NUMERICAL ANALYSIS OF HOLLING TYPE II FUNCTIONAL RESPONSE PREDATOR-PREY MODEL WITH TIME DELAY OPTIMAL SELECTIVE HARVESTING

BY

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A Thesis Submitted to the Board of Postgraduate Studies in Partial Fulfilment of the Requirements for the Award of the Degree of Master of Science in Applied Mathematics

SCHOOL OF BIOLOGICAL, PHYSICAL, MATHEMATICS, AND ACTUARIAL SCIENCES

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Declaration

This thesis is my own work and has not been presented for a degree award in any other institution.

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Acknowledgments

My special thanks goes to Dr. Titus Aminer and Prof. Benard Okelo for their support, guidance and unlimited patience during my research work. Additional thanks goes to the entire members of teaching staff of SBPMAS in JOOUST for their guidance and criticism during my presentations. Final thanks goes to my family and friends for everything they accorded me during the entire period of my study.

Dedication

To my wife Dorah and Parents Peter and Peninah ....

Abstract

Population dynamics indicate the changes in size and composition of population through time, as well as biotic and abiotic factors influencing those changes. Predator-prey (PP) relationship with harvesting and functional response involving prey refuge with Holling type I functional response (HTIFR) has been studied with recommendations on their extension to include Holling type II functional response (HTIIFR). There persists a problem in finding the numerical solution of predator-prey system having HTIIFR particularly when optimal selective harvesting (OSH) is carried out. Therefore, this study focuses on numerical analysis of predator-prey model with HTIIFR having time delay OSH. The objectives of this study includes to: Formulate a HTIIFR PP model with time delay OSH; Carry out stability analysis of the model and; Simulate the numerical outcome of the model. The methodology involves a step by step formulation of the model by considering various parameters and useful assumptions. This is followed by carrying out stability analysis of the model through local and global stability analysis techniques. Lastly, MATLAB software is utilized using Runge-Kutta technique in simulating the numerical outcomes of the model. The results of the study includes a formulated model which is both LAS and GAS when $\gamma_1 \geq 0.00604$. The numerical analysis has shown that the rate harvesting goes down due to a rise in the cost of harvesting. This study is useful in informing policy makers on how to control interaction between predator-prey environments so that extinction is controlled.

Contents

Index of Notations

Chapter 1

INTRODUCTION

1.1 Mathematical Background

The existing relationship between predator and prey species in an ecosystem play an important role both in ecology and mathematical ecology. The interaction of species in an ecosystem can lead to fluctuation in population densities due to predation [3]. The description of interaction among species and projection of the future stability of an ecosystem helps in sustaining and retaining their benefits, since the interactions have positive, negative or no effect on the interacting species [28]. The most popular mathematical model often used in describing PP interactions is the LV model [18]. The existing amount of environmental resources dictates the maximum number of species that it can carry at any particular time, that is the carrying capacity [7]. Developing mathematical models is a key approach to understanding the ecological interaction among species in which one is a predator and the other is a prey. Most numerical solutions of differential equations have received much attention where replacement of these equations by equivalent finite difference equation has been done[6]. Descriptions of the interactions between species and prediction of the future state of an ecosystem helps to maintain and sustain the benefits that we extract from nature. Developing Mathematical models is one of the key approaches applied in understanding the ecological interaction between predators and prey species. More realistic and plausible mathematical models require critical consideration of aspects [27]. Knowledge of both functional and numerical responses is required to fully understand how predators and prey interact hence providing a complete description of predator population dynamics[26]. Many species have experienced extinction while others are approaching it due to factors like; poor management of natural resources, environmental pollution, over-predation, over-exploitation among others. To protect these species from extinction, precautions like creation of reserve zones and restriction on harvesting should be put in place to allow them grow without any external disturbance [27]. The existence of reserve regions also called refuges have become a key interest to researchers in studying the predator-prey dynamics. In his work [8], Holling came up with three major types of functional responses namely; types I, II and III and the effect they have on prey killed per unit time. HTIIFRs are characterized by a decelerating intake rate, which follows from the assumption that the consumer is limited to by its capacity to process food.

1.2 Basic Concepts

In this section, we introduce elementary mathematical ideas, definitions, remarks, examples and other mathematical tools that help us in the later sections.

1.2.1 Functional responses

This shows the interaction between the rate of consumption by a single predator and prey density.

Definition 1.1. Dependence on prey

Shows how each predator is dependent on a prey or several preys.

Definition 1.2. Ratio dependence

A description of the ratio of prey to predator dependence on them.[27].

Definition 1.3. Multi-species dependence

Multi-species functional responses are functional reactions that are dependent on the abundances of multiple prey species.

1.2.2 Types of functional responses

Holling type I

It describes a linear increase in the rate of consumption for each individual predator as the number of prey rises up to a maximum point where consumption level becomes constant [1].

It is expressed as, $N = aT_s x$, where $x \geq 0$, N is the number of preys consumed and aT_s is the consumption rate of prey by a predator within a given time.

Holling type II

We consider an $T_s = T_t - bN$. Now, when this equation is combined with one for Holling type I, we come up with type II formulation given as, $N = \frac{aT_t x}{1+abx}$.

