International Journal of Innovative Research and Advanced Studies (IJIRAS)
Volume 4 Issue 7, July 2017

ISSN: 2394-4404

On The Similarities In Properties Of Essential Numerical Range And Davis-Wielandt Shell Of Hilbert Space Operators

D. O. Owego

N. B. Okelo

Omolo Ongati

Jaramogi Oginga Odinga University of Science and Technology,
Bondo-Kenya

***Abstract:** Let T be an operator on an infinite dimensional Hilbert space H. Denote the essential numerical range of the operator T by $W_e(T)$ and the Davis-Wielandt shell of the operator by $DW(T)$. We review the properties of the essential numerical range and those of the Davis-Wielandt shell. This review is aimed at striking similarities in the properties shared. The results of this study show that some of the properties shared are, for instance, unitary invariance and convexity. However, it is noted that the Davis-Wielandt shell is convex if $\dim H \geq 3$. If $\dim H = 2$, then $DW(T)$ is an ellipsoid. Thus $DW(T)$ is convex or encloses a convex set. It is also shown that both $DW(T)$ and $W(T)$ behave in the same way if T is the identity operator and scalar multiplication. This review is a contribution to operator theory and the area of numerical ranges.*

***Keywords:** Numerical range, Essential numerical range, Davis-Wielandt shell*

I. INTRODUCTION

Let H be a Hilbert space equipped with the inner product $\langle .,. \rangle$, and let $B(H)$ be the algebra of bounded linear operators acting on H. We recall that the numerical range (also known as the field of values) $W(T)$ of $T \in B(H)$ is the collection of all complex numbers of the form $\langle Tx, x\rangle$ where x is a unit vector in H.

i.e. $W(T) = \{\langle Tx, x\rangle : x \in H, \|x\| = 1\}$ See, ([2], [5], [8])

which is useful for studying operators. In particular, the geometrical properties of the numerical range often provide useful information about the algebraic and analytic properties of the operator T. For instance, $W(T) = \mu$ if and only if $T = \mu I$; $W(T)$ is real if and only if $T = T^*$, $W(T)$ has no interior points if and only if there are complex numbers, a and b with $a \neq 0$ such that $aT + bI$ is self-adjoint. Moreover, the closure of $W(T)$, denoted by $\overline{W(T)}$, always contains the spectrum of T denoted by $\sigma(T)$. See, [8]

Let $K(H)$ denote the set of compact operators on H and $\pi : B(H) \rightarrow B(H)/K(H)$ be the canonical quotient map. The essential numerical range of T, denoted by $We(T)$ is the set;

$W_e(T) = \cap \overline{W(T+K)}$ See, ([1], [2], [3])

where the intersection runs over the compact operators $K \in K(H)$.

Chacon and Chacon [3] gave some of the properties of the essential numerical range as follows:

Let $T \in B(H)$ then;

- ✓ $W_e(T)$ is a non-void compact and convex set.
- ✓ If T is an essentially normal operator, then $W_e(T) = co\left(\sigma_e(T)\right)$ and the essential numerical radius, $w_e(T) = \|T\|_e$
- ✓ $W_e(T) = 0$ if and only if T is compact

According to Chacon and Chacon [3], $W_e(T)$ is a closed subset of $\overline{W(T)}$ and the essential spectrum, $\sigma_e(T)$ is always a compact subset contained in $\sigma(T)$. The essential numerical range is unitarily invariant. For, let T be an operator on $B(H)$ then $W_e(T)$ is unitarily invariant.

Also, the essential numerical range obeys the projection property. For instance; for an operator $T \in B(H)$ we have,

$ReW_e(T) = W_e(ReT)$, where Re stands for the real part. This comes about as a result of the fact that every operator $T \in B(H)$ can be written as; $T = ReT + iIM\, T$. see, [4]

The essential numerical range is non-empty, closed and convex. The non-emptiness of the essential numerical range follows from the fact that for any orthonormal sequence $\{x_n\}_{n\geq 1}$, $(\langle Tx_n, x_n\rangle)_{n\geq 1}$ is a sequence of complex numbers bounded by $\|T\|$ and thus has a convergent subsequence. Thus $W_e(T)$ is non-empty. Moreover, it is clear that $W_e(T)$ is closed and convex being the intersection of closed, convex sets.

The Davis-Wielandt shell is a generalization of the classical numerical range and it is defined by;

$$DW(T) = \{\langle Tx, x\rangle, \langle Tx, Tx\rangle \ : \ \|x\| = 1\ ,\ \ x \in H\}$$

See, ([5], [6], [7])

The first co-ordinate is the classical numerical range, $W(T)$, while the second co-ordinate denotes the numerical range of the operator T^*T, i.e. $W(T^*T)$. So the Davis-Wielandt shell captures more information about the operator T than the numerical range of the operator. For instance, in the finite dimensional case, normality of operators can be completely determined by the geometrical shape of their Davis-Wielandt shells. However we restrict our study to an infinite dimensional Hilbert space. This is because the essential numerical range does not make sense in the finite dimensional case.

