ON SEMI-CONTINUOUS FUNCTIONS AND CONVEX OPTIMIZATION IN L^p-SPACES

BY

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Declaration

This thesis is my own work and has not been presented for a degree award in any other institution.

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Dedication

To my wife Becky, my daughters Stacy, Sheryl, Sifa and my son Asamba.

Abstract

The study of semi-continuity and optimization has garnered significant attention from mathematicians for a prolonged period. While characterization of semi-continuity has been conducted in topological spaces and Hilbert spaces, it has not been studied in L^p-spaces. Similarly, while optimality conditions for convex optimization have been established in Hilbert spaces, Hausdorff spaces and normed spaces, they have not been determined in L^p-spaces. The aim of this study was to characterize semi-continuous functions and convex optimization in L^p-spaces. The specific objectives included: characterizing lower semi-continuous (lsc) functions in L^p-spaces; characterizing upper semi-continuous (usc) functions in L^p-spaces; and establishing conditions for convex optimization in L^p-spaces. The research methodology involved the use of Fatou's Lemma and Dini's theorem to characterize lsc functions, and Beer's theorem which was used in characterizing usc functions. Technical approaches included the use of KKT conditions for optimality to establish conditions for convex optimization in L^p-spaces. The study has shown that if the epigraph of an L^p-space function is closed then , the function is lsc in the space. Additionally, it has been demonstrated that a function ϑ contained in a convex subset of an L^p-spaces $\mathcal{L}$ is usc if it is convex. The study has further shown that a Lipschitz-continuous function in a sequentially bounded subspace of a convex L^p-space $\mathcal{L}$ is lsc and has a local minimizer. It has also been proven that if $\vartheta(q)$ is Frechet differentiable in a convex L^p-space $\mathcal{L}$, then q is a stationary point of $\vartheta(q)$ and forms its local minimizer. These results have potential applications in mathematical analysis, particularly in norm approximation which is useful in image and signal processing.

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Chapter 1

INTRODUCTION

1.1 Mathematical background

In the field of mathematics, semi-continuity is a property that verifies the presence of minima or maxima within compact regions and determines if they are global or local. This is done by considering whether a function is lower semi-continuous (*lsc*), which means it contains its greatest lower bound and therefore has a local minimum. Convexity, on the other hand, is used to guarantee that an optimization method will converges to a global minimum or maximum. A function whose lower limit exists is *lsc* while the one whose upper limits exist is said to be upper semi-continuous (*usc*). Convex sets are those that contain all points on the line connecting any two points within the set, while functions are convex if a line segment connecting any two points on the function always lies above the graph of the function. Global minimizers are the values of a function's minimum over the entire domain, while local minimizers are the values that take a function to its minimum within a specific region.

Several mathematicians including Beer [5], Chen *et al* [12], Gool [22], Varagona [56], and Mirmostafaee [34], have studied semi-continuous functions and their properties. Beer [5] characterized upper semi-continuous functions in compact metric spaces and Hausdorff spaces and extended Dini's theorem to characterize sequences of upper semi-continuous functions that converge point-wise to a continuous function that converges uniformly. Gool [22] examined *lsc* functions in continuous lattices, and appied the results in potential theory. Gool's [22] work characterized *lsc* functions whose domain is a continuous lattice in compact topological spaces. Gool's [22] showed that given a continuous lattice as a domain of a *lsc* function, if its supremum exists, the the supremum is a *lsc* function. Varagona [56] examined inverse limits with *usc* bonding functions and indecomposability providing sufficient and necessary conditions for the bonding functions (ψ_i) to be a decomposable/ indecomposable continuum. Chen *et al* [12], introduced the concept of *lsc* functions from above and provided a proof that Eklands's [18] theorem holds under semi-continuous functions and convex functions in real normed linear spaces and real reflexive Banach spaces.

In their work, Correa and Hantoute [15] characterized lower semi-continuous convex relaxation functions in optimization. They demonstrated the relationship between the argmin sets of a given function and its semi-continuous convex hull using characterizations involving asymptotic functions. Correa and Hantoute [15] provided explicit formulas for Fenchel sub differential and the argmin sets of successful Legendre-Fenchel conjugates of a real-valued function. They focused on lower semi-continuous convex relaxation in optimization on infinite-dimensional real locally convex space and came up with explicit formulas for Fenchel sub-differential and the argmin sets of successful Legendre-Fenchel conjugates of functions with real values.

Mirmostafaee [34] studied *usc* and *lsc* functions of multi-valued functions in Baire spaces. Mirmostafaee's research characterized *lsc* and *usc* functions in Baire spaces, metrizable spaces and second countable spaces. Hernández and López [25] characterized semi-continuous functions in metrizable topological spaces. They showed that a function in a metrizable topological space is *lsc* if its sub level set is closed and it is *usc* if its sub level set is open. The graphical properties of semi-continuous functions were also studied where it was proved that if the hypograph of a finite function is closed then the function is *usc* and if a function's epigraph is open the function is *lsc*.

Optimization, also known as Mathematical programming, is the process of finding the smallest or largest value possible for a specific problem, given certain limitations or constraints. It has been applied in various fields including finance, epidemiology, engineering, and more. In the past, optimization problems were mainly solved using classical calculus, but in the recent decades it has become a vast independent field. Euler's Calculus of variations was one of the earliest tools for optimization, and Lagrange Multipliers were useful in identifying conditions for optimality. Hancock's optimization method involved using calculus derivatives and turning points of an objective function to find maxima and minima [24]. Ramsey[46] used derivative variations to study optimal economic growth in his well-known optimal growth theory.

During World War II, researchers focused on optimization as a way to solve large-scale planning and decision-making problems. The Simplex

Method, a classical optimization algorithm that could solve large-scale linear programming problems was developed by Dantzig. Karmarkar [28] also made important contributions to the field of linear optimization, developing analytical methods called interior point methods that could solve linear programming problems, as well as some non-linear optimization problems such as convex, semi-definite, and second-order cone problems arising in engineering and operations research.

Prior to the 1950s, researchers mainly focused on linear programming to solve optimization problems, using techniques such as the Calculus of Variations, duality theory, and interior point methods [39]. However, they encountered difficulties in formulating some problems as linear programs and realized that many real-world optimization problems are not linear [47]. As a result, research in optimization shifted towards non-linear programming in the 1950s. During this time, the all important classical KKT optimality conditions were coined, which ably solved many $NLOP$s [33]. With the generalization of the Simplex method to non-linear programs by use of the method of feasible directions, convex analysis became a popular tool for optimization, and the differentiability hypothesis was no longer widely used. Svanberg [54] used convergence of separable convex iterations to solve $NLOP$s having multiple variables. The Barzilai-Borwein [21] method was introduced as a way to solve QOPs with box constraints. However, these methods did not consider optimization in general convex functions in L^p-space or semi-continuity. In [21] the Barzilai-Borwein method is discussed. This method solved, in large scale, quadratic problems that are convex and having box constraints. In minimizing convex quadratic functions of many variables and having box constraints approximations are taken that lead to the minimizer for the QOP. Friedlander *et al* [21] applied the Barzilai and Borwein's optimization technique to solve QOPs.