Holling type III

In this type, Holling proposed functional response of the form, $N = \frac{aT_t x^k}{1+abx^k}$; if $k = 2$, where k is an integer. In general terms, we have the function for this type of functional response. $N = \frac{x^k}{a+x^k}$

1.3 Statement of the problem

Pusawidjayanti, Asmianto and Kusumasari [28] studied dynamical analysis of PP population model with HTIIFR, given by:

$$\begin{aligned}
\frac{dx}{dt} &= rx(1-\frac{x}{k}) - \frac{(1-n)mxy}{1+x} - \alpha_1 Q_1 x \\
\frac{dy}{dt} &= \frac{(1-n)cxy}{1+x} - by - \alpha_2 Q_2 y,
\end{aligned}$$

The authors suggested a further research based on addition of consideration of assumptions to see the dynamic change in an ecosystem and also the need to add maximum benefit from the process of harvesting the population [28]. Hence, our study focuses on adoption of time delay OSH in formulation of a new mathematical model. The key motivation behind our study is to consider addition of assumptions, by giving a time delay that will have effects on coexistence of predators and prey species in an ecosystem [14].

1.4 Objectives of the Study

1.4.1 Main Objective

The main objective of this study is to numerically analyze HTIIFR PP model with time delay OSH.

1.4.2 Specific Objectives

The specific objectives are to:

(i). Formulate a HTIIFR PP model with time delay OSH.

(ii). Carry out stability analysis of the model.

(iii). Simulate the numerical outcome of the model.

1.5 Significance of the Study

This study provides a numerical analysis for solving series of coupled LV competition systems of equations and many other similar systems of equations and hence this contributes knowledge in the field of mathematics and provide an avenue for further research. It is useful in analyzing population dynamics especially in fishery system and reserves to gain an understanding of the changes in population. The results of this study are useful in informing policy makers on how to control interaction between predator-prey environments so that extinction is controlled and also to enable optimal selective harvesting.

Chapter 2

LITERATURE REVIEW

2.1 Introduction

This chapter consists of literature reviewed from different research works on predator-prey equations and their numerical analysis. We have also discussed some relevant work that had been done in relation to our topic of study.

2.2 Predator-prey equations

Predator-Prey systems provided the main bench mark in this work. Intricate situations are always considered by people doing modelling to understand the situation with the aim of providing solutions to the problems which arise from the situations particualrty the population interaction cases. We note that many researchers have so far greatly studied the relationship that exists among biological species in the past few decades using varying methods [10]. The Lokta-Volterra model forms the basis of many models currently being used in analysis of population dynamics. It entails two coupled nonlinear differential equations that show the interaction between a predator and prey population as indicated;

$$\begin{aligned} \frac{dx}{dt} &= ax - bxy \\ \frac{dy}{dt} &= -cy + dxy \end{aligned} \tag{2.2.1}$$

In system 2.2.1, the constant a represents number of prey population when predators are taken to be zero and ax is growth term. The constant c represents predator population death rate when prey is absent. Finally, -cy is decay term. The xy term represents the interaction between two populations, in an open environment where the interaction is free [19].
From the model 2.2.1, a large number of prey population ensures more food to support a large predator population. Equally, it is important to note that when the predator population goes up, prey begins to die leading to a lower number of predators.

2.3 Numerical Solutions of ordinary differential equations

Numerical methods are well known in the field of science and engineering to solve various linear and nonlinear ordinary differential equations (ODEs). Ways of solving Ordinary Differential Equation (ODEs) include but not limited to Eulers method, Picards method, Taylors series method and Runge Kutta (R-K) method. These methods have gone through various stages of development with the advancement of the programming languages and for the various applications to real life applications[2].

The Runge-Kutta method is developed for various orders of convergence, that is, ranging from order 1 to order 4. In all these available versions of R-K method, the method of fourth order becomes more famous because of its convergent properties. This method is of excessive practical significance with mentioned accuracy and numerical stability in comparison with the well-known Eulers method.

The Taylors series method [8] which is well known for solving differential equations numerically becomes ineffective for the problem which involves the higher order derivatives.

In the last decade, many researchers have devoted their efforts in development of numerical methods to solve the ODEs efficiently with good accuracy. Since the advent of digital computers, most of the researchers work has been on R-K methods, and most of researcher's work has contributed to extension of the theory, and development of extended R-K methods. One merit of R-K methods as compared to other methods is that it involves no requirement of the calculations for higher order derivatives. We equally note that problems in science and engineering can be solved by reducing them to differential equations satisfying certain conditions. Analytical methods can be applied to solve different types of differential equations while others can be solved efficiently by numerical methods. The initial value problem, can be solved by any method from the methods categorized in the following groups of methods: 1. single step or pointwise methods and; 2. step by step methods.

Chapter 3

RESEARCH METHODOLOGY

3.1 Introduction

This chapter entails the description of methods and techniques which are useful in the analysis of the problem. We consider techniques for stability analysis, simulation and numerical analysis.

3.2 Consistency and stability analysis

In mathematical modelling, a chronological order is required to obtain accurate results. To achieve this, stability analysis is crucial. This goes hand in hand with consistency analysis. However, the latter is not considered in this case. In this study, we will determine Locally Asymptotically stable and Globally Asymptotically stable Equilibrium Points for the formulated model

3.3 The Runge-Kutta Methods

Numerical methods in getting solutions of ODEs can be put in two categories- Numerical integration methods and Runge-Kutta methods [22]. We consider first ODEs of the form $\frac{dy}{dx} = f(x, y)$, with conditions $y(x_0) = y_0$ where points of the domain $[x_0, x_n]$ are considered at uniform distance. The solution at the point x_{n+1} obtained by $y(x_{n+1})$ can be obtained by use of Runge-Kutta method [23].

3.3.1 First Order R-K method

The Eulers formula for first approximation to the solution of Equation **??** is given by; $y_1 = y(x_0 + h)$. It is important to note that the Eulers method is the R-K method of first order given as; $y_1 = y_0 + hy_0$.

