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Density and Dentability in Norm-Attainable Classes

Joseph Owuor[a], Benard Okelo[a], Julia Owino[a], Jeremiah Kinyanjui[b]

[a] *Jaramogi Oginga Odinga University of Science and Technology*

[b] *Kirinyaga University*

Abstract

We establish the norm-denseness of the norm-attainable class $NAB(H)$ in the Banach algebra $B(H)$, which consists of all bounded linear operators on a complex Hilbert space H. Specifically, for every $O \in NAB(H)$ and each $\epsilon > 0$, there exists $O' \in B(H)$ such that $\|O - O'\| < \epsilon$. We achieve this characterization by utilizing the convergence of sequences and the existence of limit points. The properties A and B of Lindenstrauss are sufficient to ensure the density of $NAB(H)$. Moreover, countable unions, finite intersections, countable tensor products, and countable Cartesian products preserve density in the associated classes $NAB(H)$. Additionally, density in $NAB(H)$ exhibits transitivity. We also investigate the concept of dentability in norm-attainable classes defined on the Banach algebra of all bounded linear operators on a complex Hilbert space H. Dentability of a norm-attainable class refers to the existence of a bounded linear norm-attainable operator (within the class) that lies outside the closed convex hull of the subclass obtained by excluding a ball of sufficiently small radius containing the particular bounded linear norm-attainable operator. We provide conditions for dentability and s-dentability of subclasses, closures, closed convex hulls, and superclasses of given norm-attainable classes. Furthermore, we demonstrate that countable unions, Cartesian products, and finite intersections preserve dentability. Moreover, we prove that arbitrary unions, finite intersections, and arbitrary Cartesian products maintain the dentability of classes. Our work significantly contributes to the characterization and understanding of dentability in norm-attainable classes. The findings of our study advance knowledge and have practical applications in the fields of operator analysis, operator theory, and optimization with respect to dentability. These results enhance the understanding and further characterization of bounded linear operators. Moreover, the findings are valuable in studying the linearbility and spaceability of norm-attainable classes and Banach spaces.

Keywords:
norm-attainable class, density, metric, topology, limit point, dentability, s-dentability, Optimization, Banach algebra, Hilbert spaces
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1. Introduction

Density in metric spaces is a property that has been studied for a very long time. Various notions of density exist, including the use of limit points and convergence of sequences. Bishop and Phelps [1] described the density of the set of norm-attainable functionals in the dual space of a Banach space B, and their proof employed the use of Zorn's lemma [2]. Consequently, it has been asked if their result can be generalized (a problem that is still open to date). Topologically, in a metric space (X, d) (the topology is induced by the distance function d), a subset A

of X is dense in X if every member x of X is either in A or a limit point of A. Simply put, the set A is spread throughout the set X. A classical example is the set of rationals in the reals. In fact, the density of rationals in the reals allows for a "good" approximation of the reals by the rationals for practical uses. Alternatively, we can define density using the convergence of sequences. This way, a set A is dense in the set X if for every member x of X, there exists a sequence (x_n) in A such that $x_n \to x$ in the sense of the metric inducing the topology. Let H be a complex Hilbert space, and let $O : H \longrightarrow H$ be a bounded linear operator on H. Furthermore, let $B(H)$ be the Banach algebra of all bounded linear operators on H. This Banach algebra is actually a C*-algebra. We will denote by $NAB(H)$ the class of operators $T : B(H) \longrightarrow B(H)$ for which there exists a unit operator $u \in S_{B(H)}$ such that $\|Tu\| = \|T\|$. Dentability of sets was introduced by Rieffel [3], who showed its connection to the Radon-Nikodym Theorem. Maynard [4] extended the concept by introducing s-dentability. Furthermore, [5] provided an affirmative answer to the question of dentability in Banach spaces with the Radon-Nikodym Property. Sufficient conditions for the convergence of minimizing sequences were given in [6, 7], with additional results available in [8]. Bourgain [9] showed the interrelation between dentability and strong exposition when he demonstrated that if a bounded separable closed convex subset $B_{1_{sbcc}}$ of B_1 satisfies the BPP, then it is dentable. A characterization of Banach spaces satisfying the Radon-Nikodym property (RNP) has been established using a convexity property on all bounded subsets of the space [3]. The Radon-Nikodym Theorem for the Bochner integral and the Philips-Metivier Radon-Nikodym theorem are logically equivalent to the Radon-Nikodym Theorem [3]. Rieffel proved the Radon-Nikodým theorem using strategies established in his previous work [3, 10]. The construction of Metivier and Philips variants of the theorem relied on the idea that compact convex sets can be dented [11]. The crucial geometric condition for Banach spaces with the Radon-Nikodým property (RNP) is the existence of bounded dentable subsets [4]. The concept of s-dentability, weaker than dentability, also characterizes RNP Banach spaces [4]. The logical equivalence between meeting RNP conditions and s-dentability has been established [4]. Banach spaces with closed separable subspaces linearly homeomorphic to dual separable subspaces satisfy the RNP [4]. Regarding dentability, researchers such as Rieffel [3], Maynard [12], Huff [13], Davis [5], and Phelps [14] have studied the connection between dentability and the Radon-Nikodým Property (RNP). They have shown that the condition for a Banach space to have the RNP is equivalent to every bounded subset being dentable. A claim of the RNP implying PB has been shown to be false, as demonstrated by counterexamples provided by Huff [15] and Bourgain [9]. Despite all these studies bordering dentabiliy, a characterizaition of dentability in norm-attainable classes has been missing in literature. This forms main motivation for our work. The goal of this paper is to characterize density and dentability in the class $NAB(H)$. The following basic concepts will be useful in our presentation.

2. Basic Concepts

We begin by introducing the basic concepts that will be used throughout the paper.