Moreau [38], Fenchel [20], and Rockafellar [49] are key figures in the development of convexity theory. Rockafellar [48] was the first to publish a textbook on convex analysis, which explored convexity of sets and functions applying them to mathematical programming, as well as minimax theorems, systems of inequalities, and Lagrange multipliers. Clarke [14] developed the theory of non-differentiable optimization problems, which was referred to as non-smooth optimization and the findings were applied in control theory.

The growth and development of convex optimization has been driven by demand for problem-solving and modeling in logistics and planning, advancements in computing technology, and the creation of duality theory and simplex algorithm. In recent years, it has been widely used in fields such as control systems, communication networks, signal processing, image processing, and data analysis due to the effectiveness and dependability of optimization methods, especially the interior point, in solving COPs [8]. Nesterov [40] and Nemirovski [41] realized a major breakthrough with the development of interior point methods that converge quickly to a solution for convex optimization problems. These methods have been extended to handle finite-dimensional, non-differentiable convex optimization problems, but do not account for semi-continuous functions in optimization.

The authors Lobo, Maryam, and Boyd [31] focused on using relaxation methods to solve financial optimization problems, specifically those related to portfolio allocation. These problems were formulated as convex problems with linear constraints, and the relaxation method allowed for the calculation of global solutions by providing computable upper bounds. Mitter [35] studied convex programming using duality concepts in infinite dimensional Hausdorff locally compact spaces. Mitter [35] utilized convex analysis techniques and the duality approach to investigate the conditions that must be satisfied for the duality formalism to hold, including the requirement for local compactness in the dual feasible set.

Boyd [10] worked on convex optimization problems involving the eigenvalues of graph Laplacian matrices. The study's goal focused on determining the greatest or smallest value of a function that is dependent on the eigenvalues of the graph Laplacian matrices while simultaneously satisfying constraints such as non-negativity and constraints on the total value. Boyd [10] formulated these problems as convex functions and derived interesting dual problems, sometimes providing analytical solutions and always offering efficient numerical solutions. For medium-sized problems, interior point methods were used, and for larger problems, subgradient-based methods were employed, taking into account the structure and symmetry of the problems.

Helou and De Pierro investigated the application of convex feasibility and ϵ-sub-gradient techniques to tackle non-smooth COPs that were not smooth in $\mathbb{R}^1$ in a paper published in [42]. They established a comprehensive framework incorporating various methods for resolving CCP difficulties using incremental sub-gradients. This framework allowed them to create algorithms that could effectively resolve CCP problems in large quantities, by utilizing approximate projections instead of exact Euclidean projections. The algorithms were obtained from incremental sub-gradients and aggregated incremental sub-gradients.

Zhu and Martínez [58] focused on solving multi-agent convex optimization problems in topological spaces, which involved minimizing a global objective function that was convex and subject to a convex constraint set. The constraints, which were given by convex functions and known to all agents, could be either inequalities or equalities. Zhu and Martínez [58] devised two algorithms for this optimization problem that were implemented over dynamically changing networks of topologies that were connected and allowed the agents to converge to optimal values or solutions. These algorithms used distributed primal-dual sub-gradients. For cases without equality constraints, the Lagrangian relaxation method was used, while for cases with identical local constraint sets, a penalty relaxation approach was employed to eliminate the additional equality constraints.

Several researchers, including [1, 26, 4, 44, 43], *et al* have examined the use of convex optimization in inner product spaces. Alexanderian [1] conducted a study on convex optimization in Hilbert spaces. The study used optimization tools that involve lower semi-continuous functions and convex functionals. The work focused on determining minimizers for convex programs in Hilbert spaces. The study applied the generalized Weierstrass Theorem to present conditions required for minimizers to be attained for Hilbert space convex problems. Since this assertion holds for a reflexive space such as L^2, which is a Hilbert space, it would be interesting to see whether it holds for general L^p-spaces, where $1 \leq p < \infty$, which was the aim of the current study. This study examined the applicability Alexandrian's findings to general L^p-spaces and sought to discuss properties of convex optimization in L^p-spaces. Houska and Chachuat [26] dealt with non-convex optimization problems and used a complete-search algorithm to identify feasible solutions. Bay, Grammont, and Maatouk [4] formulated interpolation problems as convex programs governed by linear constraints in Hilbert spaces. They developed an algorithm that approached a constrained interpolating function through the convergence of approximate solutions. They formulated interpolating curves/surfaces as general convex optimization problems in a Hilbert space whose constraints were given in form of linear inequalities. However, the study was limited to inner product norms, whereas in the current research, the norms were defined as L^p-norms. This change in norm structure provided a new perspective on the behavior of the functions representing the optimization problems and potentially offered new insights into determining solutions for these problems.

Okelo [44] conducted a study on optimization in Hilbert spaces posing the problems as convex functions. The results of the study showed that, if a function $\varpi : W \to \mathbb{R}$ is weakly sequentially lsc, then the function ϖ attains a minimizer on the convex set W. It was also established that if ϖ is closed, then the optimization problem $Inf_{q \in V} \psi(W)$ admits at least one global minimizer. In his study, in order to find minimizers, Okelo [44] employed the use of weak topologies. One of the key differences between Hilbert spaces and L^p-spaces is that given a sequence that is bounded it is easy to obtain a sub-sequence that converges weakly in a Hilbert space, which is not the case in L^p spaces. Therefore, this study aimed to establish optimization conditions for L^p-spaces using semi-boundedness imposed by semi-continuous functions. Offia [43], minimized COPs operating on Hilbert spaces of infinite dimensional nature using $ws - lsc$ functions. All of these studies pertain to convex optimization in Hilbert spaces, but none of them deal with convex optimization in L^p-spaces.

A small number of mathematicians, including Peypouquet [45] and Devore and Temlyakov [16], have looked into convex optimization in Banach spaces. Peypouquet [45] studied the use of convex optimization in normed spaces, particularly Banach spaces, and characterized properties such as topological duals and linear functionals in these spaces. Concepts like orthogonality and projection were analyzed in relation to convex optimization. Peypoquet [45] conducted a study on convex optimization in normed spaces. The study used optimization tools such as linear functionals, and topological duals, and the main methodology was weak topology and duality approach. The study developed optimality problems for constrained optimization in Hilbert spaces and general normed spaces. Peypoquet [45] developed optimality conditions for constrained optimization general linear spaces including Banach spaces but did not develop optimality conditions for L^p spaces even if the L^p space is also a Banach space. The current study therefore found the L^p space to be an interesting space to establish conditions for convex optimization, especially considering the underlying L^p-norm structures and the range of p i.e $1 \leq p < \infty$. Our current study sought to address this challenge by investigating convex optimization in L^p spaces, and aimed to close the gap left by Peypoquet other researchers who established optimality conditions in complete normed spaces.

Devore and Temlyakov [16] examined the application of convex optimization in Banach spaces using interior point methods and investigated recent advances in structural optimization. However, neither of these studies considered convex optimization in L^p-spaces. Unser and Aziznejad [55] worked on COPs expressing the solutions as component sums in Banach spaces. To regularize the COPs, Unser and Aziznejad conducted penalization of norms of the minima.

It is evident that limited research has been conducted on characterization of lower semi-continuous functions in L^p spaces and characterization of upper semi-continuous functions in L^p spaces. Moreover, it is clear that conditions for convex optimization have not been established in L^p spaces.

