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Research Article

Characterization of Norm Inequalities for Elementary Operators

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Abstract

Studies on the norms of the elementary operators on JB^*-algebras Prime C^*-algebras, Calkin algebras and standard operator algebras has been considered. In this paper, we characterize norm inequalities for Jordan elementary operators on C^*-algebras. The results show that if H is an infinite dimensional complex Hilbert space and B(H) the C*-algebra of all bounded linear operators on H, then for a Jordan elementary operator $U : B(H) \to B(H)$ defined by: $U(T) = PTQ + QTP$ for all $T \in B(H)$ and Pi;Qi fixed in B(H), $\|U(T)\| \leq 2\|P\|\|Q\|$. Moreover, if P_i and Q_i are diagonal operators induced by $\{\alpha_{ni}\}$ and $\{\beta_{ni}\}$respectively and H an infinite dimensional complex Hilbert space then U is bounded and $\|U\| = (\Sigma n\{\Sigma^l i_{=1}|\alpha_{ni}|^2|\beta_{ni}|^2\})^{1/2}$.

Keywords: Norm; C^*-algebra; Elementary operator; Hilbert space.

Introduction

Studies on properties of elementary operators have been of great concern to many mathematicians [1,2]. Properties such as Numerical ranges, Spectrum, Compactness and Positivity have been studied with excellent results obtained [3]. The norm property has remained an interesting area. Although the upper estimates of these norms are trivially obtainable in terms of their coefficients, estimating them from below has proved to be a challenge with different results being obtained. Entanglement [4] is a basic physical resource to realize various quantum information and quantum communication tasks such as quantum cryptography, teleportation, dense coding and key distribution [5].

Let H and K be separable complex Hilbert spaces. Recall that a quantum state is a density operator which is positive and has trace 1. From Horodecki's theorem 2 [6] Let H, K be finite dimensional complex Hilbert spaces and ρ be a state acting on $H \otimes K$. Then ρ is separable if and only if for any positive linear map $F : B(H) \to B(K)$, the operator $(F \otimes I)\rho$ is positive on $K \otimes K$. Recently [7], improved the above result and established the elementary operator criterion for infinite dimensional bipartite systems which is crucial in characterization of norms of elementary operators. Therefore, a state ρ on $H \otimes K$ is entangled if and only if there exists a positive finite rank elementary operator $F : B(H) \to B(K)$ which is not completely positive (briefly, NCP) such that $(F \otimes I)\rho$ is not positive [8]. Thus it is very important and interesting to find as many as possible positive finite rank elementary operators that are NCP, and then, to apply them to detect the entanglement of states [9]. The purpose of this paper is to give some new norm inequalities of elementary operators particularly the Jordan case and apply them to get some new examples of entangled states that cannot be detected by the PPT criterion and the realignment criterion. Recall that, a positive map Δ is said to be decomposable if it is the sum of a completely positive map Δ and the composition of a completely positive map Δ [10,11]. This paper is organized as follows: Section 1 has the introduction, Section 2 the basic concepts, Section 3 we have the main results and discussions. In section 4 we have outlined the methodology and finally section 5 has the conclusion. In the next section some concepts which are useful in the sequel are given.

Methodology

In the present study methodology involved the use of some known definitions and fundamental results as stated below:

