4.32. *Let X be a non empty set and τ_1, τ_2,...,τ_N be arbitrary topologies defined on X. Then the collection $N_\tau = \{S \subseteq X : S = (\bigcup_{i=1 A_i)}^{N} \bigcup \bigcap_{i=1}^{N} B_i) A_i, B_i \in \tau_i\}$, is N-topology if it satisfy the following axioms:*

(i) $X, \phi \in N_\tau$.

(ii). If $N = 1$ then $N\tau = \tau_1 = \tau$.

(iii). Intersection of two 2τ also implies a 2τ. Similarly, the intersection of two 3τ is also a 3τ.

Proof. For axiom (i) and (ii) are trivial topology.
To prove axiom (ii). Let $N\tau_1$ and $N\tau_2$ be two N-topologies which are defined on X. Therefore, it implies that X and $\emptyset$ are both in $N\tau_1 \bigcap N\tau_2$. Let $\{C_i\}_{i\in I} \in N\tau_1 \bigcap N\tau_2$ and $\bigcup_{i\in I} C_i \in N\tau_1$ it follows that $\bigcup_{i\in I} C_i N\tau_2$. Thus by the definition 1.12 it follows that $N\tau_1 \cap N\tau_2$ is a member of 2τ. Suppose we let $\{C_i\}_{i=1}^{N} \in N\tau_1 \bigcap N\tau_2$ then $\bigcap_{i=1}^{N} \in N\tau_1$, this implies that $\bigcap_{i=1}^{N} \in N\tau_2$. Therefore, $N\tau_1 \cap N\tau_2$ is an N-topology. □

The following remark 4.33 follows immediately.

Remark 4.33. The union of two 2τ need not to be 2τ. Likewise the union of two 3τ need not to be in 3τ.

In our next result we show some of the properties exhibited by N-topological spaces. We therefore illustrate that a function χ is continuous in N-topological spaces N_τ-open inverse in Y is N_τ-open in X. We also consider two different continuous functions that are mapping one N-topological space to another. Then it implies that the composition of these functions mapping N-topological space to another is also continuous. We give the result that follows.

Proposition 4.34. *Let $\chi : (X, \tau_1, \tau_2, ..., \tau_N) \to (Y, \tau_1, \tau_2, ..., \tau_N)$ be a continuous function if the inverse of N_τ-open subset in Y is also N_τ-open in X. Then χ is π_λ-continuous.*

Proof. Let A be N_τ-open set in Y. Then $\chi^{-1}(A)$ is clopen in X. Hence it implies that $\chi^{-1}(A) \in \pi_\lambda B(X)$, then by Proposition 4.2, a function χ is π_λ-continuous since A is N_τ-open set in Y. Then we can show that $\chi^{-1}(A)$ is a π_λ-open set in X, suppose that $\chi^{-1}(A) \neq \emptyset$ then it therefore implies that $\chi^{-1}(A)$ is a π_λ-open set in X, and if $\chi^{-1}(A) \subseteq X$, then for each $x \in \chi^{-1}(A)$, we have $\chi(x) \in A$. Since χ is π_λ-continuous then it implies that there exists a π_λ-open set B_x in X such that $x \in B_x$ and $\chi(B_x) \subseteq A$. This implies that $x \in B_x \subseteq \chi^{-1}(A)$. This therefore shows that $\chi^{-1}(A)$ is π_λ-open in X. On the other hand if we let $x \in X$ and A N_τ-open set in Y containing $\chi(x)$. Then it follows that $x \in \chi^{-1}(A)$. By hypothesis, $\chi^{-1}(A)$ is π_λ-open in X containing x, hence it suffices that $\chi(\chi^{-1}(A)) \subseteq A$. Therefore, χ is π_λ-continuous. □

Proposition 4.35. *The property of soft-$\pi - T_0$ is hereditary in tritopological spaces.*

Proof. Let Y be a soft-subspace of soft-$\pi - T_0$-space $(X, N\tau_1, N\tau_2, N\tau_3, E)$.