1.2 Basic concepts and preliminaries

In this section, the concepts of convex sets, convex functions, semi-continuous functions, global minimizers, and local minimizers are introduced. These concepts form the foundation for the subsequent analysis and discussions in this research. We have also presented fundamental preliminary results key to this study.

Definition 1.1 ([29], Definition 1.4.1)**.** Let $\{q_n\}_{n\in\mathbb{N}}$ be a sequence in an L^p-space $\mathcal{L}$. $\{q_n\}$ is said to converge to $q \in \mathcal{L}$ if for any $\xi > 0$ there is $N_\xi \in \mathbb{Z}^+$ such that $\|q_n - q\| < \xi$ whenever $n > N_\xi$. The sequence $\{q_n\}$ in L^2-space is said to converge weakly to q if, $\lim_{n\to\infty}\langle q_n, u\rangle = \langle q, u\rangle, \forall u \in L^2$.

Definition 1.2 ([44], Definition 2.2)**.** Let $\mathcal{L}$ be a normed linear space. A function $\vartheta : \mathcal{L} \to \overline{\mathbb{R}}$ is lsc if, given a sequence $\{q_n\} \in \mathcal{L}$, $\vartheta(q) \leq \liminf_{n\to\infty} \vartheta(q_n)$ for all sequences $\{q_n\} \in \mathcal{L}$ such that for each $q \in \mathcal{L}$, $q_n \to q$ strongly.

Definition 1.3 ([17], Definition 1.5)**.** Let $\mathcal{L}$ be a normed linear space. A function $\vartheta : \mathcal{L} \to \overline{\mathbb{R}}$ is *usc* if, for any sequence $\{q_n\} \in \mathcal{L}$, $\vartheta(q) \geq \limsup_{n\to\infty} \vartheta(q_n)$ for all sequences $\{q_n\} \in \mathcal{L}$ such that for each $q \in \mathcal{L}$, $q_n \to q$ strongly.

Definition 1.4 ([15], Definition 1.7)**.** Let L be a nonempty normed space. A function $\vartheta : \mathcal{L} \to \overline{\mathbb{R}}$ is said to be semi-continuous if it is either lower semi-continuous or upper semi-continuous.

Definition 1.5 ([36], Definition 3.3.1)**.** A function that is both lower semi-continuous and upper semi-continuous is said to be a continuous function.

Example 1.6. The following are examples of functions that are continuous:

(i) Trigonometric functions like $\sin x, \cos x$;

(ii) Polynomials;

(iii) Exponential functions

Definition 1.7 ([17], Definition 2.46)**.** Let $\mathcal{L}$ denote a normed linear space and $Q \neq \emptyset$ be a subset of $\mathcal{L}$. If $\forall q_1, q_2 \in Q$, and $0 \leq \eta \leq 1$ we have $\eta q_1 + (1-\eta)q_2 \in Q$ then we say that Q is convex. A function $\vartheta : Q \to \mathbb{R}$ satisfying $\vartheta(\eta q_1 + (1-\eta)q_2) \leq \eta\vartheta(q_1) + (1-\eta)\vartheta(q_2)$ for each $\forall q_1, q_2 \in Q$, and $0 \leq \eta \leq 1$ is said to be convex.

Theorem 1.8 ([3], Theorem 3.11)**.** *Let $\vartheta : \mathcal{Q} \to \mathbb{R}$ be convex on the convex set $\mathcal{Q}$. Given that the local minimum for $\vartheta(q)$ over $\mathcal{Q}$ is $q^* \in \mathcal{Q}$ then, q^* is also the global minimum of $\vartheta(q)$ over $\mathcal{Q}$.*

If we set a local minimum for $\vartheta(q)$ to be at q^*, it means that $\vartheta(q^*) \leq \vartheta(q)$ throughout the neighborhood of $q \in \mathcal{Q}$. Suppose a positive number p satisfies $q \in B[q^*; p]$ and $\exists \kappa \in (0,1]$ satisfies $q^* + \kappa(w - q^*) \in B[q^*; p], \forall w \in \mathcal{Q} : w \neq q^*$. Now, $q^* + \kappa(w - q^*) \in B[q^*; p] \cap \mathcal{L}$. Therefore, $\vartheta(q^*) \leq \vartheta(q^* + \kappa(w - q^*))$. Thus by Jensen's inequality we have

$$\vartheta(q^*) \leq \vartheta(q^* + \kappa(w - q^*)) \leq (1 - \kappa)\vartheta(q^*) + \kappa\vartheta(w).$$

Hence $\kappa\vartheta(q^*) \leq \kappa\vartheta(w)$, so $\vartheta(q^*) \leq \vartheta(w)$ implying that the minimum q^* is global.

Definition 1.9 ([3], Definition 2.8)**.** The epigraph of a function ϑ denoted as epi(ϑ) is the set of points which are greater or those that are equal to the images of ϑ i.e epi$(\vartheta) = \{(q, r) \in dom(\vartheta) \times \mathbb{R} : r \geq \vartheta(q)\}$.
The hypograph of ϑ denoted as hypo(ϑ) is therefore given by hypo$(\vartheta) = \{(q, r) \in dom(\vartheta) \times \mathbb{R} : r < \vartheta(q)\}$.

Remark 1.10. The function ϑ is convex if and only if epi(ϑ) is convex and a function is *usc* if and only if hypo(ϑ) is closed.

Definition 1.11 ([30], Definition 1.5)**.** Let $(\mathcal{L}, \mathfrak{X}, \mu)$ be a measure space and a number p be given such that $1 \leq p < \infty$. Then an L^p space which consists of measurable functions is defined as $L^p(\mathcal{L}, \mathfrak{X}, \mu) = \{\vartheta : \vartheta \text{is measurable}(and) \int_{\mathcal{L}} |\vartheta|^p d\mu < \infty\}$. The L^p-norm of $\vartheta \in L^p(\mathcal{L})$ is given by

$$\|\vartheta\|_p = \left(\int_{\mathcal{L}} |\vartheta|^p d\mu\right)^{\frac{1}{p}}.$$

The following proposition asserts that $(\mathcal{L}, \|.\|)$ is indeed an $L^p(\mathcal{L}, \mathfrak{X}, \mu)$-

space.

Proposition 1.12. *Let $L^p(\mathcal{L}, \mathfrak{X}, \mu)$ denote an L^p-space on a measure space $(\mathcal{L}, \mathfrak{X}, \mu)$ with $1 \leq p < \infty$. Let $\|.\|_p$ be an L^p norm. Then $(\mathcal{L}, \|.\|_p)$ is an $L^p(\mathcal{L}, \mathfrak{X}, \mu)$-space on $(\mathcal{L}, \mathfrak{X}, \mu)$.*

Proof. We need to show that $\|.\|_p$ is a norm on $L^p(\mathcal{L}, \mathfrak{X}, \mu)$. Let ϑ be a measurable function in $\mathcal{L}$. From the definition of L^p norm, we have, $\|\vartheta\|_p = \left(\int_{\mathcal{L}} |\vartheta|^p d\mu\right)^{\frac{1}{p}}$. We proceed to show that $\|\vartheta\|_p$ satisfies the three axioms for a norm below:

(i). $\|\vartheta\|_p \geq 0$ and $\|\vartheta\|_p = 0 \iff \vartheta = 0$. By absoluteness property, $|\vartheta|^p \geq 0$ implying that $\int_{\mathcal{L}} |\vartheta|^p d\mu \geq 0$, hence $\left(\int_{\mathcal{L}} |\vartheta|^p d\mu\right)^{\frac{1}{p}} \geq 0$ showing that $\|\vartheta\|_p \geq 0$.
Let $\|\vartheta\|_p = 0$, then $\left(\int_{\mathcal{L}} |\vartheta|^p d\mu\right)^{\frac{1}{p}} = 0$ implying that $|\vartheta|^p = 0$ hence $\vartheta = 0$.
Conversely suppose $|\vartheta|^p = 0$, then $\int_{\mathcal{L}} |\vartheta|^p d\mu = 0$ implying that $\left(\int_{\mathcal{L}} |\vartheta|^p d\mu\right)^{\frac{1}{p}} = 0$. Thus, $\|\vartheta\|_p = 0$.