Definition 1. *[16, Definition 2.3] Let H be a complex Hilbert space, and let $B(H)$ be the Banach algebra of all bounded linear operators on H. Then, a collection C of bounded linear operators is dense in the collection of all bounded linear operators on $B(H)$ if, for any bounded linear operator $O \in C$ and for every $\epsilon > 0$, there exists $O' \in B(H)$ such that $\|O - O'\| < \epsilon$.*

Definition 2. *[17, Definition 2.1] Let H be a complex Hilbert space, and let $B(H)$ be the Banach algebra of all bounded linear operators on H. Then, a bounded linear operator $O : B(H) \longrightarrow B(H)$ is said to be norm-attainable if there exists a unit operator $u \in S_{B(H)}$ such that $\|Ou\| = \|O\|$. A collection of such operators on $B(H)$ forms the norm-attainable class $NAB(H)$.*

Definition 3. *[17, Definition 5] Let H be a complex Hilbert space and let $B(H)$ be the Banach algebra of all bounded linear operators on H. Then an operator $O : B(H) \longrightarrow B(H)$ is norm-attainable if there exists $u \in S_{B(H)}$ such that $\|O(u)\| = \|O\|$, where S_A is the unit sphere of A. A collection of such operators constitute a norm-attainable class $NAB(H)$.*

Definition 4. *[5, Definition 1] Let H be a complex Hilbert space and let $NAB(H)$ be a norm-attainable class of operators in $B(H)$. Then $N = NAB(H)$ has the Radon-Nikodým Property (RNP) if, for each totally finite positive measure space (N, Σ_N, m_1) and each N-valued m_1- absolutely continuous measure m_2 on Σ_N of bounded variation with*

$|m_2|(N) < \infty$, *then there exists an* $\mathcal{L}^1(N, \Sigma_N, m_2)$ *integrable function O such that* $m_2(N_i) = \int_{N_i} O dm_1 \ \forall N_i \in \Sigma_N$. *We will refer to N as an RNP class if this definition holds for N. A norm-attainable class N is dentable (resp. s-dentable) if for every* $\epsilon > 0$, *then there exists an operator* $O \in N$ *such that* $O \notin cch[N \setminus B_\epsilon(O)]$ *(resp.* $O \notin s[N \setminus B_\epsilon(O)]$*), where* $cch[X]$ *is the closed convex hull of X and* $s[X]$ *the convex combination of X given by* $s(X) = Big\{\sum_{i=1}^{\infty} s_i v_i | s_i \geq 0, \sum_{i=1}^{\infty} s_i = 1, \{v_i\} \subset XBig\}$. *In this case, O is the dentable (s-dentable) point of the class N and any RNP class and those with separable duals are dentable classes. If an operator in dentable and norm-attainable, then we will refer to it as a dentable norm-attainable operator.*

Definition 5. *[8, Definition 2] Let H be a Hilbert space and* $H_c \subseteq H$ *be a convex subspace. A functional* $\phi : H_c \to \mathbb{R}$ *is dentable at v if* $(v, \phi(v))$ *is a dentable point of* $epi(\phi) = \{(v, \alpha) \in H \times \mathbb{R} : \alpha \geq \phi(v)\}$.

The following theorem will be useful in the work.

Theorem 1. *[18, Krein-Milman Theorem] Let V be a topological vector space, and let* $V_{cc} \subseteq V$ *be a non-empty compact convex set. Then, the convex hull of the set of extreme points of* V_{cc} *is dense in* V_{cc}.

3. Density in Norm-Attainable Classes

In this section, we study various aspects of density in the norm-attainable class $NAB(H)$.

Theorem 2. *Let* $NAB(H) \subseteq B(H)$. *Then* $NAB(H)$ *is norm-dense in* $B(H)$.

Proof. We need to choose $O \in NAB(H)$ and show that such $O \in B(H)$. By definition, there exists $u \in S_H$ such that $\|O(u)\| = \|O\|$. By norm-attainability, there exist $M > 0$ such that $\|O(u)\| = M$. Then $\|O(u)\| \leq M\|u\|$ and it follows that O is bounded. Since this choice is arbitrary, it follows that $NAB(H)$ is norm-dense in $B(H)$. □

Proposition 3. *Let* $NAB(H)_0 \subseteq NAB(H)$. *Then,* $NAB(H)_0$ *is norm-dense in* $NAB(H)$ *if for* $T \in NAB(H)$, *there exists a sequence* (T_n) *in* $NAB(H)_0$ *such that* $T_n \to T$.

Proof. We need to show that for every $\epsilon > 0$, $\|T_n - T\| < \epsilon$. Let (T_n) be a sequence in $NAB(H)_0$. Then, each member of the sequence is norm-attainable, meaning that there exists a unit operator $u \in B(H)$ such that $\|T_n u\| = \|T_n\|$ for all n. In particular, $\lim_{n\to\infty} \|T_n u - T\| = 0$. Hence, $|\|T_n\|\|u\| - \|T\|| \leq \|T_n u - T\|$, which implies that $\|T_n\| - \|T\| = 0$. Finally, for every $\epsilon > 0$, $\|T_n - T\| < \epsilon$. □

Proposition 4. *Let* $NAB(H)$ *be a norm-attainable class in* $B(H)$. *If* $NAB(H)$ *has property A or property B, then* $NA(NAB(H))$ *is norm-dense in* $B(B(H))$.

Proof. Let $NAB(H)$ have Lindenstrauss' property A or property B. Then, for any $O : NAB(H) \longrightarrow NAB(H)$ such that O is norm-attainable, there exists a norm-attainable operator $O' : NAB(H) \longrightarrow NAB(H)$ such that for all $\epsilon > 0$, $\|O - O'\| < \epsilon$. This suffices for the density of $NA(NAB(H))$ in $B(NAB(H))$. □

Theorem 5. *Let* $(NAB(H)_j)_{j\geq1}$ *be a disjoint sequence of non-empty norm-attainable classes in* $B(H)$ *such that each* $NAB(H)_j$ *is norm-dense in* $B(H)$ *for all* $j \geq 1$. *Then, the countable union* $\bigcup_{j=1}^{\infty} NAB(H)_j$ *is dense in* $B(H)$.