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Definition 2.1. A C^*-algebra is a Banach algebra over the field of complex numbers together with a map $*: A \to A$ having the following properties [12]:
(1). it is an involution, for every x ϵ A; $x^{**} = (x^*)^* = x$
(2). for all x, y ϵ A; $(x + y)^* = x + y$, $(xy) = y^*x^*$
(3). for every λ ϵ C and every x ϵ A; $(\lambda x) = \bar{\lambda} x^*$
(4). for all x ϵ A $\|x^*x\| = \|x\| \|x^*\|$.
Definition 2.2 **[**5, Definition 2.1.8**]**. A Banach algebra is a Banach space *B(H)* which is at the same time an algebra in which for all A,B ϵ *B(H)* one has $\|AB\| \leq \|A\| \|B\|$. Banach *-algebra is a Banach algebra with an involution.
Definition 2.3 **[**4, Definition 2.1**]**. A norm on a vector space *V* is a nonnegative real valued function $\|.\|$: $V \to R$ satisfying the following properties:
i). $\|v\| \geq 0 \ \forall v \in V$.
ii). $\|v\| = 0$ iff $v = 0$.
iii). $\|\lambda v\| = |\lambda| \|v\| \ \forall \lambda \in \mathbf{C}$ and $v \in V$.
iv). $\|v+w\| \leq \|v\| + \|w\|$.
Definition 2.4 **[**5, Definition 2.1.3**]**. A Hilbert space is a complete inner product space.
Definition 2.5 **[**9, Section 2**]**. Let *H* be an infinite dimensional complex Hilbert space and *B*(*H*) the *C**-algebra of all bounded linear operators on *H*. $W_{P,Q} : B(H) \to B(H)$ is called an elementary operator if it is expressible in the form: $W(T) = \sum_{i=1}^{n} P_i\ T\ Q_i : \forall T \in B(H)$ and P_i, Q_i fixed in *B*(*H*) and we define the particular elementary operators:
i). The left multiplication operator, $L_P : B(H) \to B(H)$ defined by: $L_P(T) = PT \ \forall T \in B(H)$
ii). The right multiplication operator, $R_Q : B(H) \to B(H)$ defined by: $R_Q(T) = TQ \ \forall T \in B(H)$
iii). The generalized derivation (implemented by P,Q)defined by: $\delta_{P,Q} = PT - TQ$
iv). The basic elementary operator (implemented by P,Q)defined by $M_{P.Q}(T) = PTQ; \ \forall T \in B(H)$
v). The Jordan elementary operator (implemented by P,Q) defined by
$U_{P,Q}(T) = PTQ + QTP; \ \forall T \in B(H)$…..1

For a successful completion of the present paper, background knowledge of Functional analysis, the operator theory, especially normal operators, self adjoint operators, hyponormal operators on a Hilbert space, numerical range and the spectrum of operators on a Hilbert space is vital. We have stated some known fundamental principles which shall be useful in our research. The methodology involved the use of known inequalities and techniques like Cauchy -Schwartz inequality, Minkowski's inequality, parallelogram law and the polarization identity. Lastly, we used the technical approach of tensor products in solving the stated problem. Also the methodology involved the use of known inequalities and techniques like the polarization identity.

Result and discussions

The norm of the Jordan elementary operator *U*(*T*) acting on *B*(*H*) is determined in the present section.
Proposition 3.1. Let H be an infinite dimensional complex Hilbert space and B(H) the C*-algebra of all bounded linear operators on H. For a Jordan elementary operator $U : B(H) \to B(H)$ defined by: $U(T) = PTQ + QTP$ for all T ϵ B(H) and Pi;Qi fixed in B(H),
Then: $\|U(T)\| \leq 2\|P\| \|Q\|$……2
Proof. From definition of supremum norm;
$\|U(T)\| = \sup\{PTQ + QTP : \|T\| = 1\}$
$= \sup\{\|PTQ\| + \|QTP\| : \|T\| = 1\}$
$\leq \{\|P\| \|T\| \|Q\| + \|Q\| \|T\| \|P\| : \|T\| = 1\}$
$= \|P\| \|Q\| + \|Q\| \|P\|$
$= 2\|P\| \|Q\|$
Lemma 3.2. Given linearly independent P;Q ϵ B(H), we can find $R_1;R_2 \in B(H)$, $\delta_1, \delta_2 > 0$ and $S \in C\backslash\{0\}$ so that $P\otimes Q + Q\otimes P = R_1\otimes R_1 + R_2\otimes R_2$, $R_1 = (SP+S^{-1}Q)/\sqrt{2}$, $R2 = i(SP-S^{-1}Q)/\sqrt{2}$
and $\|P\otimes Q + Q\otimes P\|_H = \|\delta_1 R_1 R_1^* + \delta_2 R_2 R_2^*\| = \|\delta_1^{-1} R^*_1 R_1 + \delta_2^{-1} R^*_2 R_2\|$…………………3
Proof. Since the Haagerup norm infimum for $W = P\otimes Q + Q\otimes P$ is realized via a representation $W = P_1\otimes Q_1 + P_2\otimes Q_2$ and moreover by scaling P_i to λP_i and Q_i to $\lambda^{-1} Qi$ for a suitable λ.
We arrange that $\|W\|_H = \|P_1 P^*_1 + P_2 P^*_2\| = \|Q_1 Q^*_1 + Q_2 Q^*_2\|$……………………...…4
we adopt a convenient matrix notation
$W = [P,Q] \odot [Q, P]^t = [P_1, P_2] \odot [Q_1,Q_2]^t$ for the two tensor product expression above and note that all possible representations of *W* take the form
$W = [P'_1, P'_2] \odot [Q'_1,Q'_2]t = ([P_1, P_2]\,\alpha) \odot (\alpha^{-1}[Q_1,Q_2]^t)$ ……5
For a 2 × 2 invertible scalar matrix α, we also use the transpose notation for the linear operation on the tensor product that sends $P_1\otimes Q_1$ to $P_1\otimes Q_1$. Then we have