There are distinct soft-points e_A and e_B with $e_A \neq e_B \in Y$. Since $Y \subseteq X$ then it implies that $e_A, e_B \in X$ and so $(X, N\tau_1, N\tau_2, N\tau_3, E)$ is a soft-$\pi - T_0$-space. By separation axioms, there exists soft- π-open sets (F_1, E), (F_2, E) such that $e_A \in (F_1, E)$ or e_B does not exists in (F_1, E) and $e_B \in (F_2, E)$, e_A does not belongs to (F_2, E). Hence it follows that $(F_1, E) \cap Y = (F_{1Y}, E)$ is a soft-π-open set in Y and $e_A \in (F_{1Y}, E)$, e_B is not a member of (F_{1Y}, E). On the other hand, we can show that e_A does not exists in (F_2, E), $e_B \in (F_2, E)$ then e_A does not exists (F_{2Y}, E) and $e_B \in (F_{2Y}, E)$. Hence Y is a soft-$\pi - T_0$. □

Lemma 4.36. *Given that $(X, N\tau_1, N\tau_2, ..., \tau_N)$ is a normal N-topological space then the property of T_4 is both topological and hereditary.*

Proof. Suppose that $(X, N\tau_1, N\tau_2, ..., \tau_N)$ is a normal space it therefore implies that there exist two disjoint closed sets say a and b where $a \neq b$. Also there are two disjoint open sets say U and V such that $a, b \in X$. Then this suffices that $a \in U$, whereas b does not exists in U similarly a is not a member of V but $b \in V$. Normal bitopological space implies T_2 space. Therefore, it suffices that $a, b \in X$ with $a \neq b$. This follows that there exists $U \in \tau_1 \cup \tau_2$ such that $a \in U$, whereas b does not exists in U. On the other hand a is not a cardinality of V but $b \in V$, hence normal spaces have topological property. We show that normality and hereditary properties are the same. By Proposition 4.18, $\chi : (X, N\tau_1, N\tau_2, ..., \tau_N) \rightarrow (Y, N\tau_1', N\tau_2', ..., \tau N')$ and χ is homeomorphic since it is a bijective function. Let $M \subseteq X$. Therefore, this follows that if $(X, N\tau_1, N\tau_2, ..., \tau_N)$ is a normal space then M is also normal, by employing the conditions for normality, a subspace of X is also normal. considering disjoint open sets U and V we have $U \in N_{\tau_1}$ and

$V \in N_{\tau_2}$ such that $a \in U$, $b \in V$ and $U \cap V = \emptyset$. Hence $(M, N_{\tau_{M1}}, N_{\tau_{M2}})$ is normal. Therefore, $U \in N_{\tau_1}$ and $V \in N_{\tau_2}$ so $U \cap M \in N_{\tau_{M1}}$ and $V \cap M \in N_{\tau_{M2}}$. This closely follows that $a \in U \cap M$ likewise $b \in V \cap M$. Hence $(M, N\tau_{M1}, N\tau_{M2}, ..., N\tau_{MN})$ has both normality and topological properties induced from $(X, N\tau_1, N\tau_2, ..., \tau_N)$. □

Theorem 4.37. *The property of soft-π_λ-closed is hereditary in N-normal topological space.*

Proof. We have two disjoint points (F_1, E) and (F_2, E) which are soft-π_λ-closed subsets of $(Y, N\tau_1', N\tau_2', ..., \tau N')$. Then there exists soft-π-closed subsets (H, E) and (V, E) in $(X, \tau_1, \tau_2, \tau_3, E)$ such that $(F_1, E) = Y \cap (H, E)$ and $(F_2, E) = Y \cap (V, E)$, since y is soft-π-closed in $(X, \tau_1, \tau_2, \tau_3, E)$. Since (F_1, E) and (F_2, E) are soft-π_λ-closed in $(X, \tau_1, \tau_2, \tau_3, E)$. When we employ the conditions for normality $(X, \tau_1, \tau_2, \tau_3, E)$ is soft-π_λ-normal then it implies that there exists soft-π-open set (F_3, E), (F_4, E)
in $(X, \tau_1, \tau_2, \tau_3, E)$ such that $(F_1, E) \subseteq (F_2, E)$, $(F_3, E) \subseteq (F_4, E)$ and $(F_3, E) \cap (F_4, E) = \phi$. However, $(F_1, E) \subseteq y \cap (F_3, E)$, $(F_2, E) \subseteq y \cap (F_4, E)$ where $y \cap (F_3, E)$, $y \cap (F_4, E)$ are soft-disjoint soft-π_λ-open subsets in Y. Therefore, $(Y, N\tau_1, N\tau_2, N\tau_3, E)$ is a soft-π_λ-normal soft-subspace. □

In this third objective, the results show that continuity and separation axioms via the notion of ij-continuity can be naturally extended to N-topological spaces.