3.3.2 Second-Order R-K method

The modified Eulers formula for numerical solution of an ODE Equation **??** is the R-K method of second order or midpoint method.

3.3.3 Third-Order R-K formula

The formula for the third order R-K method follows analogously from the second order.

3.3.4 Fourth-Order R-K formula

This method is the most commonly used Runge-Kutta (R-K) method known as the classical Runge-Kutta method defined as $y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$. Similarly, the value of y_2 in the second interval is obtained by replacing x_0 by x_1 and y_0 by y_1 in the above set of formulae. In general, this process can continue upto finding y_n.

3.3.5 Fifth-Order R-K formula

The fifth-order R-K method was introduced by Kutta [Geeta Arora], [4] but since there were errors in the presentation of his results, it was then partly corrected and it gave rise to two different formulations of the now fifth-order R-K method given by

$$y_1 = y_0 + \frac{h}{90}(7k_1 + 32k_2 + 12k_4 + 32k_5 + 7k_6).$$

3.4 Numerical analysis

Studies carried out analytically can not be complete without verifying the formulated model numerically. Therefore, there is need to carry out simulations of the dynamical behaviour of the system using Runge-Kutta iteration methods discussed in the section above. We use Forward RK Method and Backward RK Method.To carry out this procedure, we choose the values of the parameters following ecological observations which are realistic although they are hypothetical in nature in a fishery set up by considering three species Phytoplankton, Zooplankton and Fish. The most important variables for analysis are Optimal Selective Harvesting and sales of Fish.

Chapter 4

RESULTS AND DISCUSSION

4.1 Introduction

In this section, we formulate the model under study and establish the existence of its equilibrium points. The local asymptotic stability of the model is then discussed and finally numerical solution is investigated.

4.2 Model Formulation

Our aim is to carry out a systemic formulation of the model of a two species PP interaction, where $x(t)$ and $y(t)$ denote the population densities of prey species and predator species respectively at any time t. Now, the generalized PP model is given by:

$$\begin{aligned}\frac{dx}{dt} &= ax - xp(x)y,\\ \frac{dy}{dt} &= -dy + \alpha xp(x)y,\end{aligned} \tag{4.2.1}$$

where a, d, α and $xp(x)$ represents the specific growth rate of prey population in the absence of predator, natural death rate of predators in the absence of prey, conversion factor and response function respectively.
If we then assume that the prey population grows logistically in the absence of predators with a growth rate r and carrying capacity k, then Equation 4.2.1 changes to:

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1 - \frac{x}{k}\right) - xp(x)y,\\ \frac{dy}{dt} &= \alpha xp(x)y - dy.\end{aligned} \tag{4.2.2}$$

Let the functional response function $xp(x)$ be expressed in the form of $xp(x) = \frac{mx}{1+x}$ corresponding to a HTIIFR. Then the system of Equation 4.2.2 becomes:

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1 - \frac{x}{k}\right) - \frac{mxy}{1+x},\\ \frac{dy}{dt} &= \frac{\alpha mxy}{1+x} - dy.\end{aligned} \tag{4.2.3}$$

If for economical purpose, we only let the predator species be subjected to harvesting, then the System 4.2.3 changes to:

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right)-\frac{mxy}{1+x},\\ \frac{dy}{dt} &= \frac{\alpha mxy}{1+x}-dy-q_2E_2y,\end{aligned} \tag{4.2.4}$$

where q_2 is the catchability coefficient and $0 < E_2(t) < E_{max}$ is the harvesting effort of the predator species.

Now, if we introduced a time delay constant ($\tau \geq 0$) in the harvesting term, then the system of equations 4.2.4 extends to:

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right)-\frac{mxy}{1+x},\\ \frac{dy}{dt} &= \frac{\alpha mxy}{1+x}-dy-q_2E_2y(t-\tau),\end{aligned} \tag{4.2.5}$$

which is the HTIIFR model with a time delay predator harvesting.

Similarly, if we assume that only the prey species are selectively harvested for their economical value, then System 4.2.5 can as well be written as:

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right)-\frac{mxy}{1+x}-q_1E_1x,\\ \frac{dy}{dt} &= \frac{\alpha mxy}{1+x}-dy.\end{aligned} \tag{4.2.6}$$

Introducing the time delay constant ($\tau \geq 0$) in the harvesting term leads to the required HTIIFR model with only prey harvesting given by:

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right)-\frac{mxy}{1+x}-q_1E_1x(t-\tau),\\ \frac{dy}{dt} &= \frac{\alpha mxy}{1+x}-dy.\end{aligned} \tag{4.2.7}$$

The Systems 4.2.5 and 4.2.7 are formulated under the following assumptions:

(i). The prey species grow logistically in the absence of the predators.

(ii). The predator feeds on the prey according to a HTIIFR.

(iii). Prey species find enough food at all times.

(iv). Only one of the species is subjected to harvesting hence selective harvesting.

(v). The catch rate function $q_i E_i$ is based on the catch-per-unit-effort.

(vi). Harvesting of species begin to occur after a certain age or size.

The meanings of the parameters used in the formulated models are explained as per the table below:

Parameter	**Meaning**
$x(t)$	Population density of prey species at time t
$y(t)$	Population density of predator species at time t
r	Intrinsic growth rate of prey species
k	Carrying capacity for prey species
m	Capturing rate of predator on prey
α	Conversion rate of prey to predator
d	Natural death rate of predator in the absence of prey
q_2	Catchability coefficient of predator
q_1	Catchability coefficient of prey
E_1	Harvesting effort of prey
E_2	Harvesting effort of predator
τ	Time delay constant

4.3 Existence and stability of equilibrium points

We independently establish the existence and stability of EPs for the two models; System 4.2.5 and System 4.2.7.