$W = W^t = [Q_1,Q_2] \odot [P_1, P_2]^t = ([P_1, P_2]\,\alpha) \odot ([Q_1,Q_2](\alpha^{-1})^t)^t$ ………6

from $[Q_1,Q_2] = [P_1, P_2]\,\alpha$ and $[P_1, P_2]\,\alpha^t = [Q_1,Q_2]$ together with linear independence we get $\alpha = \alpha^t$ symmetric. We can now express $\alpha = U \Delta U^t$ where U is a unitary matrix and Δ is a diagonal matrix with positive diagonal entries $\delta_1^{-1}, \delta_2^{-1}$.

Take $[P'_1, P'_2]U;\ [Q'_1,Q'_2] = [Q_1,Q_2](U^{-1})^t$ so that $W = [P'_1, P'_2] \odot [Q'_1,Q'_2]^t$

$\|W\|_H = \|(P'_1)(P'_1)^*+(P'_2)(P'_2)^*\| = \|(Q'_1)^*(Q'_1) + (Q'_2)^*(Q'_2)\|$……….7

And

$[P'_1; P'_2] = [P1;P2]U\Delta = [P_1; P_2]\,\alpha\,(U^{-1})^t = [Q'_1;Q'_2]$. In other words, $P'_i\,\delta_i^{-1} = Q'_i$ $(i = 1; 2)$.

Now take $R_i = \sqrt{\delta_i}\,Q'_i$ and we then have $W = R_1,R_1 \odot R_2,R_2]$ together with

$\|W\|_H = \|\delta_1 R_1 R^*_1 + \delta_2 R_2 R^*_2\| = \|\delta_1^{-1} R_1 R^*_1 + \delta_2^{-1} R_2 R^*_2\|$…………8

It now remains to relate R_1,R_2 to P,Q as claimed. Thus if we put $P' = (R_1 - iR_2)/\sqrt{2}$ and $Q' = (R_1 + iR_2)/\sqrt{2}$ we have

$W = P' \otimes Q' + Q' \otimes P' = [P',Q'] \odot [Q', P']^t = [P,Q] \odot [Q, P]^t$…………9

Thus there exists a $S \in$ C with $P' = SP$ and $Q' = S^{-1}Q$.

Theorem 3.3. $\|U(T)\| \geq 2\|P\|\|Q\|$

Proof. Suppose P,Q are linearly dependent i.e $P = \lambda Q$ such that $U(T) = 2\lambda P \times P$, then $\|U\| = 2\|P\|\|Q\|$. So we deal only with the case of independent P,Q.