THEOREM [PROPERTIES OF THE DAVIS-WIELANDT SHELL] (SEE, [7] THEOREM 2.1)

Let $T \in B(H)$, then;

- ✓ $(\mu, r) \in DW(T)$ if and only if there is an orthonormal pair of vectors such that, $Tx = \mu x + \sqrt{r - |\mu|^2}$
- ✓ The set $DW(T)$ is bounded. In particular, $DW(T) \subseteq \wp(T)$ with $\wp(T) = \{(\mu, r) \in \mathbb{C} \times [0, \infty) : |\mu|^2 \leq r \leq \|T\|^2\}$
- ✓ $DW(T) = DW(UTU^*)$ for any unitary operator $U \in B(H)$
- ✓ For any $\alpha, \beta \in \mathbb{C}$, the Davis-Wielandt shell denoted by $DW(\alpha T + \beta I) = \{(\alpha\mu + \beta, |\alpha|^2 v + 2Re(\alpha\bar{\beta}\mu) + |\beta|^2) \ : \ (\mu, v) \in DW(T)\}$
- ✓ Suppose $T \in B(H)$ is the direct sum of $T_1 \oplus \ldots \oplus T_m$, then; $conv\{DW(T_1) \cup \ldots \cup DW(T_m)\}$
- ✓ The set $DW(T)$ is closed if $dim\, H$ is finite

II. MAIN RESULTS

The Essential numerical range and the Davis-Wielandt Shell of Hilbert space operators share a variety of properties as illustrated below;

UNITARY INVARIANCE

Both the Essential Numerical Range and the Davis-Wielandt Shells of Hilbert Space operators are invariant under unitary similarity. Let $K(H)$ denote the set of compact operators on H and $\pi : B(H) \to B(H)/K(H)$ be the canonical quotient map. The essential numerical range of T, denoted by $W_e(T)$ is the set;

$$W_e(T) = \cap\, \overline{W(T + K)}$$

where the intersection runs over the compact operators $K \in K(H)$ See, ([1], [2], [3]) . Clearly;

$W_e(T) = W_e(UTU^*)$ for any unitary operator $U \in B(H)$.

This is because the Essential Numerical Range is the numerical range of the coset containing T in the Calkin algebra. Thus, since the numerical range is invariant under unitary similarity, we infer that the Essential Numerical Range is also unitarily invariant.

The Davis-Wielandt shell is a generalization of the classical numerical range and it is defined by;

$$DW(T) = \{\langle Tx, x\rangle, \langle Tx, Tx\rangle \ : \ \|x\| = 1\ ,\ \ x \in H\}$$

See, ([5], [6], [7])

The Davis-Wielandt shell is invariant under unitary similarity

$$DW(T) = DW(UTU^*)$$

for any unitary operator $U \in B(H)$. To prove this, we need to show that both the first and second co-ordinate satisfy unitary invariance.

For the first co-ordinate, we have;

$$\begin{aligned}\langle (U^*TU)x, x\rangle &= \langle T(Ux),\ (Ux)\rangle \\ &= \langle Tx, x\rangle \\ &= W(T)\end{aligned}$$

Consequently, for the second co-ordinate, we have;

$$\begin{aligned}\langle (U^*TU)x,\ (U^*TU)x\rangle &= \langle (TU)x,\ (UU^*TU)x\rangle \\ &= \langle (TU)x,\ (TU)x\rangle \\ &= \langle Tx,\ Tx\rangle \\ &= W(T^*T)\end{aligned}$$

The above proofs follow from the fact that;

$$\begin{aligned}\langle Ux,\ Ux\rangle &= \langle x,\ (U^*U)x\rangle \\ &= \langle x,\ x\rangle \\ &= 1\end{aligned}$$

Thus $DW(T) = \{\langle (UTU)x,\ x\rangle, \langle (UTU)x,\ (UTU)x\rangle : x \in H,\ \|x\| = 1\}$

CONVEXITY

The essential numerical range is defined as $W_e(T) = \cap_{K\in K(H)} \overline{W(T + K)}$. Therefore, since each $\overline{W(T + K)}$ is convex by the Toeplitz-Hausdorff theorem, $\overline{W(T + K)}$ is convex as well. Consequently, $\cap_{K\in K(H)} \overline{W(T + K)}$ is convex as well. Therefore, $W_e(T)$ is convex. (See, [9])

Consequently, suppose $T \in B(H)$ with $\dim H \geq 3$, then $DW(T)$ is convex. The Davis-Wielandt shell of a Hilbert space operator is convex if $\dim H \neq 2$, and an ellipsoid if $\dim H = 2$. Thus the Davis-Wielandt shell is either a convex set or encloses a convex set. (See, [7])

SCALAR MULTIPLICATION

For any $\alpha, \beta \in \mathbb{C}$, the Davis-Wielandt shell denoted by

$$DW(\alpha T + \beta I) = \{(\alpha\mu + \beta, |\alpha|^2 v + 2Re(\alpha\bar{\beta}\mu) + |\beta|^2) \ : \ (\mu, v) \in DW(T)\}$$