4.3.1 Model with prey selective harvesting

We consider the System 4.2.7 below:

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right)-\frac{mxy}{1+x}-q_1E_1x(t-\tau),\\ \frac{dy}{dt} &= \frac{\alpha mxy}{1+x}-dy.\end{aligned}$$

4.3.2 Existence of equilibrium points of System 4.2.7

Letting $r_1 = r - q_1 E_1(t-\tau)$ from system 4.2.7, leads to:

$$\begin{aligned} \frac{dx}{dt} &= r_1 x - \frac{rx^2}{k} - \frac{mxy}{1+x}, \\ \frac{dy}{dt} &= \frac{\alpha mxy}{1+x} - dy. \end{aligned} \tag{4.3.1}$$

We therefore show the existence of the predator extinction EP $E_*(x_*, 0)$ by letting $y = 0$ in System 4.3.7 to get $r_1 x - \frac{rx^2}{k} = 0$, $\Rightarrow r_1 = \frac{rx}{k}$ and hence $x_* = \frac{kr_1}{r}$. Therefore, the predator extinction EP exists and is given by:

$$E_*(x_*, 0) = E_*(\frac{kr_1}{r}, 0).$$

For the existence of coexistence EP $E^*(x^*, y^*)$, we solve the algebraic equations:

$$r_1 x - \frac{rx^2}{k} - \frac{mxy}{1+x} = 0 \tag{4.3.2}$$

$$\frac{\alpha mxy}{1+x} - dy = 0 \tag{4.3.3}$$

From Equation 4.3.3, we have:

$$\alpha mx = d + dx, \Rightarrow (\alpha m - d)x = d,$$

implying that $x^* = \frac{d}{\alpha m - d}$ which when substituted into Equation 4.3.2 leads to:

$$\begin{aligned} r_1 - \frac{rx^*}{k} &= \frac{my}{1+x^*} \\ \Rightarrow my &= (1+x^*)(r_1 - \frac{rx^*}{k}). \end{aligned}$$

Hence,

$$y^* = \frac{(1+x^*)(r_1 - \frac{rx^*}{k})}{m},$$

and therefore, the coexistence EP is given by;

$$E^*(x^*, y^*) = E\left[\frac{d}{\alpha m - d}, \frac{(1+x^*)(r_1 - \frac{rx^*}{k})}{m}\right],$$

which exists whenever $\alpha m > d$ and $r_1 > \frac{rx^*}{k}$.

4.3.3 Local Stability Analysis of System 4.2.7

We establish the local stability of the EP by constructing a Jacobian matrix for which the nature of the eigenvalues of the constructed matrices helps determine the stability. From System 4.2.7, we have:

$$\begin{aligned} r_1 x - \frac{rx^2}{k} - \frac{mxy}{1+x} &= g_1(x,y), \\ \frac{\alpha mxy}{1+x} - dy &= g_2(x,y). \end{aligned}$$

Partially differentiating g_1 and g_2 with respect to x and y separately results into a Jacobian matrix of the System 4.2.7 expressed as:

$$J = \begin{pmatrix} r_1 - \frac{2rx}{k} - \frac{(1+x)my - mxy}{(1+x)^2} & -\frac{mx}{1+x} \\ \frac{(1+x)\alpha my - \alpha mxy}{(1+x)^2} & \frac{\alpha mx}{1+x} - d \end{pmatrix}. \tag{4.3.4}$$

The Jacobian matrix 4.3.4 evaluated at the vanishing EP $E_0(0,0)$ is given as:

$$J(E_0) = \begin{pmatrix} r_1 & 0 \\ 0 & -d \end{pmatrix},$$

which implies that the eigenvalues $\mu_1 = r_1$ and $\mu_2 = -d$. Therefore, E_0 is LAS whenever $r_1 < 0$, otherwise it is a saddle point.

Similarly, the Jacobian matrix evaluated at the predator extinction point $E_*(x_*, 0)$, is given as:

$$J(E_*) = \begin{pmatrix} r_1 - \frac{2rx}{k} & -\frac{mx}{1+x} \\ 0 & \frac{\alpha mx}{1+x} - d \end{pmatrix},$$

from which the eigenvalues μ_1 and μ_2 are given by $\mu_1 = r_1 - \frac{2rx}{k}$ and $\mu_2 = \frac{\alpha mx}{1+x} - d$.

Therefore, E_* is LAS if:

$$r_1 < \frac{2rx}{k}, \tag{4.3.5}$$

$$\frac{\alpha mx}{1+x} = d, \tag{4.3.6}$$

otherwise unstable.

Finally, the coexistence EP is LAS if the real roots of the equation $|J(E^*) - \mu I| = 0$ are all negative. Therefore,

$$J(E^*) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

where the entries

$b_{11} = r_1 - \frac{2rx}{k} - \frac{(1+x)my - mxy}{(1+x)^2}$, $b_{12} = -\frac{mx}{1+x}$, $b_{21} = \frac{(1+x)\alpha my - \alpha mxy}{(1+x)^2}$ and $b_{22} = \frac{\alpha mx}{1+x} - d$.