By applying Lemma 3.2, $\|U\| = \|P \otimes Q + Q \otimes P\|_H$ and using the fact that the norm of a 2 ×2 positive matrix(the maximum of the eigenvalues) is at least half of the trace, we obtain

$\|U\| \geq \frac{1}{2}(\delta_1\|R_1\|^2 + \delta_2\|R_2\|^2)$, which implies

$\|U\| \geq \frac{1}{2}(\delta_1^{-1}\|R_1\|^2 + \delta_2^{-1}\|R_2\|^2)$

We deduce that,

$$\begin{aligned}\|U\| &= \frac{1}{4}((\delta_1+\delta_1^{-1})\|R_1\|^2 + (\delta_2+\delta_2^{-1})\|R_2\|^2)\\ &\geq \frac{1}{2}(\|R_1\|^2+\|R_2\|^2)\\ &= \frac{1}{2}trace(R^*_1R_1+R^*_2R_2)\\ &= \frac{1}{2}trace((SP)^*(SP)+(S^{-1}Q)^*(S^{-1}Q))\\ &= \frac{1}{2}(\|SP\|^2 + \|S^{-1}Q\|^2)\\ &= \|SP\|\|S^{-1}Q\| = \|P\|\|Q\|\end{aligned}$$

Corollary 3.4. $\|U(T)\| = 2\|P\|\|Q\|$

Proof. The proof follows as a direct consequence of Proposition 3.1 and Theorem 3.3.

Theorem 3.5. Let U : B(H) → B(H) be a Jordan elementary operator defined by $UT = \Sigma^l_{i=1} P_i\,T\,Q_i$: $\forall$ T $\in$ B(H) and P_i,Q_i fixed in B(H) where P_i and Q_i are diagonal operators induced by $\{\alpha_{ni}\}$and $\{\beta_{ni}\}$respectively and H an infinite dimensional complex Hilbert space then U is bounded and $\|U\| = (\Sigma n\{\Sigma^l i_{=1}|\alpha_{ni}|^2|\beta_{ni}|^2\})^{1/2}$

Proof. We compute the mapping U using R_n as the elements of $B(H)$, we have that $UR_n = \Sigma^l_{i=1}P_i R_n Q_i$.Now suppose $l = 2$, we obtain $UR_n = P_1R_nQ_1 + P_2R_nQ_2$ This shows that U is a diagonal operator with $UR_n = \Sigma_n\Sigma^l_{i=1}\alpha_{ni} R_n\beta_{ni}$. Thus in general, we have that $UR_n = \Sigma^l_{i=1}\alpha_{ni}R_n\beta_{ni}$ which implies that U is a diagonal operator with diagonals $\{\Sigma^l_{i=1}\alpha_{ni}\,\beta_{ni}\}$ By assuming that $l = 2$ we show that this operator is bounded.

$$\begin{aligned}\|URn\|^2 &= \|\Sigma_n \Sigma^l_{i=1}\alpha_{ni}R_n\beta_{ni}\|^2\\ &\leq \Sigma_n \Sigma^l_{i=1}|\alpha_{ni}\,\beta_{ni}|^2\|R_n\|^2\\ &= \Sigma_n \Sigma^l_{i=1}|\alpha_{ni}|^2|\beta_{ni}|^2\end{aligned}$$

But since $\{\alpha_{ni}\}$ and $\{\beta_{ni}\}$ are bounded for all n, it implies that their infinite summation is bounded and hence the operator U is bounded since taking the suprimum of both sides of the above equation we have $\|U\| = (\Sigma_n\{\Sigma^l i_{=1}|\alpha_{ni}|^2|\beta_{ni}|^2\})^{1/2}$ For the norm of U, we consider the following calculation

$$\begin{aligned}\Sigma_n\Sigma^l i_{=1}|\alpha_{ni}|^2|\beta_{ni}|^2 &= |\Sigma_n \Sigma^l_{i=1}\alpha_{ni}\,\beta_{ni}|^2\|R_n\|^2\\ &= \|\Sigma_n\Sigma^l i_{=1}\alpha_{ni} R_n \beta_{ni} R_n\|^2\\ &= \|UR_n\|^2\\ &\leq \|U\|^2\|R_n\|^2\end{aligned}$$