(ii). $\|\kappa\vartheta\|_p = |\kappa|\|\vartheta\|_p, \forall \kappa \in \mathbb{K}$. Now $\|\kappa\vartheta\|_p = \left(\int_{\mathcal{L}} |\kappa\vartheta|^p d\mu\right)^{\frac{1}{p}} = \left(\int_{\mathcal{L}} |\kappa|^p |\vartheta|^p d\mu\right)^{\frac{1}{p}} = |\kappa| \left(\int_{\mathcal{L}} |\vartheta|^p d\mu\right)^{\frac{1}{p}}$. Thus, $\|\kappa\vartheta\|_p = |\kappa|\|\vartheta\|_p$.

(iii). $\|\vartheta + \varphi\|_p \leq \|\vartheta\|_p + \|\varphi\|_p, \forall \vartheta, \varphi \in \mathcal{L}$. Let $\vartheta, \varphi \in \mathcal{L}$, then $\|\vartheta + \varphi\|_p = \left(\int_{\mathcal{L}} |\vartheta + \varphi|^p d\mu\right)^{\frac{1}{p}}$. So by Minkowski's inequality we have $\left(\int_{\mathcal{L}} |\vartheta + \varphi|^p d\mu\right)^{\frac{1}{p}} \leq \left(\int_{\mathcal{L}} |\vartheta|^p d\mu\right)^{\frac{1}{p}} + \left(\int_{\mathcal{L}} \varphi|^p d\mu\right)^{\frac{1}{p}}, \forall p \geq 1$. This is equivalent to $\|\vartheta + \varphi\|_p \leq \|\vartheta\|_p + \|\varphi\|_p$.
Since all axioms have been satisfied we conclude that $\|.\|_p$ is a norm on $L^p(\mathcal{L}, \mathfrak{X}, \mu)$ and therefore the ordered pair $(\mathcal{L}, \|.\|_p)$ is an

$L^p(\mathcal{L}, \mathfrak{X}, \mu)$-space on $(\mathcal{L}, \mathfrak{X}, \mu)$.

$\square$

Remark 1.13. Without loss of generality,

(i). This study takes $(\mathcal{L}, \|\vartheta\|_p)$ to be an L^p-space.

(ii). Throughout this study $\mathcal{L}$ denotes an L^p-space unless it is stated otherwise.

Definition 1.14 ([44], Definition 2.5)**.** A point $q^* \in \mathbb{R}^n$ is termed as a global minimizer of the program $\min_{q\in Q} \vartheta(q)$, for $Q \subseteq \mathbb{R}^n$ if $\vartheta(q^*) \leq \vartheta(q), \forall q \in Q$ and $q^* \in Q$.

Definition 1.15 ([44], Definition 2.6)**.** A point $q^* \in \mathbb{R}^n$ is termed as a local minimizer of the program $\min_{q\in Q} \vartheta(q)$, for $q^* \in Q$ and there exists $\xi > 0$ such that $\vartheta(q^*) \leq \vartheta(q), \forall q \in Q$ whenever $q \in Q$ satisfies $\|q - q^*\| \leq \xi$.

Definition 1.16 ([2], Definition 4.3.2)**.** Let $a \in \mathbb{R}^n$ be a vector and $\mathcal{L} \neq \emptyset$ be a set. Then a is a feasible direction at $q \in \mathcal{L}$ if $\exists \eta_0 > 0$ satisfying $(q + \eta a) \in \mathcal{L}, \forall \eta \in [0, \eta_0]$.

Definition 1.17 ([37], Definition 2.20)**.** Let $Q \subseteq \mathcal{L}$ denote a set of constraints where $\mathcal{L}$ is an L^p-space. Letting Q be regular, the function $\vartheta : \mathcal{L} \to \mathbb{R} \cup (-\infty, \infty)$ is said to be Lipschitz-continuous if $\exists \mathcal{F} \in \mathbb{K} : \forall r \in \mathcal{L}, \|\vartheta(q + r) - \vartheta(q)\| \leq \mathcal{F}\|(q + r) - q\|$.

Definition 1.18 ([6], Definition 3.1)**.** Let $(\mathcal{L}, \|.\|_p)$ be an L^p-space. Suppose $G \subseteq \mathcal{L}$ is open. Then $\vartheta : G \to \overline{\mathbb{R}}$ is Gâteaux differentiable at $q \in G$ if $\forall r \in G$, $\vartheta'(q, r) = \lim_{\kappa \to 0} \frac{\vartheta(q+\kappa r)-\vartheta(q)}{\kappa}, \forall \kappa \in \mathbb{K}$. Here, $\vartheta'(q, r)$ is termed as Gâteaux-variation of q with respect to r.
Given that $\forall r \in \mathcal{L}, \vartheta'(q, r) = \lim_{\kappa \to 0} \frac{\vartheta(q+\kappa r)-\vartheta(q)}{\kappa}$ is uniform in $q \in G$ then ϑ is uniformly Gâteaux differentiable on G.

Definition 1.19 ([6], Definition 3.2)**.** A function ϑ from a subset G of an L^p-space $\mathcal{L}$ is Fréchet differentiable at $q \in G$ if a bounded operator $P : \mathcal{L} \to \mathcal{L}$ exists that satisfies $\lim_{\|\kappa\| \to 0} \frac{\|\vartheta(q+\kappa r)-\vartheta(q)-P\kappa\|_p}{\|\kappa\|_p} = 0$. We define the Fréchet derivative of q with respect to r by $\vartheta'(q, r) = \lim_{\kappa \to 0} \frac{\vartheta(q+\kappa r)-\vartheta(q)-P\kappa}{\kappa}$

Definition 1.20 ([20], Definition 3.7)**.** A convex function ϑ from a normed linear space $\mathcal{Q}$ to the extended real line $\overline{\mathbb{R}}$ is called proper if $\vartheta(q) > -\infty, \forall q \in \mathcal{Q}$ or $\vartheta(q_0) < +\infty$ for some $q_0 \in \mathcal{Q}$.

Definition 1.21 ([13], Definition 4.3)**.** A function ϑ from a normed linear space $\mathcal{Q}$ to the extended real line $\overline{\mathbb{R}}$ is called coercive if $\vartheta(q) \to +\infty$ as $\|q\|_{\mathcal{Q}} \to +\infty$ for every $q \in \mathcal{Q}$. If the limit of a continuous function ϑ is given as $\lim_{\|q\| \to \infty} \vartheta(q) = \infty$ then, ϑ is coercive.