Evaluating $|J(E^*) - \mu I| = 0$ leads to

$$|J(E^*) - \mu I| = \begin{vmatrix} b_{11} - \mu & b_{12} \\ b_{21} & b_{22} - \mu \end{vmatrix} = 0,$$

Which simplifies to $(b_{11} - \mu)(b_{22} - \mu) - b_{12}b_{21} = 0$, implying that

$$\mu^2 - (b_{11} + b_{22})\mu + b_{11}b_{22} = 0;$$

whose real roots will all be negative if by Routh-Hurwitz criterion; $-(b_{11} + b_{22}) > 0$ and $b_{11}b_{22} - b_{12}b_{21} > 0$, which hold whenever Condition 4.3.11 and Condition 4.3.12 are satisfied. Hence, the equilibrium point $E^*(x^*, y^*)$ is LAS if Condition 4.3.11 and Condition 4.3.12 hold, otherwise unstable.

4.3.4 Global Stability Analysis of System 4.2.7

If the equilibrium points $E_*(x_*, 0)$ and $E^*(x^*, y^*)$ are both LAS, then they are GAS, if given a Lyapunov function $\varpi(x, y) > 0$, the $\frac{d\varpi}{dt} \leq 0$.
Indeed, let

$$\varpi(x, y) = \left(x - x_* - x_* \ln \frac{x}{x_*}\right) + \frac{1}{\alpha}y, \qquad (4.3.7)$$

for the predator extinction EP $E_*(x_*, 0)$. Then, the time derivative of $\varpi(x, y)$ in the direction of the solution of System 4.2.7 is given by

$$\begin{aligned}
\frac{d\varpi}{dt} &= \frac{\partial \varpi}{\partial x}.\frac{dx}{dt} + \frac{\partial \varpi}{\partial y}.\frac{dy}{dt}, \\
&= \left(\frac{x - x_*}{x}\right)\left[r_1 x - \frac{rx^2}{k} - \frac{mxy}{1+x}\right] + \frac{1}{\alpha}\left(\frac{\alpha mxy}{1+x} - dy\right). \\
&= (x - x_*)\left[-\frac{r}{k}(x - x_*) - \frac{my}{1 + (x - x_*)}\right] + \frac{m(x - x_*)y}{1 + (x - x_*)} - \frac{dy}{\alpha}. \\
&= -\frac{r}{k}(x - x_*)^2 - \frac{my(x - x_*)}{1 + (x - x_*)} + \frac{m(x - x_*)y}{1 + (x - x_*)} - \frac{dy}{\alpha}. \\
\frac{d\varpi}{dt} &= -\frac{r}{k}(x - x_*)^2 - \frac{dy}{\alpha} < 0.
\end{aligned}$$

Therefore, $E_*(x_*, 0)$ is GAS.

Similarly, the Lyapunov function for the positive EP $E^*(x^*, y^*)$ is given by;

$$\varpi(x, y) = \left(x - x^* - x^* \ln \frac{x}{x^*}\right) + \frac{1}{\alpha}\left(y - y^* - y^* \ln \frac{y}{y^*}\right). \qquad (4.3.8)$$

The time derivative of Equation 4.3.8 in the direction of the solution of System 4.2.7 leads to;

$$\begin{aligned}
\frac{d\varpi}{dt} &= \frac{\partial \varpi}{\partial x}.\frac{dx}{dt} + \frac{\partial \varpi}{\partial y}.\frac{dy}{dt}, \\
&= \left(\frac{x - x^*}{x}\right)\left[r_1 x - r\frac{x^2}{k} - \frac{mxy}{1+x}\right] + \frac{1}{\alpha}\left(\frac{(y - y^*)}{y}\right)\left(\frac{\alpha mxy}{1+x} - dy\right). \\
&= (x - x^*)\left[-\frac{r}{k}(x - x^*) - \frac{m(y - y^*)}{1 + (x - x^*)}\right] + \frac{1}{\alpha}(y - y^*)\left[\frac{\alpha m(x - x^*)}{1 + (x - x^*)} - d\right]. \\
&= -\frac{r}{k}(x - x^*)^2 - \frac{m(y - y^*)(x - x^*)}{1 + (x - x^*)} + \frac{m(x - x^*)(y - y^*)}{1 + (x - x^*)} - \frac{d(y - y^*)}{\alpha}. \\
\frac{d\varpi}{dt} &= -\frac{r}{k}(x - x^*)^2 - \frac{d(y - y^*)}{\alpha} < 0.
\end{aligned}$$

Hence, $E^*(x^*, y^*)$ is GAS.

4.3.5 Model with predator selective harvesting

Consider the System 4.2.5 below;

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right)-\frac{mxy}{1+x},\\ \frac{dy}{dt} &= \frac{\alpha mxy}{1+x}-dy-q_2E_2y(t-\tau),\end{aligned} \tag{4.3.9}$$

4.3.6 Existence of EPs

Consider the System 4.2.5. If we let $r_2 = d + q_2E_2(t-\tau)$, then model 4.2.5 becomes;

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right)-\frac{mxy}{1+x},\\ \frac{dy}{dt} &= \frac{\alpha mxy}{1+x}-r_2y.\end{aligned}$$

We then show the existence of the trivial EP $E_0(0,0)$, the extinction of predator EP $E_1(x_1,0)$ and finally the coexistence EP $E_2(x_2,y_2)$. Therefore, the existence of $E_0 = E_0(0,0)$ is trivial and demonstrating the existence of $E_1(x_1,0)$ involves letting $y = 0$ in Equation 4.3.1 to get;

$$rx-\frac{rx^2}{k}=0, \Rightarrow r=\frac{rx}{k},$$

and hence $x = rk$.

Therefore, the predator extinction point is given by $E_1(x_1,0) = E_1(rk,0)$.