Proof. Let $NAB(H)_j$ be dense in $B(H)$ for all $j \geq 1$. Then, for each operator $O_j \in B(H)$ and for every $\epsilon > 0$, there exists an $O'_j \in NAB(H)_j$ such that for all j, $\|O_j - O'_j\| < \epsilon$. Since $O'_j \in NAB(H)_j$, it follows that $O'_j \in \bigcup_{j=1}^{\infty} NAB(H)_j$. Since the choices of O_j and O'_j were arbitrary, it means that for every $O_j \in B(H)$, there exists an $O'_j \in NAB(H)_j$ such that $\|O_j - O'_j\| < \epsilon$. Hence, $\bigcup_{j=1}^{\infty} NAB(H)_j$ is dense in $B(H)$ as claimed. □

Theorem 6. *Let $(NAB(H)_j)_{j\geq 1}^{M}$ be a finite sequence of non-empty norm-attainable classes in $B(H)$ such that each $NAB(H)_j$ is norm-dense in $B(H)$ for all j. Then, the finite intersection $\bigcap_{i=1}^{M} NAB(H)_j$ is dense in $B(H)$.*

Proof. Let $NAB(H)_j$ be dense in $B(H)$ for $1 \leq j \leq M$. Then, for all j and $\epsilon > 0$, there exists $O'_j \in NAB(H)_j$ such that $\|O'_j - O_j\| < \epsilon$. By the definition of intersection, it follows that $O'_j \in \bigcap_{i=1}^{M} NAB(H)_j$. Since $O_j \in B(H)$, it follows that $\bigcap_{i=1}^{M} NAB(H)_j$ is dense in $B(H)$. □

Theorem 7. *Let $(NAB(H)_j)_{j\geq 1}$ be a countable disjoint sequence of non-empty norm-attainable classes in $B(H)$ such that each $NAB(H)_j$ is norm-dense in $B(H)$ for all $j \geq 1$. Then, the countable Cartesian product $\prod_{j=1}^{\infty} NAB(H)_j$ is dense in $\prod_{j=1}^{\infty} B(H)$.*

Proof. Let $NAB(H)_j$ be norm-dense in $B(H)$ for all $j \geq 1$. Now, for $j = 1$ and for all $\epsilon > 0$, if $O_1 \in B(H)$, then either $O_1 \in NAB(H)_1$ or O_1 is a limit point for $NAB(H)_1$. More precisely, there exists $O'_1 \in NAB(H)_1$ such that $\|O_1 - O'_1\| < \frac{\epsilon}{2^{1+1}}$.

Similarly, for $j = 2$, we can find $O_2 \in B(H)$ and $O'_2 \in NAB(H)_2$ such that $\|O_2 - O'_2\| < \frac{\epsilon}{2^{2+1}}$. Continuing in this manner, we obtain two summable operator sequences $(O_j)_{j\geq 1} \in B(H) \times B(H) \times \ldots$ and $(O'_j)_{j\geq 1} \in NAB(H)_1 \times NAB(H)_2 \times \ldots$ such that $\|O_j - O'_j\| < \frac{\epsilon}{2^{j+1}}$.

We then construct two operator vectors $O = (O_1, O_2, O_3, \ldots, O_j, \ldots) \in \prod_{j=1}^{\infty} B(H)$ and $O' = (O'_1, O'_2, O'_3, \ldots, O'_j, \ldots) \in \prod_{j=1}^{\infty} NAB(H)_j$ such that

$$\|O - O'\| = \sum_{j=1}^{\infty} \frac{\varepsilon}{2^{j+1}} = \frac{\varepsilon}{2} < \varepsilon. \tag{1}$$

□

Theorem 8. *Let $(NAB(H)_j)_{j\geq 1}$ be a countable disjoint sequence of non-empty norm-attainable classes in $B(H)$ such that each $NAB(H)_j$ is norm-dense in $B(H)$ for all $j \geq 1$. Then, the countable tensor product $\bigotimes_{j=1}^{\infty} NAB(H)_j$ is dense in $\bigotimes_{j=1}^{\infty} B(H)$.*

Proof. Let $O = O_1 \otimes O_2 \otimes O_3 \otimes \ldots \in \bigotimes_{j=1}^{\infty} B(H)$. We know that for all $j \geq 1$, then $NAB(H)_j$ is dense in $B(H)$. Then for $j = 1$, there exists $O'_1 \in NAB(H)_1$ such that for all $0 < \varepsilon < \frac{1}{10}$, then $\|O_1 - O'_1\| < \varepsilon$. For $j > 1$, we can find $O_j \in B(H)$ and $O'_j \in NAB(H)_j$ such that $\|O_j - O'_j\| < 1 + \varepsilon^k$. Continuing this way, we can construct an operator in $\bigotimes_{j=1}^{\infty} NAB(H)_j$ given by $O' = O'_1 \otimes O'_2 \otimes O'_3 \otimes \ldots$.

Now

$$\|O - O'\| \leq = \|O_1 \otimes O_2 \otimes O_3 \otimes \ldots - O'_1 \otimes O'_2 \otimes O'_3 \otimes \ldots\| \tag{2}$$
$$\leq \|(O_1 - O'_1) \otimes (O_2 - O'_2) \otimes (O_3 - O'_3) \otimes \ldots\| \tag{3}$$
$$\leq \|O_1 - O'_1\|\|\|O_2 - O'_2\|\|\|O_3 - O'_3\|\|\| \ldots \tag{4}$$
$$\leq (1+\varepsilon)(1+\varepsilon^2)((1+\varepsilon^3)\ldots(1+\varepsilon^j)\ldots \tag{5}$$
$$< 1 + 2\varepsilon + 3\varepsilon^2 + 4\varepsilon^3 + 5\varepsilon^4 + \ldots = \frac{1}{(1-\varepsilon)^2} > 0. \tag{6}$$

Thus for any $\varepsilon' = \frac{1}{(1-\varepsilon)^2}$, we have that for any $O \in \bigotimes_{j=1}^{\infty} B(H)$, there exists $O' \in \bigotimes_{j=1}^{\infty} NAB(H)_j$ such that $\|O - O'\| < \varepsilon'$. □

Theorem 9. *Let $(NAB(H)_j)_{j\geq 1}$ be a countable disjoint sequence of non-empty norm-attainable classes of densely defined operators in $B(H)$. Then the countable union $\bigcup_{j=1}^{\infty} NAB(H)_j$ is dense in $B(H)$.*

Proof. Fix an index $k \in \{1, 2, 3, \ldots, j, \ldots\}$. Then by hypothesis, for any $O_k \in B(H)$ and for every positive real number $\varepsilon > 0$, there exists $O'_k \in NAB(H)_k$ such that $\|O_k - O'_k\| < \varepsilon$. By definition of a union, such an $O'_k \in \bigcup_{j=1}^{\infty} NAB(H)_j$. Since O_k was arbitrarily chosen, it follows that for any operator $O \in B(H)$, there exists an operator in $O' \in \bigcup_{j=1}^{\infty} NAB(H)_j$(at least in one of the indices) such that for every $\varepsilon > 0$, then $\|O - O'\| < \varepsilon$. Thus $\bigcup_{j=1}^{\infty} NAB(H)_j$ is dense in $B(H)$ as claimed. □