Remark 1.22 ([13], Definition 4.4)**.** A coercive function acting on a closed set must attain a global minimum.

In this study we considered convex optimization problems (COPs) in the following functional form:

$$Min_{q \in \mathcal{Q}} \vartheta(q) \tag{1.2.1}$$

subject to

$$\begin{aligned} \varphi_i(q) &\leq 0, 1 \leq i \leq t &(1.2.2)\\ \varrho_j(q) &= 0, 1 \leq j \leq v, &(1.2.3) \end{aligned}$$

where $q \in \mathcal{Q}$ is the optimization variable and the set $\mathcal{Q}$ is convex;
the COP 1.2.1 is convex;
the inequality constraints 1.2.2 are convex functions
and the equality constraints 1.2.3 are convex functions.
A solution to the COP 1.2.1 satisfying 1.2.2 and 1.2.3 is any point $q^* \in \mathcal{Q}$ attaining

$$\inf\{\vartheta(q) : q \in \mathcal{Q}\}.$$

An optimization variable $q \in \mathbb{R}^n$ is feasible only if $q \in \mathcal{Q}$ satisfies all constraints in 1.2.2 and 1.2.3
A feasible solution q^* is termed as globally optimal if $\vartheta(q) \geq \vartheta(q^*)$ holds for every feasible q. A vector $\overline{q}$ is said to be locally optimal if it is feasible and if $\exists \eta \geq 0 : \vartheta(q) \geq \vartheta(\overline{q})$ for any feasible q satisfying $\|q - \overline{q}\| \leq \eta$.

Definition 1.23. Let $\mathcal{Q}$ be a directed set and let $\leq$ be a partial order relation in $\mathcal{Q}$ such that $\forall q_1, q_2 \in \mathcal{Q}$, there is $p \in \mathcal{Q}$ such that $q_1 \leq p$ and $q_2 \leq p$. Then a Moore-Smith sequence is a function which assigns every element $q \in \mathcal{Q}$ a unique number q_j for each $j \in \mathcal{Q}$.

In the following section the problem of the study is stated.

1.3 Statement of the problem

Characterization of lower semi-continuous functions and upper semi-continuous functions has been done in variuous spaces for instance topological spaces, Hilbert spaces and normed spaces. However, limited research on *lsc* and *usc* functions in L^p-spaces has been done in terms of their characterizations. Hernández and López [25] characterized semi-continuous functions in metrizable topological spaces. They showed that a function in a metrizable topological space is *lsc* if its sub level set is closed and it is *usc* if its sub level set is open. Hernández and López [25] left an open problem for research in characterization of upper and lower semi-continuous functions in real valued function spaces which include the L^p spaces. Hence, this formed the basis of this study. While Gool [22] used continuous lattices in characterizing *lsc* functions in topological spaces this study employed the use of Moore-Smith sequences in characterizing upper and lower semi-continuous functions in L^p-spaces. Convex optimization conditions in Hilbert spaces and general normed spaces have been established. Peypouquet [45] established conditions for convex optimization in normed spaces and showed that if a convex function is *lsc* then it attains a minimizer on a bounded set in a normed space. Peypoquet [45] used Lagrange multipliers and Fenchel conjugates to establish optimality conditions, while this study employed the use of KKT optimality conditions. Since scanty research has been directed to establishing conditions for convex optimization in L^p spaces, the L^p-space therefore formed an interesting space for convex optimization since we used measurable functions instead of vectors and the finite L^p norm structures of these functions presented excellent boundedness property appropriate for constrained optimization. Thus, in this study we considered semi-continuous functions and convex optimization in L^p spaces.

1.4 Objectives of the study

1.4.1 Main objective

The main objective of this study is to characterize lower and upper semi-continuous functions and convex optimization in L^p-spaces.

1.4.2 Specific objectives

The specific objectives of the study include:

(i) Characterize lower semi-continuous functions in L^p spaces.

(ii) Characterize upper semi-continuous functions in L^p spaces.

(iii) Establish conditions for convex optimization in L^p spaces.

1.5 Significance of the study

Results from this study are a contribution of knowledge in the field of functional analysis particularly in measure theory. They may also be applicable in mathematical analysis specifically in norm approximation which plays a vital role in noise reduction in communication channels, image and signal processing, data compression and storage. In addition it will form a basis for future research.

Chapter 2

LITERATURE REVIEW

2.1 Introduction

We review literature on semi-continuous functions, linear optimization, non linear optimization and convex optimization.

2.2 Semi-continuous functions

Many Mathematicians have investigated *lsc* and *usc* functions including [5], [12], [15], [22] and [34]. Beer [5] applied Stone Approximation of *usc* mappings on metric spaces and investigated the compactness property in metric spaces. He extended Dini's theorem to characterize sequences of *usc* functions that converge uniformly. Beer [5] illustrated that, for an interior point z, (z, α) forms the epigraph of ϱ whenever $\alpha > \varrho(z)$. This showed that the interior of the convex function ϱ is *usc* throughout its domain. It was proved that, the supremum of the sequence of affine functions represent a *lsc* convex function. Below are some of Beer's main findings:

Lemma 2.1 ([5], Lemma 1.3)**.** *If a function g is usc on a compact metric space (M, d). Let $U(M)$ denote the set of all bounded usc functions on M and $\overline{g}$ denote the closure of the function g. Define an upper λ-parallel function of g as $g^+_\lambda(z) = \sup z : (z, r) \in B_\lambda(\overline{g}), \forall z \in M, \forall r \in \mathbb{R}$. Then $\forall z \in M$, $\lambda > 0$, the function g^+_λ is in $U(M)$.*

Lemma 2.1 Beer [5] characterized upper semi-continuity in compact metric spaces but this study characterized semi-continuity of functions in L^p spaces.

Theorem 2.2 ([5], Theorem 1)**.** *Let Ω be a lattice of usc functions and M be a compact metric space. Define a metric d_3 by $d_3 = D(hypo\psi, hypo\Omega)$.*

(i). For a usc function ψ, $\exists\{\psi_n\} \in \Omega$ tending to ψ from above in the metric d_3

(ii). The sequence $\{\psi_n\}$ can only converge uniformly to ψ if ψ is a continuous function.

Theorem 2.2 characterizes upper semi-continuous functions in terms of sub-lattices that are upper dense on compact metric spaces. Beer [5] characterized upper semi-continuous functions in compact metric spaces and Hausdorff spaces using Dini's theorem and Stone Approximation theorem. Sequences of upper semi-continuous functions were characterized in terms of sub-lattices. The current study characterized lower and upper semi-continuous functions and the sequences were in terms of the Moore-Smith sequences.

Gool [22] studied lower semi-continuity and continuous lattice. The results obtained extended some analytical properties of *lsc* functions to *lsc* functions whose domain is a continuous lattice. Some results were applied in potential theory to describe solutions of systems of differential equations through generalizing semi-continuity of such functions. Some of Gool's findings are:

Theorem 2.3 ([22], Theorem 4.4). *For a topological space* (Y, τ), *if a continuous lattice* $T \subset Y$ *exists, then:*

(i) if $T \subset lsc(Y)$, *then the supremum of* T *is lsc in* Y.

(ii) if ψ *and* ϱ *are lsc functions then* $\psi \wedge \varrho$ *is lsc in* Y.