Finding $E_2(x_2, y_2)$ requires that we solve $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$ which implies that,

$$r - \frac{rx}{k} - \frac{my}{1+x} = 0, \tag{4.3.10}$$

$$\frac{\alpha m x}{1+x} - r_2 = 0. \tag{4.3.11}$$

From Equation 4.3.3, $(\alpha m - r_2)x = r_2 \Rightarrow x_1 = \frac{r_2}{\alpha m - r_2}$.
Substituting x_1 in Equation 4.3.2 gives;

$$\frac{my}{1+x} = \left(r - \frac{rk}{k}\right), \Rightarrow y_2 = \frac{(1+x_2)(r - \frac{rx_2}{k})}{m}.$$

Therefore the coexistence EP;

$$E_2(x_2, y_2) = E_2\left[\frac{r_2}{\alpha m - r_2}, \frac{(1+x_2)(r - \frac{rx_2}{k})}{m}\right]$$

exists if $\alpha m > r_2$ and $r > \frac{rx_2}{k}$.

4.3.7 Local Stability Analysis of System 4.2.5

At this point, we investigate the stability of the Model 4.2.5 around each of the EPs $E_0(0,0)$, $E_1(x_1, 0)$ and $E_2(x_2, y_2)$. Therefore,
Let

$$rx - \frac{rx^2}{k} - \frac{mxy}{1+x} = f_1(x, y),$$

$$\frac{\alpha mxy}{1+x} - r_2 y = f_2(x, y).$$

The Jacobian matrix of the System 4.2.5 is obtained by partially differentiating f_1 and f_2 with respect to x and y respectively. Hence,

$$J = \begin{pmatrix} r - \frac{2rx}{k} - \frac{(1+x)my - mxy}{(1+x)^2} & -\frac{mx}{1+x} \\ \frac{(1+x)\alpha my - \alpha mxy}{(1+x)^2} & \frac{\alpha mx}{1+x} - r_2 \end{pmatrix}. \tag{4.3.12}$$

The Jacobian matrix 4.3.12 evaluated at $E_0(0,0)$ is given by;

$$J(E_0) = \begin{pmatrix} r & 0 \\ 0 & -r_2 \end{pmatrix},$$

from which we have the eigenvalues given as $\lambda_1 = r$ and $\lambda_2 = -r_2$; which implies that $J(E_0)$ is only stable whenever $r < 0$, otherwise a saddle point.

The Jacobian matrix evaluated at the predator extinction EP $E_1(x_1, 0)$, is given by;

$$J(E_1) = \begin{pmatrix} r - \frac{2rx}{k} & -\frac{mx}{1+x} \\ 0 & \frac{\alpha mx}{1+x} - r_2 \end{pmatrix},$$

from which the eigenvalues λ_1 and λ_2 are given as $\lambda_1 = r - \frac{2rx}{k}$ and $\lambda_2 = \frac{\alpha mx}{1+x} - r_2$. Therefore, E_1 is LAS if:

$$r < \frac{2rx}{k}, \tag{4.3.13}$$

$$\frac{\alpha mx}{1+x} < r_2. \tag{4.3.14}$$

Finally, let the $J(E_2)$ be denoted by the matrix;

$$J(E_2) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where the entries

$a_{11} = r - \frac{2rx}{k} - \frac{(1+x)my-mxy}{(1+x)^2}$, $a_{12} = -\frac{mx}{1+x}$, $a_{21} = \frac{(1+x)\alpha my-\alpha mxy}{(1+x)^2}$ and $a_{22} = \frac{\alpha mx}{1+x} - r_2$.

The eigenvalues μ_i of $J(E_2)$ are determined by solving the auxiliary equation $|J(E_2) - \mu I| = 0$.

Hence,

$$|J(E_2) - \mu I| = \begin{vmatrix} a_{11} - \mu & a_{12} \\ a_{21} & a_{22} - \mu \end{vmatrix} = 0,$$

which simplifies to $(a_{11} - \mu)(a_{22} - \mu) - a_{12}a_{21} = 0$, which upon expansion and factorization yields; $\mu^2 - (a_{11} + a_{22})\mu + a_{11}a_{22} = 0$.

According to Routh-Hurwitz criterion, $J(E_2)$ has all negative real roots if $-(a_{11} + a_{22}) > 0$ and $a_{11}a_{22} - a_{12}a_{21} > 0$, which holds whenever the Condition 4.3.5 and Condition 4.3.6 hold. Hence, $E_2(x_2, y_2)$ is LAS if Condition 4.3.5 and Condition 4.3.6 are satisfied, otherwise unstable.

4.3.8 Global Stability Analysis of System 4.2.5

Suppose that the predator extinction EP $E_1(x_1, 0)$ is GAS, then we need to show that a positive definite function $\varpi(x, y)$;

$$\frac{d\varpi}{dt} = \frac{\partial\varpi}{\partial x}.\frac{dx}{dt} + \frac{\partial\varpi}{\partial y}.\frac{dy}{dt},$$

is a negative definite function.

Therefore, let the positive definite function be given as;

$$\varpi(x, y) = \left(x - x_1 - x_1 \ln \frac{x}{x_1}\right) + \frac{1}{\alpha}y. \tag{4.3.15}$$

The derivative of $\varpi(x, y)$ with respect to time is expressed as;

$$
\begin{aligned}
\frac{d\varpi}{dt} &= \frac{\partial\varpi}{\partial x}.\frac{dx}{dt} + \frac{\partial\varpi}{\partial y}.\frac{dy}{dt}, \\
&= \left(\frac{x - x_1}{x}\right)\left[rx(1 - \frac{x}{k}) - \frac{mxy}{1+x}\right] + \frac{1}{\alpha}\left(\frac{\alpha mxy}{1+x} - r_2 y\right). \\
&= (x - x_1)\left[-\frac{r}{k}(x - x_1) - \frac{my}{1 + (x - x_1)}\right] + \frac{m(x - x_1)y}{1 + (x - x_1)} - \frac{r_2 y}{\alpha}. \\
&= -\frac{r}{k}(x - x_1)^2 - \frac{my(x - x_1)}{1 + (x - x_1)} + \frac{m(x - x_1)y}{1 + (x - x_1)} - \frac{r_2 y}{\alpha}. \\
\frac{d\varpi}{dt} &= -\frac{r}{k}(x - x_1)^2 - \frac{r_2 y}{\alpha} < 0.
\end{aligned}
$$

Hence, $E_1(x_1, 0)$ is GAS.