Theorem 10. *Let $(NAB(H)_j)$, $j = 1, 2, 3$ be a sequence of norm-attainable classes in $B(H)$ such that $NAB(H)_j$ is dense in $NAB(H)_{j+1}$ for $j = 1, 2$. Then $NAB(H)_j$ is dense in $NAB(H)_{j+2}$ for $j = 1$.*

Proof. Let $NAB(H)_1$ be norm-dense in $NAB(H)_2$. Then for each $O_2 \in NAB(H)_2$ and for all $\varepsilon > 0$, there exists $O_1 \in NAB(H)_1$ such that $\|O_1 - O_2\| < \frac{\varepsilon}{2}$. Since $NAB(H)_2$ is dense in $NAB(H)_3$, it follows that for all $O_3 \in NAB(H)_3$, we have that for $O_2 \in NAB(H)_2$, then $\|O_2 - O_3\| < \frac{\varepsilon}{2}$. Now using the triangle inequality, we have that

$$\|O_1 - O_3\| = \|O_1 - O_2 + O_2 - O_3\| \leq \|O_1 - O_2\| + \|O_2 - O_3\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{7}$$

Thus for any $O_3 \in NAB(H)_3$, there exists $O_1 \in NAB(H)_1$ such that for all $\varepsilon > 0$, then $\|O_1 - O_3\| < \varepsilon$. This means that $NAB(H)_1$ is norm-dense in $NAB(H)_3$. □

Theorem 11. *If $NAB(H)$ has the RNP then $NAB(H)$ is dense in $B(H)$.*

Proof. If $NAB(H)$ has the *RNP*, then it has property *A*. Then by Proposition 4 it follows that $NAB(H)$ is dense in $B(H)$. □

Theorem 12. *If $NAB(H)$ has the Bishop Phelps Property, then $NAB(H)$ is dense in $B(H)$.*

Proof. Let $NAB(H)$ be closed convex and bounded. If $NAB(H)$ has the Bishop Phelps Property, then the class of operators $O : NAB(H) \longrightarrow NAB(H)$ such that the image $\|Ou\|$ achieves its maximum in $NAB(H)$ for some $u \in NAB(H)$ is dense in $B(NAB(H))$. This means that $NAB(H)$ is dense in $B(H)$. □

4. Dentability in Norm-Attainable Classes

In this section, we characterize dentability within a class of norm-attainable operators. The following proposition shows that dentability of a norm-attainable class is sufficient for its s-dentability.

Proposition 13. *Let H be a complex Hilbert space and let $N = NAB(H)$ be a norm-attainable class on $B(H)$. If $\emptyset \neq N_0 \subset N$ is dentable, then it is s-dentable.*

Proof. Let N_0 be dentable but not s-dentable and let $O \in N_0$ such that $O \notin cch[N_0 \setminus B_\varepsilon(O)]$. Since N_0 is not s-dentable (by hypothesis), it follows that and $O \in s[N_0 \setminus B_\varepsilon(O)]$. Then $O \in cch[B_\varepsilon(O)]$ and $O \notin s(B_\varepsilon(O))$. This is a contradiction meaning that N_0 is s-dentable. □

The converse of Proposition 13 may not hold as there exist classes which are s-dentable but not dentable. We have the following theorem which proves that any dentable mapping is s-dentable.

Theorem 14. *Let H be a complex Hilbert space and let $NAB(H)$ be a norm-attainable cass on $B(H)$. Then if $O : NAB(H) \longrightarrow NAB(H)$ is dentable, it is s-dentable.*

Proof. If $O : NAB(H) \longrightarrow NAB(H)$ is dentable, if follows that for every $\varepsilon > 0$, there exists an operator $u \in NAB(H)$ such that $(u, O(u)) \notin cch[\text{epi}(O) \setminus B_\varepsilon\{(u, O(u))\}]$, where $\text{epi}(F)$ is the epigraph of the operator F. We know from Proposition 13 that s-dentability of O follows from its dentability. Thus O is s-dentable. □

Remark 1. *The converse of Theorem 14 does not hold. A good example is a norm-attainable class on $B_{C[0,1]}$ which has s-dentable point but does not have any dentable point.*

In the next theorem, we prove that dentability for the closed convex hull of a non-empty subclass is sufficient for the dentability that subclass.

Theorem 15. *Let H be a complex Hilbert space and let $NAB(H)$ be a norm-attainable class on $B(H)$. If $N_0 \subset NAB(H)$ is dentable, then so is the closed convex hull of N_0.*

Proof. Let N_0 be a dentable subclass. Then for all $\varepsilon > 0$, there exists a norm-attainable operator $O \in N_0$ such that $O \notin cch[N_0 \setminus B_\varepsilon(O)]$. Then $O \notin cch(N_0)$ and $O \in cch(B_\varepsilon(O))$. Since $B_\varepsilon(O) \subseteq cch(B_\varepsilon(O))$, then there exists ε' such that $0 < \varepsilon' < \varepsilon$ and $O \in B_{\varepsilon'}(O)$. We can choose an operator $O' \in B_{\varepsilon'}(O)$ such that $O' \in cch(N_0)$. Thus $O' \notin cch[chh[N_0] \setminus B_{\varepsilon'}(O)]$. In particular, we have that $O' \notin cch[chh[N_0] \setminus B_{\varepsilon'}(O')]$ and thus $cch[N_0]$ is dentable as claimed. □

We have the following corollary from Theorem 15.

Corollary 1. *Let H be a complex Hilbert space and let $NAB(H)$ a norm-attainable class on $B(H)$. For any $\emptyset \neq N_0 \subset NAB(H)$, if $cch(N_0)$ is not dentable, then neither is N_0.*

Proof. Let N_0 be not dentable. Then for all $\varepsilon > 0$, there exists $O \in cch(N_0)$ such that $O \in cch[cch(N_0) \setminus B_\varepsilon(O)]$. In particular, $O \in cch[cch(N_0)]$ and $O \notin cch[B_\varepsilon(O)]$. Now for any $\delta \in (0, \varepsilon)$ we know that $B_\delta(O) \subseteq cch[B_\delta(O)]$, it follows that if $O \notin cch[B_\delta(O)]$, then $O \notin B_\delta(O)$. Let $O' \in N_0$ such that $O' \notin B_\delta(O)$. Then $O' \in N_0$ implies that $O' \in cch[N_0]$. Putting together $O' \in N_0$ and $O' \notin B_\delta(O)$ implies that $O' \in [N_0 \setminus B_\delta(O')]$. Now since $[N_0 \setminus B_\delta(O')] \subseteq chh[N_0 \setminus B_\delta(O')]$, it follows that $O' \in chh[N_0 \setminus B_\delta(O')]$ and N_0 is not dentable as claimed. □

We state and prove the following lemma that shall be used in the proof of Corollary 2.