Theorem 2.3 characterizes *lsc* functions in a topological space. It shows that the supremum of a lattice of *lsc* functions in a topological space is *lsc*.

Lemma 2.4 ([22], Lemma 5.2). *Let* Y *be a topological space and* $lsc(Y)$ *denote a collection of all lower semi-continuous functions on* Y. *Let* $NUM(Y)$ *be a collection of functions on* Y *and let* $T \subset Y$ *be a continuous lattice. Suppose* $lsc(Y) = \{NUM(Y) : g = \sup\{h \in Y : h \leq g\}\}$. *Then* Y *is completely regular when* $T \neq 1$.

Lemma 2.4 gives an investigation on *lsc* functions in topological spaces using continuous lattices.

In [15] Correa and Hantoute related the argmin sets of a given function and its semi-continuous convex hull using characterizations involving asymptotic functions. They came up with explicit formulas for Fenchel sub-differential and the argmin sets of successful Legendre-Fenchel conjugates of functions with real values. The two results below represent Correa and Hantoute's work:

Proposition 2.5 ([15], Proposition 1)**.** *Let V be a real Hausdorff locally convex space (HLCS.) Given a function $\psi : V \to \overline{\mathbb{R}}$, $v \in V$ and $L \subset V^*$ be given. Then, $\partial_L^r \psi(v) = L \cap \partial\psi(v)$ in each of the following:*

(i) If $\exists F^* \in cl(L \cap dom\psi^*)$, then it holds that

$$\psi(v) \leq \liminf_{z \rightharpoonup v} (\psi(z) - \langle z - v, F^* \rangle)$$

(ii) $L = V^*$, $int(dom\psi^*) \neq \emptyset$, and ψ is weakly lsc.

Proposition 2.5 characterizes lower semi-continuous convex relaxation in optimization on infinite-dimensional real locally convex space.

Theorem 2.6 ([15], Theorem 3)**.** *Let a function $\psi : V \to \bar{\mathbb{R}}$ be a positively homogeneous such that $V^* \in \Im(\psi)$. Then for every $v^* \in V^*$ the statements below hold:*

(i) $\partial\psi^*(v^*) = \overline{co}((\partial^r\psi)^{-1}(v^*))$;

(ii) if $int(dom\psi^*) \neq \emptyset$, and ψ is weakly lsc, then $\partial\psi^*(v^*) = co((\partial^r\psi)^{-1}(v^*))$;

(iii) if in (ii) we assume $V = \mathbb{R}^n$ then $\partial\psi^*(v^*) = co((\partial\psi)^{-1}(\psi^*))$.

Theorem 2.6 characterizes semi-continuity with respect to positive homogeneity and boundedness of a function. This result is limited to infinite dimensional real locally convex spaces and not general L^p spaces. Correa and Hantoute [15] investigated lower semi-continuous convex relaxation in optimization on infinite-dimensional real locally convex space but this study carried out a characterization of lower and upper semi-continuous functions in L^p spaces. Convex conjugates and sub-differentials were used but this study employed the use of L^p norms

Mitter [35] studied convex lower semi-continuous functions in Hausdorff spaces. The following proposition represents key notions of Mitter's optimization using lower semi-continuous functions as a tool:

Proposition 2.7 ([35], Proposition 2)**.** *Let V be a $HLCS$ and $\psi : V \to \overline{\mathbb{R}}$ be lsc function that is convex. Define the recession function ψ_∞ by $\psi_\infty = \sup_{r \in Dom\psi^*} \langle z, r \rangle, \forall z \in Dom\psi$. Then,*

(i). $\psi_\infty(z) = \min\{r \in R : (z, r) \in (epi\psi)_\infty\}$

(ii). $\psi_\infty(z) = \sup_{a \in Dom\psi} \sup_{t>0}[\psi(a + tz) - \psi(a)]/t$

(iii). $\psi_\infty(z) = \sup_{t>0}[\psi(a + tz) - \psi(a)]/t \ \forall a \in Dom\psi$

(iv). $\psi_\infty(z) = \sup_{a \in Dom\psi}[\psi(a + z) - \psi(a)]$

(v). $\psi_\infty(z) = \sup_{w \in Dom\psi^*} \langle z, w \rangle$.

In (i)., since the set $\{(epi\psi)_\infty\}$ is closed, then $\psi_\infty(z)$ attains its minimum if $epi\psi)_\infty \neq +\infty$.

Proposition 2.7 gives an investigation semi-continuous functions and convex optimization in Hausdorff locally convex infinite dimensional spaces and not semi-continuous functions and convex optimization in L^p spaces. Mitter's [35] involved characterization of lsc functions in Hausdorff spaces and applying the result in convex optimization in Hausdorff spaces. The current study characterized lower and upper semi-continuity of functions in L^p-spaces and used the result in establishing conditions for convex optimization in L^p-spaces.

2.3 Optimization

In Mathematics optimization is the process of finding the optimal value (minimum or maximum) of a function which represents the best solution of a given problem depicted by the function. It simply refers to finding the minimizer or maximizer of the function.

2.3.1 Linear optimization

Linear programming has been studied by many mathematicians including [24] and [28]. It involves minimizing a linear function representing an optimization problem while considering linear inequalities constraints and linear equalities constraints. Most methods and algorithms that are used to solve non-linear optimization problems and convex programs were first developed to solve LP problems. Dantzig's Simplex method that solved LP problems in a large scale was later extended to solve a variety of $NLOP$s. Hancock [24] studied maxima and minima and linear optimization but did not consider convex optimization nor use convex analysis tools in optimization.

A defining milestone in linear optimization was the invention of polynomial-time algorithm by Karmarkar [28]. It emerged to be better than the ellipsoid algorithm in terms of running time. The following are some of Karmarkar's main results:

Theorem 2.8 ([28], Theorem 2)**.** *Either (i)* $C^T z^{(k+1)} = 0$ *or (ii)* $\psi(z^{(k+1)}) \leq \psi(z^{(k)}) - \delta$ *such that* δ *depends on* σ*. A particular selection that works: If* $\sigma = \frac{1}{4}$*, then* $\delta \geq \frac{1}{8}$.

Theorem 2.8 shows how the polynomial time interior-point method for LP, developed by Karmarkar [28], represents underlying conceptual sequence of operations in solving linear optimization problems.

Theorem 2.9 ([28], Theorem 3)**.** *Let* Ω' *be a null space. Define* Ω'' *to be the transformed space of* Ω' *by* $\Omega'' = \{\alpha : B(q_0 - \alpha) = 0\}$*. Then* $\exists b' \in B(q_0, \alpha r) \cap \Omega'' :$
either (i) $C'^T b' = 0$
or (ii) $\psi'(b') \leq \psi'(q_0) - \delta$ *where* δ *is a constant.*

Theorem 2.9 proves the existence minimizers for a linear optimization problem. These interior point methods developed by Karmarkar [28] to solve linear optimization problems were widely applied in engineering and operations research. Later these methods were later extended to solving convex optimization problems. Karmarkar's work was exclusively on linear optimization, and it was only the finite dimensional real spaces that were considered. It was the interest of our study to establish conditions for convex optimization in L^p-spaces.