Taking the suprimum of both sides and since $\|R_n\| = 1$, we obtain $\Sigma_n|\Sigma^l i_{=1}\alpha_{ni}|^2|\beta_{ni}|^2 \leq \|U\|$

Generalizing this to an arbitrary $T \in B(H)$, we observe that for $T = \Sigma_n T_nR_n$, then; $UT = \Sigma_n T_n UR_n = \Sigma_n \Sigma^l_{i=1}T_n\alpha_{ni} R_n\beta_{ni}$ so that

$$\begin{aligned}\|UT\|^2 &= \Sigma_n|\Sigma^l_{i=1}T_n\alpha_{ni} R_n \beta_{ni}|^2\\ &= \Sigma_n|T_n|^2|\Sigma^l_{i=1}\alpha_{ni} R_n \beta_{ni}|^2\\ &\leq \Sigma_n\{\Sigma^l_{i=1}|\alpha_{ni}|^2|\beta_{ni}|^2\Sigma_n|T_n|^2\}\end{aligned}$$

Which implies that $\|UT\|^2 \leq \Sigma_n\Sigma^l_{i=1}|\alpha_{ni}|^2|\beta_{ni}|^2\|T\|^2$ Taking the suprimum over n on both sides with $\|T\| = 1$, we obtain [13] $\|U\|^2 \leq \Sigma_n\{\Sigma^l_{i=1}|\alpha_{ni}|^2|\beta_{ni}|^2\}$. And with our earlier result we conclude that $\|U\| = (\Sigma_n\{\Sigma^l_{i=1}|\alpha_{ni}|^2|\beta_{ni}|^2\})^{1/2}$

Example 3.6. Consider the operator $U : V_2(\mathrm{C}) \to V_2(\mathrm{C})$ defined by $U(X) = P_1TQ_1 + P_2TQ_2$

Taking $T=\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, $P_1=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $Q_2=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $P_2=Q_1=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

$$UT=\begin{bmatrix} A & B \\ -C & -D \end{bmatrix}+\begin{bmatrix} -A & B \\ -C & D \end{bmatrix}=\begin{bmatrix} 0 & 2B \\ -2C & 0 \end{bmatrix},$$

it suffices to show that $\|U\| = 2$, so let $(\alpha_{11}, \alpha_{12}) = (1,-1)$, $(\alpha_{12}, \alpha_{22}) = (1; 1)$, $(\beta_{11}, \beta_{21}) = (1; 1)$; $(\beta_{12}, \beta_{22}) =(-1; 1)$ Therefore, $\Sigma^2_{i=1}|\alpha_{1i}|^2|\beta_{1i}|^2 = 2$ and $\Sigma 2i=1|\alpha_{2i}|^2|\beta_{2i}|^2 = 2$ implying that $\|U\|= ((\Sigma_2\{\Sigma^2_{i=1}|\alpha_{ni}|^2|\beta_{ni}|^2\})^{1/2}=2$.

Conclusions

Although the present paper have considered a 2-dimensional space, the result above for norms of Jordan elementary operator acting on C*-algebra is true for an infinite dimensional Hilbert space by induction method [14]. Hilbert space operators have been studied by many mathematicians including Hilbert, Weyl, Neumann, Toeplitz, Hausdorff, Rhaly, Mecheri, Shapiro among others [15,16]. These operators are of great importance since they are vital in formulation of principles of mathematical analysis and quantum mechanics. The operators include normal operators, posinormal operators, hyponormal operators, normaloid operators among others. It is interesting to study localization of the spectrum, description and structure of essential spectrum, block diagnalization, invariant subspaces, and the structure and utility of quadratic numerical ranges but not the numerical range of elementary operators [17].

Conflict of interest

Authors declare there are no conflicts of interest.

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