Lemma 1. *Let H be a complex Hilbert space and let C be the C-* algebra of all bounded linear operators on H and let $NAB(H)$ be a norm-attainable class on $B(H)$. Then $\emptyset \neq N_0 \subseteq NAB(H)$ is not dentable, if and only if there exists $\varepsilon > 0$ such that for any norm-attainable operator $O \in N_0$, then $N_0 \subseteq cch[N_0 \setminus B_\varepsilon(O)]$.*

Proof. We have two directions to prove.
$\Longrightarrow$ Let N_0 be a not dentable and non-empty subclass in $NAB(H)$. Then there exists $O \in N_0$ such that for every $\varepsilon > 0$, we have $O \in cch[N_0 \setminus B_\varepsilon(O)]$. Since $O \in N_0$ implies that $O \in cch[N_0 \setminus B_\varepsilon(O)]$ (by the non-dentability of N_0), and the choice of O is arbitrary, it follows that for any $\varepsilon > 0$, then $N_0 \subseteq cch[N_0 \setminus B_\varepsilon(O)]$.
$\Longleftarrow$ Let $O \in N_0$ and let $\varepsilon > 0$, such that $N_0 \subseteq cch[N_0 \setminus B_\varepsilon(O)]$. By definition of subclasses, any such $O \in N_0$ will be a member of $cch[N_0 \setminus B_\varepsilon(O)]$. This means that there exists $O \in N_0$ such that $O \in cch[N_0 \setminus B_\varepsilon(O)]$, which means that N_0 is not dentable. □

The following corollary arises from Lemma 1.

Corollary 2. *Let H be a complex Hilbert space and let $NAB(H)$ be a norm-attainable class on $B(H)$. If $\emptyset \neq N_0 \subseteq NAB(H)$ is closed and convex, then the consequent of Lemma 1 is that there exists $O \in N_0$ such that for any $\varepsilon > 0$, then*

$$N_0 = cch[N_0 \setminus B_\varepsilon(O)]. \tag{8}$$

Proof. We have to prove the inclusions and conclude equality of the two norm-attainable classes using the principle of extensionality. By definition, we know that $N_0 \setminus B_\varepsilon \subseteq N_0$ and that $N_0 = cl(N_0)$ if N_0 is closed (where $cl(N_0)$ is the closure of N_0). By taking closures for the convex hulls, we have $cch[N_0 \setminus B_\varepsilon(u)] \subseteq cch[N_0] = N_0$, where the last part of the statement is true by the convexity of N_0 in the hypothesis. Thus

$$cch[N_0 \setminus B_\varepsilon(O)] \subseteq N_0 \tag{9}$$

Conversely, if N_0 is not dentable, then by Lemma 1, there exists an operator $u \in N_0$ such that for any $\varepsilon > 0$, then

$$N_0 \subseteq cch[N_0 \setminus B_\varepsilon(O)]. \tag{10}$$

The two inclusions in Equations 9 and 10 imply the equality claimed in Corollary 2. □

Theorem 16. *Let H be a complex Hilbert space and $NAB(H)$ be a norm-attainable class on $B(H)$. If $N_s \subset NAB(H)$ is a relatively norm-compact convex subclass, then it N_s dentable.*

Proof. Fix a relatively compact convex subset $N_s \subseteq NAB(H)$ and pick any of its extreme points $e \in cl(N_s)$ (points of $cl(N_s)$ which cannot be expressed as a nontrivial convex combination of points in N_s and $cl(A)$ is the closure of A). Since N_s is relatively compact, it follows that $cl(N_s)$ is compact. The for each $\varepsilon > 0$, it is clear that $e \notin cl[cl(N_s) \setminus B_\varepsilon(e)]$, where By Theorem 1 (Krein Milman Theorem) and Theorem 15, $e \notin cch[cl(N_s) \setminus B_\varepsilon(e)]$. Now by definition, we have that $cch[cl(N_s) \setminus B_\varepsilon(e)] = cch[N_s \setminus B_\varepsilon(e)]$. Thus $e \notin cch[N_s \setminus B_\varepsilon(e)]$ and N_s is dentable as desired. □

The following Corollary arises from Theorem 16.

Corollary 3. *Let H be a complex Hilbert space and let $NAB(H)$ be a norm-attainable class on $B(H)$. If $N_c \subseteq NAB(H)$ is any norm-compact subclass of $NAB(H)$, then N_c is dentable.*

Proof. Compactness is sufficient for relative compactness and the result follows from Theorem 16. □

Theorem 17. *Let H be a complex Hilbert space and let $NAB(H)$ be a norm-attainable class on $B(H)$. If $N_s \subseteq NAB(H)$ is any subclass of $NAB(H)$ such that $cch[N_s]$ is dentable, then so is N_s.*

Proof. Let $cch[N_s]$ be dentable. Then for any $\varepsilon > 0$, there exists $O \in N_s$ such that $O \notin cch[cch[N_s] \setminus B_\varepsilon(O)]$. In particular, $O \notin cch[cch[N_s]] = cch[N_s]$ and $O \in B_\varepsilon(O)$. Since $N_s \subseteq cch[N_s]$, if $O \notin cch[N_s]$, then $O \notin N_s$. Now choose $\delta \in (\varepsilon, \infty)$ and let $O' \in N_s$ such that $O' \in B_\delta(O)$. Hence $O' \notin N_s \setminus B_\delta(O)$. By taking closed convex hulls, we have that $O' \in N_s$ such that for $\delta \in (\varepsilon, \infty)$, then $O' \notin chh[N_s \setminus B_\delta(O')]$. This proves that N_s is dentable. □

We have the following corollary.

Corollary 4. *Let H be a complex Hilbert space and let $NAB(H)$ be a norm-attainable class on $B(H)$. If $N_s \subseteq NAB(H)$ is any subclass such that $cch[N_s]$ is s-dentable, then so is N_s.*

Proof. This is clear by Proposition 13 and Theorem 17. □

We have some results on dentability of classes and subclasses. The following theorem relates the dentability of the closure of a class to that of the class.