2.3.2 Non-linear optimization

Non-linear optimization involves minimizing (or maximizing) optimization problems with non linear objective functions. A key development in non-linear optimization was realized through the coining of Kurush-Kuhn-Tucker KKT optimality conditions for non-linear programming by Kurush, Kuhn and Tucker.

Sharma and Hunachew [53] generalized the Simplex Method to non-linear programs by presenting the methods of feasible directions and developed an algorithm method of finding feasible directions at each iteration, and optimizing along the feasible direction. Sharma and Hunachew [53] worked on methods of finding the feasible directions in solving non-linear optimization particularly quadratic programming using improving directions. Their main findings include:

Lemma 2.10 ([53], Lemma 1.2)**.** *Suppose z minimizes the $NLOP$*
min $\psi(z)$
subject to '$Cz \leq d$ and $Pz = p$.
Supposing C^T can be decomposable into (C_1^T, C_2^T) and that d^T is decomposable into (d_1^T, d_2^T), let $C_1 z = d_1$ and $C_2 z < d_2$. Then there is vector $\boldsymbol{a} \neq 0$ forming the feasible direction at z iff $C_1\boldsymbol{a} \leq 0$ and $P\boldsymbol{a} = 0$. Furthermore, $\boldsymbol{a}$ is said to be an improving feasible direction if $\nabla\psi(z)^T\boldsymbol{a} < 0$.

Lemma 2.10 shows that at each iteration in finding feasible directions, the Zoutendijk method generates an improving feasible direction and then optimizes along that direction. The optimization problem under consideration is quadratic.

Lemma 2.11 ([53], Lemma 1.3). *Suppose z minimizes the $NLOP$, min $\psi(z)$; subject to $Cz \leq d$ and $Pz = p$.*
Let z is a feasible solution satisfying $C_1 z = d_1$ and $C_2 z = d_2$ such that $C^T = (C_1^T, C_2^T)$ and $d^T = (d_1^T, d_2^T)$, then z is a KKT point if and only if $\inf_z \psi(z) = 0$.

Lemma 2.11 shows that if the minimal objective function value is zero, then z is a KKT point. This objective function is quadratic. In our study we considered convex objective functions.
Sharma and Hunachew [53] compared the functionality of the Zoutendijk's algorithm and the Successive Quadratic Programming (SQP) technique to solve $NLOP$s but this study considered COPs in L^p-spaces.
Svanberg [54] worked on non-linear optimization and introduced special class of methods of optimization with global limits. In this way a variety of inequality-constrained $NLOP$s were efficiently solved. The optimization methods were applicable to problems consisting of very many variables even when the Hessian matrices forming the NLP functions and the feasible set are dense because of the use of separable approximations. It was observed that each iteration point that it generated yields minima strictly lower than the *Karush-Kuhn-Tucker* points. The following are some important findings obtained by Svanberg [54]:

Lemma 2.12 ([54], Lemma 7.3). *$\forall i = 0, 1, ..., m$, $\exists \rho_i^{max} < \infty$ satisfying $\rho_i^{(r)} \leq \rho_i^{max}$ for any outer iteration r.*

In Lemma 2.12 it is observed the iterations are of finite number. This method is used to find solutions to inequality constrained non-linear optimization problems but not convex optimization problems whose con-

straints are convex functions.
Svanberg [54] therefore worked on approximations of COPs that are conservatively separable. Global convergence was theoretically proved and these methods were demonstrated to work numerically.
However, Svanberg's [54] researched focused on solving $NLOP$s using Hessian matrices but not the study of semi-continuous functions and convex optimization in L^p spaces as in the current study.

Friedlander, Martinez and Raydan [21] used the method developed by Barzilai and Borwein in minimizing convex quadratic functions of many variables and having box constraints. The main optimization problem which was solved was stated in the form:

$$Min\psi(z) \equiv \frac{1}{2}z^T A_z - b^T z.$$

The work of Friedlander, Martinez and Raydan [21] includes the result given below:

Lemma 2.13 ([21], Lemma 2.1)**.** *Consider the optimization problem min* $\psi(z) = \frac{1}{2}z^T Az - b^T z$ *where* A *is a positive definite square matrix and* $b \in ran(A)$*. Then* $\widehat{z}$ *is a minimizer of* $\psi(z)$ *if and only if there is* $\alpha_j, 1 \leq j \leq \ell - 1$*, satisfying*

$$\widehat{z} = \alpha_1 v_1 + ... + \alpha_{\ell-1} v_{\ell-1} + \frac{\beta_\ell}{\lambda_\ell} + ... + \frac{\beta_n}{\lambda_n} v_n$$

where $\beta_\ell, \ell \leq j \leq n$ *are defined by* $b = \beta_\ell v_\ell + \beta_{\ell+1} v_{\ell+1} + ... + \beta_n v_n$.

Lemma 2.13 characterizes the minimizers of the convex quadratic function $\psi(z)$. This result is based on convex quadratic functions only but not the general convex functions.

Friedlander, Martinez and Raydan [21] solved convex quadratic optimization problems which are box constrained but not general convex functions in L^p spaces. Barzilai-Borwein Method was used and not semi-continuous functions.

2.3.3 Convex optimization

In convex optimization the objective functions together with constraint sets must be convex. Convexity guarantees that a global optimum coincides with a local optimum thus the convex optimum value is unique. Again the minimum/maximum of convex problems is convex. Many optimization problems can easily be formulated as convex functions. Convex optimization is now prevalent in use compared to other optimization methods owing to the fact that it is easy to attain a global or local minimum (or maximum) of its function. Thus many studies have been conducted in convex optimization.

Rockafeller [50] characterized convexity in relation to continuity and differentiability of functions that are convex. Systems of inequalities, Lagrange multipliers and minimax theorems were investigated in relation to convexity. Rockafeller [50] defined inner products of convex sets as the external points of the duality theorem. This established a basis for a generalization that convex bi-functions are similar to linear transformations. The results Rockafeller obtained are summarized as follows:

Theorem 2.14 ([49], Theorem 7.1)**.** *If* $\psi : \mathbb{R}^n \to \overline{\mathbb{R}}$ *is a function, then the following conditions are equivalent:*
i). ψ *is lsc throughout* $\mathbb{R}^n$*.*
ii). $\{z \in \psi(z) \leq \eta, \forall z \in \mathbb{R}^n\}$ *is closed* $\forall \eta \in \mathbb{R}$*.*
iii). The set epi(ψ) *is a closed set in* $\mathbb{R}^{n+1}$*.*

In Theorem 2.14 *lsc* functions are applied in the study of convexity of functions. The functions were acting on $n-$tuples of real numbers.

Theorem 2.15 ([49], Theorem 16.3)**.** *Let* $P : \mathbb{R}^n \to \mathbb{R}^m$ *be a linear transformation. For a convex function* $\psi \in \mathbb{R}^n$*, we have*

$$(P\psi)^* = \psi^* P^*.$$

For any function $\varphi \in \mathbb{R}^m$*, one has*

$$((\overline{\varphi})P)^* = \overline{(P^* \varphi^*)}.$$

If $\exists z : Pz \in ri(dom\varphi)$*, then*

$$(\varphi P)^*(z^*) = \inf\{\varphi^*(w^*) : P^* w^* = z^*\},$$

reaching the infimum at z^**.*

Theorem 2.15 shows that with functional operations two linear transformations become dual to each other.
Rockafellar [49] investigated convexity of sets and functions operating on $\mathbb{R}^n$. Rockafellar devoted this work towards convex analysis in real spaces. Boyd [10] worked on convex optimization of graph Laplacian eigenval-

ues. The optimization objective involved choosing the edge weights of a graph and then minimizing/maximizing some function of the eigenvalues the Laplacian matrix associated with it. The constraints of these weights included non negativity and given total values. Boyd [10] equivalently posed this problem as concave functions maximized over a convex set. Fundamental optimality conditions were presented and interesting dual problems derived. In some cases analytical solutions to the problems were given and in all cases numerical solutions were efficiently computed. For medium sized optimization problems interior point methods were employed and for larger problems sub-gradient-based methods were used while putting into consideration the structure and symmetry of the problems in solving them.