Similarly, establishing if $E_2(x_2, y_2)$ is GAS requires that for any positive definite function $\varpi(x, y) > 0$, then;

$$\frac{d\varpi}{dt} = \frac{\partial\varpi}{\partial x}.\frac{dx}{dt} + \frac{\partial\varpi}{\partial y}.\frac{dy}{dt} < 0.$$

Therefore, the Lyapunov function of $E_2(x_2, y_2)$ is given by;

$$\varpi(x, y) = \left(x - x_2 - x_2 \ln\frac{x}{x_2}\right) + \frac{1}{\alpha}\left(y - y_2 - y_2 \ln\frac{y}{y_2}\right). \qquad (4.3.16)$$

The time derivative of Equation 2 is expressed as;

$$\begin{aligned}
\frac{d\varpi}{dt} &= \frac{\partial\varpi}{\partial x}.\frac{dx}{dt} + \frac{\partial\varpi}{\partial y}.\frac{dy}{dt},\\
&= \left(\frac{x-x_2}{x}\right)\left[rx(1-\frac{x}{k}) - \frac{mxy}{1+x}\right] + \frac{1}{\alpha}\left(\frac{(y-y_2)}{y}\right)\left(\frac{\alpha mxy}{1+x} - r_2 y\right).\\
&= (x-x_2)\left[-\frac{r}{k}(x-x_2) - \frac{m(y-y_2)}{1+(x-x_2)}\right] + \frac{1}{\alpha}(y-y_2)\left[\frac{\alpha m(x-x_2)}{1+(x-x_2)} - r_2\right].\\
&= -\frac{r}{k}(x-x_2)^2 - \frac{m(y-y_2)(x-x_2)}{1+(x-x_2)} + \frac{m(x-x_2)(y-y_2)}{1+(x-x_2)} - \frac{r_2(y-y_2)}{\alpha}.\\
\frac{d\varpi}{dt} &= -\frac{r}{k}(x-x_2)^2 - \frac{r_2(y-y_2)}{\alpha} < 0.
\end{aligned}$$

Hence, $E_2(x_2, y_2)$ is GAS.

In the next section, we tackle the last objective. We carry out the numerical analysis of the model to illustrate the dynamical behaviour of the system. This is done through computer simulations due to lack of real data.

4.4 Numerical Simulation

Studies carried out analytically can not be complete without verifying the formulated model numerically using MATLAB software. We therefore carry out simulations of the dynamical behaviour of the system using Runge-Kutta iteration methods discussed in Chapter three. We choose the parameters following ecological observations which are realistic although they are hypothetical in nature. The parameter values are as follows: $r = 2.05, k = 121, \beta = 0.59, \alpha = 3.98, \gamma = 0.48, \beta_0 = 0.4, d =$

$0.03598, \rho = 0.99, s = 0.25, s_1 = 0.25, \delta = 0.65, q = 0.015, E = 0.39, \gamma_1 = 0.09$.

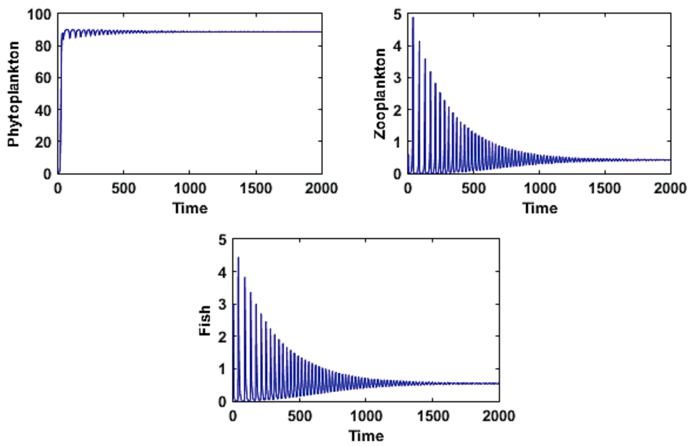

Figure 4.4.1: Population densities of Phytoplankton, Zooplankton and Fish over time evolution

The hypothetical set of values of the parameters has been used in drawing Figure 4.4.1. We can see from this figure that we have LAS equilibrium point from the interior illustrating the coexistence of PZF.

Consider the numerical analysis of the OSH which can be solved using the parameters as indicated: $r = 2.15, k = 101.5, \beta = 0.58, \alpha = 1.01, \gamma = 0.66, \beta_1 = 0.51, d = 0.3501, \rho = 0.195, \gamma_1 = 0.62, s = 0.49, s_1 = 0.37, \delta = 0.019, q = 0.019$. We have used the FRKM to solve the System 4.2.5 within a specified time bound. We follow the procedure by the use of BRKM to solve the optimal selective harvesting problem in System 4.2.7 Finally, the OSH results are displayed with consideration to sale of fish, the cost involving harvests and discount rate (δ_1) due to time delay respectively.