Theorem 18. *Let H be a complex Hilbert space and let $N = NAB(H)$ be a norm-attainable class on $B(H)$. If the closure $cl(N)$ of N is dentable, then so is N.*

Proof. Let $cl(N)$ be dentable and choose $O \in N$ such that for any $\varepsilon > 0$, then $O \notin cch[cl(N) \setminus B_\varepsilon(O)]$. Since $N \subseteq cl(N)$, it follows that whenever $O \in N$, then $O \in cl(N)$ and furthermore, $N \setminus B_\varepsilon(O) \subseteq cl(N) \setminus B_\varepsilon(O)$. Now if $O \notin cl(N) \setminus B_\varepsilon(O)$, then $O \notin N \setminus B_\varepsilon(O)$. Then by taking closed convex hulls, we have that for any $\delta \in (\varepsilon, \infty)$, there exists $O \in N$ such that $O \notin cch[N \setminus B_\delta(O)]$. Thus N is dentable as claimed. □

Remark 2. *For the proof to work, we must choose a point in N. This is because $N \subseteq cl(N)$ and it is possible to have $O \notin N$ but $O \in cl(N)$. Hence N may be dentable but $cl(N)$ may fail to be dentable.*

We have the following corollaries.

Corollary 5. *Let H be a complex Hilbert space and let $NAB(H)$ be a norm-attainable class on $B(H)$Let N_1 and N_0 be non-empty subclasses in $NAB(H)$ such that $N_0 \subseteq N_1$. If N_1 is dentable, then N_0 is dentable.*

Proof. Let $O \in N_0$ such that for every $\varepsilon > 0$, we have that $O \notin cch[N_1 \setminus B_\varepsilon(O)]$ (by dentability of N_1). We also have that $N_0 \subseteq N_1$ implies that $cch[N_0 \setminus B_\varepsilon(O)] \subseteq cch[N_1 \setminus B_\varepsilon(O)]$. Now since $O \notin cch[N_1 \setminus B_\varepsilon(O)]$, it is easy to see that $O \notin cch[N_0 \setminus B_\varepsilon(O)]$ and this means that N_0 is dentable. □

Remark 3. *Dentability is reverse-transitive by set inclusion if we choose a dentable point from the intersection of the three classes. That is for any subclasses $\emptyset \neq N_0 \subseteq N_1 \subseteq N_2$ such that N_2 and N_1 are dentable, then N_0 is also dentable. The proof of Corollary 5 fails if we choose a dentable point of N_1 which is not in N_0. This is to say that a dentable point of a superclass may fail to exist in the subclass.*

We now shift attention to dentability of sums, scalar multiples and products of mappings between norm-attainable classes.

Theorem 19. *Let H be a complex Hilbert space and let $NAB(H)$ be a norm-attainable class on $B(H)$. Let $O_1 : NAB(H) \longrightarrow NAB(H)$ and $O_2 : NAB(H) \longrightarrow NAB(H)$ be positive and monotone norm-attainable operators such that $O_1 \leq O_2$. Then if O_1 and O_2 are dentable, so is their sum $O_1 + O_2$.*

Proof. Since $O_1 \leq O_2$, then $O_1 \leq O_1 + O_2$ and $\text{epi}(O_1 + O_2) \subseteq \text{epi}(O_2) \subseteq \text{epi}(O_1)$. By dentability of O_1, there exists $h_1 \in NAB(H)$ such that for every $\epsilon > 0$, then $(h_1, O_1(h_1)) \notin \text{cch}[\text{epi}(O_1) \setminus B_\varepsilon(h_1, O_1(h_1))]$. In particular, for any $\delta \in (\varepsilon, \infty)$, then there exists $h_2 \in N_1$ such that $h_2 \in B_\varepsilon(h_1)$ and

$$\text{epi}(O_1 + O_2) \setminus B_\delta(h_2, (O_1 + O_2)(h_2)) \subseteq \text{epi}(O_1) \setminus B_\delta(h_2, O_1(h_2)). \tag{11}$$

By taking closed convex hulls in the Inclusion 11, we have that $(h_2, (O_1 + O_2)(h_2)) \notin cch[\text{epi}(O_1 + O_2) \setminus B_\delta(h_2, (O_1 + O_2)(h_2))]$. Thus the sum is dentable as claimed. □

Remark 4. *From the proof of Theorem 19, it is clear that only one of the summand operators need to be dentable. In deed, Theorem 19 is sufficient for dentability of a difference of dentable operators.*

Theorem 20. *Let H be a Hilbert space and $NAB(H)$ be a norm-attainable class on $B(H)$. Further let $O : NAB(H) \longrightarrow NAB(H)$ be a positive and monotone norm-attainable operator and let λ be a complex scalar. If O is dentable, so is $|\lambda|O$.*

Proof. We prove for positive scalar multiples and the negative ones can be done analogously. There are two cases to consider. On the one hand, if $|\lambda| > 1$, then $O < |\lambda|O$ and $\text{epi}(|\lambda|O) \subseteq \text{epi}(O)$. Since O is dentable, then for any $\varepsilon > 0$, there exists $h_1 \in NAB(H)$ such that $(h_1, O(h_1)) \notin cch[\text{epi}(O) \setminus B_\varepsilon(h_1, O(h_1))]$. Furthermore, from $\text{epi}(|\lambda|O) \subseteq \text{epi}(O)$, it follows that

$$\text{cch}[\text{epi}(|\lambda|O) \setminus B_\varepsilon(h_1, O(h_1)] \subseteq \text{cch}[\text{epi}(O) \setminus B_\varepsilon(h_1, O(h_1)]. \tag{12}$$

In particular,

$$\text{cch}[\text{epi}(|\lambda|O) \setminus B_\varepsilon(h_1, |\lambda|O(h_1)] \subseteq \text{cch}[\text{epi}(O) \setminus B_\varepsilon(h_1, |\lambda|O(h_1)]. \tag{13}$$

Now for every $\delta \in (\lambda, \infty)$, then $(h_1, \lambda O(h_1)) \notin \text{cch}[\text{epi}(O) \setminus B_\delta(h_1, \lambda O(h_1)]$ implies that $(h_1, \lambda O(h_1)) \notin \text{cch}[\text{epi}(\lambda O) \setminus B_\delta(h_1, \lambda O(h_1)]$ and $|\lambda|O$ is dentable for $|\lambda| > 1$. On the other hand, if $0 < |\lambda| < 1$ then $|\lambda|O < O$and $\text{epi}(O) \subseteq \text{epi}(O)$. Take $h_2 \in NAB(H)$ such that $h_2 \in B_\varepsilon(h_1)$. Then for any $\delta \in (|\lambda|, \infty)$, we have $(h_2, |\lambda|O(h_2)) \notin \text{cch}[\text{epi}(|\lambda|O) \setminus B_\delta(h_2, |\lambda|O(h_2)]$. Thus in both cases, $|\lambda|O$ is dentable. □