Chapter 5

CONCLUSION AND RECOMMENDATIONS

5.1 Introduction

We draw conclusion based on our objectives and the results obtained in this chapter. We also give recommendations that can help in tackling further research in this area of study on continuity of functions on bitopological spaces.

5.2 Conclusion

We summarize our work by highlighting the results obtained in our study as per the problem stated in Section 1.3 of this work. Our objectives were to characterize notion of ij-continuity in bitopological spaces, establish the separation method for bitopological spaces through ij-continuity and determine extensions of continuity and separation axioms in N-topological space as stated in Section 1.4. In chapter 1, we gave mathematical background, basic definitions and concepts which we found very essential to our study. In chapter 2, we have done literature review on continuity of functions on both topological and bitopological spaces which were related to our topic of study. For instance, studies on Strong Continuity in Topological Spaces by Nourman [53], Mappings and Pairwise Continuity On Pairwise Lindelöf Bitopological Spaces by Adem and Zabidin [2] and Separation Axioms for Bitopological Spaces by Arya and Nour [12] among others. In chapter 3, we have outlined the methodologies used in obtaining our results.

In chapter 4, we have given out the results of our study. For objective one, we have characterized various notions of continuity in bitopological spaces, we showed that suppose that if $\chi : (X, \tau_1, \tau_2) \to (Y, \tau_1^{'}, \tau_2^{'})$ be an open function then a subset W of X is said to be π_λ-open if and only if it is semi-closed and an intersection of π_λ-open sets in X. Moreover, χ is therefore said to be π_λ-continuous. We also showed that if a function $\chi : X \to Y$ is $ij - \pi_\lambda$-continuous and if for each open set X_0 of X we have $\eta \in X$, such that $\chi \mid_{X_0}: X_0 \to Y$ is said to be π_d-continuous. On composition of functions we have shown that if we have the functions $\chi_1 : X \to Y$ be π_λ-continuous and $\chi_2 : Y \to Z$ be π_d-continuous. Therefore, $\chi_2 \circ \chi_1$ is $ij - \pi_d$ continuous.

For objective two on establishing separation technique for bitopological spaces we have shown that they exhibit both topological and heredity properties. Next, we have shown that T_0 space implies T_1 which also implies T_2 space and the converse is true. We have indicated in our results that suppose bitopological spaces (X, τ_1, τ_2) and $(Y, \tau_1^{'}, \tau_2^{'})$ are T_0, T_1, and

T_2, spaces then the properties of T_0, T_1, and T_2, are both hereditary and topological. We have therefore established that T_0-Kolmogorov space, T_1-Fretchét space, T_2-Housdorff space, $T_{\frac{5}{2}}$-Urysohn space and T_4-Normal Housdorff space can be used in bitopological spaces through the notion of ij-continuity as separation criteria for bitopological spaces via the notion of ij-continuity. For the third objective on determining extensions of separation axioms in N-topological spaces we have shown that continuity in bitopological spaces can be naturally extended up to N-topological space as shown in Proposition 4.35, Lemma 4.36 and Theorem 4.37. Finally, we have also shown continuity in N-topological spaces as shown in Proposition 4.32 and Proposition 4.34. Therefore, these results indicate that continuity in N-topological spaces and separation axioms that are used to separate N-topological spaces can be naturally extended to N-topological spaces.

5.3 Recommendations

Continuity of bitopological spaces and other N-topological spaces is a very interesting area of study in mathematics and has not been fully exhausted so far. In our case we considered only semi-continuity, weak continuity and strong continuity of bitopological spaces. We therefore recommend that further research can be directed to other aspects of continuity such as local continuity, fuzzy continuity and global continuity in bitopological spaces and N-topological spaces. Secondly, in our study through ij-continuity we have established separation criteria for bitopological spaces. Therefore, our recommendation is that further research can be done to establish separation criteria in a fuzzy bitopological spaces. Lastly, we showed extensions of continuity and separation axioms in discrete N-topological spaces. We therefore recommend that more research should be carried out to show extensions of of continuity and separation axioms in local, fuzzy and global N-topological spaces.

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