Since $DW(T) = \{\langle Tx, x\rangle, \langle Tx, Tx\rangle \ : \ \|x\| = 1\ ,\ \ x \in H\}$, we have the first co-ordinate as;

$$\langle (\alpha T + \beta I)x, x\rangle = \langle \alpha Tx + \beta Ix, x\rangle$$

$$= \langle \alpha Tx, x\rangle + \langle \beta Ix, x\rangle$$
$$= \alpha\langle Tx, x\rangle + \beta\langle Ix, x\rangle$$
$$= \alpha\mu + \beta$$

For the second co-ordinate, we have;

$$\langle(\alpha T + \beta I)x,\ (\alpha T + \beta I)x\rangle = \langle \alpha Tx + \beta Ix,\ (\alpha T + \beta I)x\rangle$$
$$= \langle \alpha Tx,\ (\alpha T + \beta I)x\rangle + \langle \beta Ix,\ (\alpha T + \beta I)x\rangle$$
$$= \langle \alpha Tx, \alpha Tx + \beta Ix\rangle + \langle \beta Ix, \alpha Tx + \beta Ix\rangle$$
$$= \langle \alpha Tx, \alpha Tx\rangle + \langle \alpha Tx, \beta Ix\rangle + \langle \beta Ix, \alpha Tx\rangle + \langle \beta Ix, \beta Ix\rangle$$
$$= \alpha\bar{\alpha}\langle Tx, Tx\rangle + \alpha\bar{\beta}\langle Tx, x\rangle + \beta\bar{\alpha}\langle x, Tx\rangle + \beta\bar{\beta}\langle x, x\rangle$$
$$= |\alpha|^2 + 2Re\left(\alpha\bar{\beta}\mu\right) + |\beta|^2$$

As for the essential numerical range, for all $\alpha, \beta \in \mathbb{C}$, we have;

$$W_e(T) = W_e(\alpha T) + W_e(\beta I)$$
$$= \alpha W_e(T) + \beta W_e(I)$$
$$= \alpha W_e(T) + \beta$$

Thus both the essential numerical range and the Davis-Wielandt shell behave in the same way to complex scalar multiplication.

IDENTITY OPERATOR

If $T = I$ then the essential numerical range behaves in the same way as the first and second co-ordinates of the Davis-Wielandt shell. That is, for the essential numerical range, we have;

$$W_e(I) = 1$$

As for the Davis-Wielandt shell, we have the first co-ordinate of the shell as;

$$\langle Ix, x\rangle = \langle x, x\rangle$$
$$= 1$$

As for the second co-ordinate, we have;

$$\langle Ix, Ix\rangle = \langle I^*Ix, x\rangle$$
$$= \langle x, x\rangle$$
$$= 1$$

Thus the essential numerical range, the first and the second co-ordinate of the Davis-Wielandt shell behave in the same way to identity operator.

III. CONCLUSION

Both the essential numerical range and the Davis-Wielandt shell of an operator T in an infinite dimensional Hilbert space are convex. However, it is noted that the convexity of the Davis-Wielandt shell holds only for $\dim H \geq 3$. $DW(T)$ is an ellipsoid if $\dim H = 2$. The essential numerical range and the Davis-Wielandt shell are both unitarily invariant. Besides, they both behave in the same way to identity operator and scalar multiplication.

ACKNOWLEDGEMENTS

We wish to acknowledge the National Commission for Science, Technology and Innovation (NACOSTI) for providing partial funding towards this research.

REFERENCES

[1] Bakic D., Compact operators, the essential spectrum and the essential numerical range, Mathematical Communications, (3)1998, 103 – 108
[2] Baraa M., Essential numerical range of elementary operators, Proceedings of Amer. Math. Soc. Vol 133(2004), No. 6. pg. 1723 – 1726
[3] Chacon G. A., Chacon G. R., Some properties of Composition Operators on the Dirichlet space, Acta Math. Univ. Comenianae, vol 1 (2000) pg. 7 Vol.2, (1989).
[4] Christophe J., Compressions and Pinchings, Journal of Operator Theory, vol.50, 2003, 211-220
[5] Davis C, The shell of a Hilbert Space Operator II, Acta Sci. Math. (Szeged), 31 (1970), 310 – 318
[6] Poon Y. T, Li C. K., Spectrum, Numerical Range and Davis-Wielandt Shells of Normal Operators, Acta Sci. Math. (Szeged)
[7] Poon Y. T., Li C. K., N-Sing Sze, Davis-Wielandt Shells of Operators, Acta Sci. Math. (Szeged)
[8] Rao D., Gustafson K., Numerical Range (Springer-Verlag, 1977).
[9] Stout F. Q., Schur products of operators and the essential numerical range, Trans. Amer. Math. Soc vol 264 (1981), pg. 39 - 47