Remark 5. *Observe that the case when $|\lambda| = 1$ is a trivial case. That is, if O is dentable, then its scalar multiple with a scalar $|\lambda| = 1$ is still O which is dentable by hypothesis.*

Theorem 21. *Let H be a complex Hilbert space and let $NAB(H)$ be a norm-attainable class on $B(H)$. Let $\{O_1, O_2\}$ be a sequence of positive and monotone norm-attainable operators $O_i : NAB(H) \longrightarrow NAB(H), i = 1, 2$ such that $O_1 \leq O_2$. If O_1 and O_2 are dentable, then so is O_1O_2.*

Proof. We consider two cases, that is, inside and outside the unit ball. If $1 < O_1 \leq O_2 \leq O_1O_2$, then it follows that $\text{epi}(O_1O_2) \subseteq \text{epi}(O_2) \subseteq \text{epi}(O_1)$. Then by dentability of O_1 and O_2, the dentability of the product follows as seen in the proof of Theorem 19. If $0 < O_1 \leq O_2 < 1$, then for some $\Lambda > \frac{1}{\|O_1\|} > 1$, if follows that $0 < O_1O_2 \leq O_1 \leq O_2 \leq 1 \leq \Lambda O_2$. By dentability of O_2, it follows from Theorem 20 that ΛO_2 is dentable. Furthermore, we know that $\text{epi}(\Lambda O_2) \subset \text{epi}(O_1O_2)$ which implies that

$$cch[\text{epi}(\Lambda O_2) \setminus B_\varepsilon(h_1, \lambda O_2(h_1))] \subset cch[\text{epi}(O_1O_2) \setminus B_\varepsilon(h_1, \Lambda O_2(h_1))] \tag{14}$$

Now choose $h \in B_\varepsilon(h_1)$ such that for any $\delta \in (\varepsilon, \infty)$, then

$$(h, O_1O_2(h)) \notin cch[\text{epi}(O_1O_2) \setminus B_\delta(h_1, O_1O_2(h))]. \tag{15}$$

Then O_1O_2 is dentable as desired. □

Remark 6. *By the proof of Theorem 21, the dentability of one of O_1 and O_2 is sufficient for that of the product O_1O_2. Moreover, Theorem 20 and Theorem 21 can be used to prove that the quotient of dentable operators is dentable.*

We now study the dentability of unions, intersections and cartesian products of norm-attainable classes.

Theorem 22. *Let H be a complex Hilbert space and let $NAB(H)$ be a norm-attainable class on $B(H)$. Further, let $\{NAB(H)_j\}_{j\geq 1}$ be a countable sequence of disjoint norm-attainable classes on $B(H)$. If $NAB(H)_j$ is dentable for all $j \in \mathbb{N}$, then the countable union $N = \bigcup_{j=1}^{\infty} NAB(H)_j$ is dentable.*

Proof. Fix a disjoint countable sequence of norm-attainable classes $\{NAB(H)_j\}$ on $B(H)$. Since each of them is dentable, it follows that for some $k \in \mathbb{N}$, there exist $h_k \in NAB(H)_k, k = 1, 2, 3, \ldots$ such that for $\varepsilon > 0$, then $h_k \notin cch[NAB(H)_k \setminus B_\varepsilon(h_k)]$. We know that $NAB(H)_k \subseteq N$, which implies that $NAB(H)_k \setminus B_\varepsilon(h_k) \subseteq N \setminus B_\varepsilon(h_k)$. Take closed convex hulls on both sides to obtain $cch[NAB(H)_k \setminus B_\varepsilon(h_k)] \subseteq cch[N \setminus B_\varepsilon(h_k)]$. Now using the disjointness of the countable sequence and dentability of N_k (there nothing special about N_k as any of the classes would do the job) and if $h \notin NAB(H)_k$, then $h \notin N$. Then $h_k \notin cch[N \setminus B_\varepsilon(h_k)]$ and N is dentable. □

Remark 7. *From the proof of Theorem 22, one only needs to choose a dentable point for one of the subclasses and this works for the disjoint union. This proof fails when the union is not of disjoint classes.*

Theorem 23. *Let H be a complex Hilbert space such that $NAB(H)$ is a norm-attainable class on $B(H)$. Further, let $\{NAB(H)_i\}_{1\leq i\leq M}$ be a finite sequence of norm-attainable classes on $B(H)$. If $NAB(H)_i$ is dentable for all $i \in \{1, 2, \ldots, M\}$, then the finite intersection $N = \bigcap_{i=1}^{M} NAB(H)_i$ is dentable.*

Proof. Fix a finite sequence $\{NAB(H)_i\}_{1\leq i\leq M}$ of dentable norm-attainable classes on $B(H)$. Since $NAB(H)_i$ is dentable for all i, there exists an index $1 \leq k \leq M$ such that $h_k \in NAB(H)_k$ and h_k is a dentable point for $NAB(H)_k$. This implies that for some $\varepsilon > 0$, we have that $h_k \notin cch[NAB(H)_k \setminus B_\varepsilon(h_k)]$. Using the fact that $N \subseteq NAB(H)_k$, it follows that $N \setminus B_\varepsilon(h_k) \subseteq NAB(H)_k \setminus B_\varepsilon(h_k)$. Now take convex closures on the inclusion to yield $cch[N \setminus B_\varepsilon(h_k)] \subseteq cch[NAB(H)_k \setminus B_\varepsilon(h_k)]$. Clearly $h_k \notin cch[N \setminus B_\varepsilon(h_k)]$. Thus $N = \bigcap_{i=1}^{M} NAB(H)_i$ is dentable. □

Theorem 24. *Let H be a complex Hilbert space and let $NAB(H)$ be a norm-attainable class on $B(H)$. Further, let $\{NAB(H)_j\}_{j\geq 1}$ be a countable sequence of norm-attainable classes on $B(H)$. If $NAB(H)_j$ is dentable for all $j \in \mathbb{N}$, then the countable Cartesian product $P = \prod_{j=1}^{\infty} NAB(H)_j$ is dentable.*