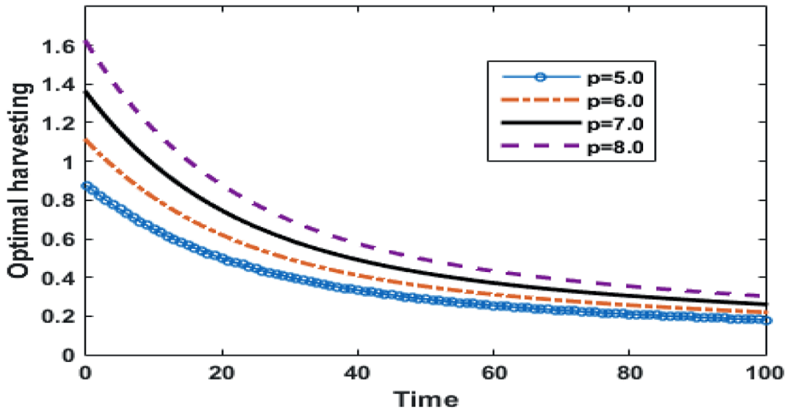

Figure 4.4.2: OSH of fish and to sales.

It is observed from Figure 4.4.2 that an increase in the sales of fish leads to an increase in the OSH rate of fish. This increase happens gradually.

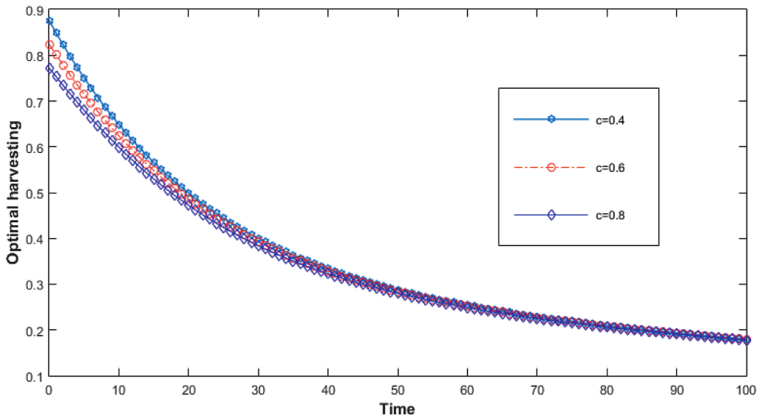

Figure 4.4.3: OSH of fish and costs of harvest.

From Figure 4.4.3, it is clear that as the cost of harvests of fish goes up, two things happen. Firstly, the OSH of fish gradually goes down. Consequently, the gradual downward trend leads it to the EP. From Figure 4.4.4, we observe that as the annual time delayed discount rate

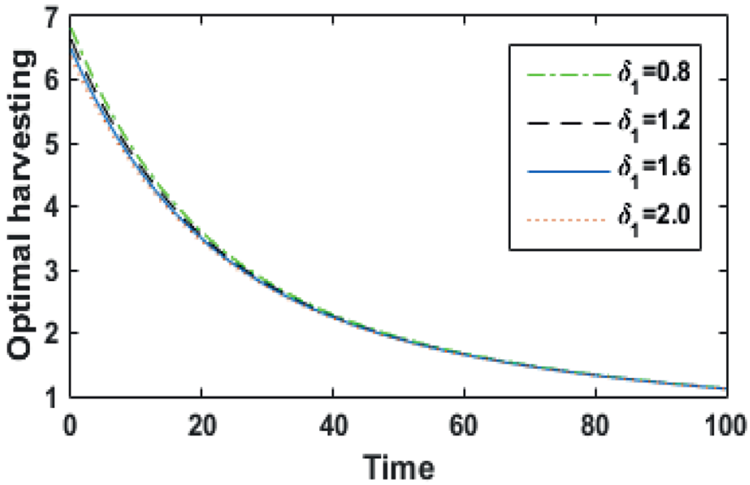

Figure 4.4.4: OSH of fish and production cost discount rate (δ_1).

goes up for the sales, OSH of fish also goes up. These changes also happen gradually under time delay.

Chapter 5

CONCLUSION AND RECOMMENDATIONS

5.1 Introduction

Studies involving numerical analysis of models cannot be complete unless asystematic approach is given. In this study, we have carefully followed a step by step analysis to come up with the results. Therefore, in this last chapter, we give a summary of our work. The conclusion and recommendations of this study is given in this chapter based on the objectives in Section 1.4.

5.2 Conclusion

The study had three specific objectives. We give our conclusion as follows:

(i). T We have formulated the model in Equations 4.2.5 and 4.2.7 by incorporating optimal selective harvesting.

(ii). The second objective was to carry out stability analysis of the model. We have done both local and global analysis of the model. is both locally and globally asymptotically stable when $\gamma_1 \geq 0.00604$. Therefore, it can be concluded that the formulated model attains stability when more zooplanktons are eaten fish.

(iii). Finally for the third objective, we have done numerical simulations of the model and the results from the figures are making practical sense and so this model can be implemented in a fishery system.

5.3 Recommendations

The world is dynamic and portrays a dynamical system with problems which arise everyday that requires mathematical modelling that can help in understanding these dynamics. Therefore, studies on mathematical modelling of these aspects cannot be exhausted. This requires that we give recommendations that can enable other researchers to carry out studies in modelling that will help in finding solutions to these problems world over. Hence, we recommend that:

(i). A study can be carried out on a territory model with Holling type III functional response with optimal selective harvesting but without time delay.

(ii). Bifurcation, convergence and stability analysis can be done on the model developed in (i) above in order to determine convergence patterns and stability conditions for the model.

(iii). Numerical analysis can be done through simulation to unveil the applications of the model in real life situations in other setups other than the fisheries system.

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