Proof. Fix a countable sequence $\{NAB(H)_j\}_{j\geq 1}$ of dentable norm-attainable classes on $B(H)$. By dentability of $NAB(H)_j$ for all j, we have that there exists $h_j \in NAB(H)_j$ for $\varepsilon_j > 0$ such that $h_j \notin cch[NAB(H)_j \setminus B_{\varepsilon_j}(h_j)]$. The map that associates $(h_1, h_2, \ldots) \longmapsto NAB(H)_1 \times NAB(H)_2 \times \ldots = P$ is bijective. Choose $\varepsilon = \max\{\varepsilon_j\}$. Then we have $(h_1, h_2, \ldots) \notin cch[P \setminus B_\varepsilon(h_1, h_2, \ldots)]$ and P is dentable. □

5. An application to optimization

Consider an optimization problem of the form:
Minimize $f(x)$
subject to $g_i(x) \leq 0$ for $i = 1, 2, \ldots, m$
$h_j(x) = 0$ for $j = 1, 2, \ldots, p$
Where:
$f(x)$ is the convex objective function, $g_i(x)$ are the inequality constraints, $h_j(x)$ are the equality constraints, and$x$ is the vector of variables. We take the domain Ω of $f(x)$ to be convex, closed and bounded. To minimize $f(x)$, we define a minimizing sequence (x_n) in Ω such that $f(x_n) \longrightarrow \inf f(\Omega)$. Then $\lim x_n = x$ will be a dentable point of $f(x)$, which minimizes f. This approach can also be used on the dual problem. By extending Theorem 19 to more than two operators, more than one optimization problems can be solved once by ensuring that at least one of the objective functions is dentable. This alludes to superposition of solutions to different convex optimization problems. However, it is expected that to obtain sharper bounds, all the objective functions must necessarily be convex and dentable.

6. Conclusions

In conclusion, we have presented various key findings regarding dentability and density in norm-attainable classes. We have shown that dentability of a norm-attainable class NAC implies its s-dentability, and the dentability of the closed convex hull of NAC ensures the dentability of NAC itself. Conversely, if a class lacks dentability, its closed convex hull also lacks dentability. This relationship highlights the importance of dentability in understanding the structure of norm-attainable classes.

We have demonstrated that both relatively norm-compact classes and norm-compact classes are dentable, expanding our understanding of dentability in different subclasses. Furthermore, we have established conditions for the dentability of superclasses and subclasses, providing insight into the dentability properties of related classes.

Additionally, we have investigated the preservation of dentability under various operations. We have proven that dentability is preserved under summation, scalar multiplication, and product for positive and monotone dentable operators in norm-attainable classes. This result highlights the robustness of dentability in the context of these operations.

Moreover, we have shown that dentability of classes is preserved under countable unions, finite intersections, and countable Cartesian products. This preservation property extends the understanding of dentability to composite classes formed through these set operations.

Furthermore, our study has focused on characterizing density in the norm-attainable class $NAB(H)$ within the Banach algebra $B(H)$ of all bounded linear operators on a complex Hilbert space H. We have established the characterization of density using concepts such as the convergence of sequences and the existence of limit points. Notably, properties A and B of Lindenstrauss have been shown to be sufficient for the density of $NAB(H)$. Additionally, we have demonstrated that countable unions, finite intersections, countable tensor products, and countable Cartesian products preserve density in associated classes. The transitivity of density in $NAB(H)$ further enhances our understanding of the distribution of norm-attainable operators.

Overall, our findings contribute significantly to the characterization and understanding of dentability and density in norm-attainable classes. These results have implications in the fields of operator analysis, operator theory, and optimization, providing valuable insights into the properties of bounded linear operators. Furthermore, they aid in the study of linearbility and spaceability of norm-attainable classes and Banach spaces, further advancing our comprehension of these fundamental mathematical concepts.

References

[1] E. Bishop, R. R. Phelps, A proof that every Banach space is subreflexive, Bulletin of the American Mathematical Society 67 (1) (1961) 97–98.
[2] K. Conrad, Zorn's lemma and some applications, Expository papers.
[3] M. Rieffel, Functional Analysis: Dentable subsets of Banach spaces, with application to a Radon-Nikodỳm theorem, Thompson Book, Co. Washington, 1967.
[4] H. B. Maynard, A geometrical characterization of Banach spaces with the Radon-Nikodym property, Transactions of the American Mathematical Society 185 (1) (1973) 493–500.
[5] W. Davis, R. Phelps, The Radon-Nikodym property and dentable sets in Banach spaces, Proceedings of the American Mathematical Society 45 (1) (1974) 119–122.
[6] C. G. Looney, Convergence of minimizing sequences, Journal of Mathematical Analysis and Applications 61 (3) (1977) 835–840.
[7] C. G. Looney, Locally uniformly quasi-convex programming, SIAM Journal on Applied Mathematics 28 (4) (1975) 881–884.
[8] C. G. Looney, Dentable functions and radially uniform quasi-convexity, International Journal of Mathematics and Mathematical Sciences 1 (4) (1978) 533–539.
[9] J. Bourgain, On dentability and the Bishop-Phelps property, Israel Journal of Mathematics 28 (4) (1977) 265–271.
[10] M. Rieffel, The Radon-Nikodym theorem for the Bochner integral, Transactions of the American Mathematical Society 131 (2) (1968) 466–487.
[11] R. Phillips, On weakly compact subsets of a Banach space, American Journal of Mathematics 65 (1) (1943) 108–136.
[12] H. B. Maynard, A Radon-Nikoỳm theorem for operator-valued measures, Transactions of the American Mathematical Society 173 (1) (1972) 449–463.
[13] R. Huff, Dentability and the radon-nikodỳm property, Duke Mathematical Journal 41 (1) (1974) 111–114.
[14] R. R. Phelps, Dentability and extreme points in Banach spaces, Journal of functional analysis 17 (1) (1974) 78–90.
[15] R. Huff, On non-density of norm-attaining operators , Rev. Roumaine Math. Pures Appl 25 (1) (1980) 239–241.
[16] P. Mogotu, N. Okelo, O. Ongati, On denseness of similarity orbits of norm-attainable operators, International Journal of Advances in Applied Mathematics and Mechanics 6 (4) (2019) 14–21.
[17] N. Okelo, On norm-attainable operators in banach spaces, Journal of function Spaces 2020 (2020) 1–6.
[18] M. Krein, D. Milman, On extreme points of regular convex sets, Studia Mathematica 9 (1940) 133–138.