Boyd [10] worked on COP using eigenvalues of Laplacian matrices. The COPs that were considered were represented as concave functions. The current study solved COPs in L^p-spaces which were formulated as convex functions.

Mitter [35] worked on convex programming using duality concepts. Conditions for the duality formalism to hold were set. One essential condition was that the dual feasible set must satisfy local compactness. The convex sets which are locally compact have a non-empty interior that corresponds to polar sets. Mitter [35] used convex analysis techniques and duality approach in Hausdorff locally compact spaces($HLCS$). Some of Mitter's findings are as follows:

Theorem 2.16 ([35], Theorem 2)**.** *Suppose $\psi : \mathcal{L} \to \overline{\mathbb{R}}$ is a convex function on $(\mathcal{L})$ such that $\forall z_0 \in \mathcal{L}, \psi(z_0) < \infty$. Then the statements below are equivalent:*

(i). $\partial\psi(z_0) \neq \emptyset$.

(ii). $\exists$*o-nbhd*$N : \inf_{z\in N} \psi'(z_0; z) > -\infty$.

(iii). $\exists$*o-nbhd*N, $\eta > 0$ *satisfying* $\inf_{z\in N} \frac{\psi(z_0+rz)-\psi(z_0)}{r} > -\infty$ *for any* $0 < r < \eta$.

(iv). $\lim_{z\to 0} \inf \psi'(z_0; z) > -\infty$.

(v). $\lim_{z\to 0^+} \frac{\psi(z_0+rz)-\psi(z_0)}{r} > -\infty$.

Theorem 2.16 examines continuity and boundedness of convex functions in Hausdorff locally convex spaces and not convex optimization in L^p space.

Theorem 2.17 ([35], Theorem 6)**.** *Suppose* $Y_0 = Y(z_0) \geq \inf_{z_0} \sup_z \ell(z_0, z)$ *is finite. Define* Q_0 *by* $Q_0 = Y^*(z_0)$ *where* $Y^*(z_0) = \sup_z\{\langle z_0, z\rangle - Y(z)\}$. *Then the statements below are equivalent:*

(i). Q_0 *has solutions.*

(ii). $\partial Y(z_0) \neq \emptyset$.

(iii). $\exists \widehat{z} \in G : Y_0 = \langle z_0, \widehat{z} \rangle - Y^*(-G^* \widehat{z}, \widehat{z})$

Should $Y(.)$ be convex, then each statement above is equivalent to:

(iv). $\exists$*o-nbhdN* $: \inf_{z \in N} Y'(z_0; z) > -\infty$

(v). $\liminf_{z \to 0} Y'(z_0; z) > -\infty$

(vi). $\lim_{z \to 0^+} \inf_{r \to 0} \frac{Y(z_0+rz)-Y_0}{r} \equiv \sup_{N=o-nbhd} \inf_{r>0} \inf_{z \in N} \inf_{e \in U} \frac{Q(e,Le+z_0+rz)-Y_0}{r} > -\infty$

If $PY(.)$ is convex and $\mathcal{L}$ is a normed space, then the above are equivalent to:

(vii). There exists $\varepsilon > 0, M > 0 : Q(e, Le + e_0 + z) - Y_0 \geq -M|z| \forall z \in U, |z| \leq \varepsilon$.

Moreover if (1) is true then $\widehat{y}$ solves D_0 if and only if $\widehat{q} \in \partial(z_0)$, and $\widehat{e}$ is a solution for Y_0 if and only if there is a $\widehat{q}$ satisfying any of the conditions $(1'), (3')$ *below. The following statements are equivalent:*

(i). *$\widehat{e}$ solves Y_0, $\widehat{q}$ solves Q_0, and* $Y_0 = Q_0$

(ii). $F(\widehat{u}, L\widehat{u} + x_0 = \langle z_0, \widehat{q} \rangle) - G^*(-L^*\widehat{q}, \widehat{q})$.

(iii). $(-L^*\widehat{q}, \widehat{q}) \in \partial(\widehat{e}, L\widehat{e} + z_0)$.

Theorem 2.17 summarizes results on duality approach to solving convex optimization problems. This work was centered on Hausdorff locally convex spaces and not in L^p spaces and not study convex optimization in L^p spaces.

Mitter [35] conducted a study on convex optimization in Hausdorff locally convex spaces. This study utilized the duality approach and the optimization tools were conjugate functions. The findings proved that the optimal value changed as constraints' perturbations changed following feasible directions. This holds if conditions for duality, compactness are met by the feasible set which is convex. Our current study on convex optimization took into account the semi-continuity aspect and is based on L^p spaces. Instead of the duality approach and conjugate functions, our study focused on the use of semi-continuous functions and L^p-norms to investigate the properties of convex optimization in these spaces.

Neto and De Pierro [42] worked on non-smooth convex optimization in Euclidean spaces using methods for convex feasibility together with $\epsilon-$sub-gradient methods. A new Unified framework that solved constrained COPs was developed involving incremental sub-gradients. Applying this unifying framework method new algorithms which easily computed large-scale optimization problems especially the ones in tomographic imaging

were developed and proved. Approximate projection was used in place of Euclidean projection in order to efficiently handle large-scale optimization problems. The algorithms were derived from incremental sub-gradients and also from aggregated incremental sub-gradients. The convergence properties for these algorithms were proved. The convex optimization problem was stated as below:
Find $z \in \mathbb{R}^q$:

$$z \in \arg\min \psi(z) = \sum_{i=1}^{m} \psi_i(z) : z \in \mathcal{M},$$

An algorithm for the above problem was given as:

$$z_{k+\frac{1}{2}} = \mathfrak{O}_{\psi}(\eta_k, z_k);$$

$$z_{k+1} = \mathfrak{F}_{\mathcal{M}}(z_{z+\frac{1}{2}}),$$

The following are some of the results obtained:

Proposition 2.18 (42, Proposition 2.2)**.** *Suppose the sequence* $\{\eta_k\}$ *satisfies* $\eta_k \to 0, \eta \geq 0, \sum_{k=0}^{\infty} \eta_k = \infty$, *then we have*

$$\lim_{k\to\infty} \inf \psi(z_k) \leq \inf_{z\in\mathcal{M}} \psi(z).$$

Neto and De Pierro [42] studied constrained convex optimization in Euclidean spaces and not constrained convex optimization in L^p spaces. Incremental sub-gradient methods and property of boundedness were used and not semi-continuous functions in L^p spaces.

Zhu and Martínez [58], the multi-agent COPs are solved in topological spaces. Specifically, they minimized a function that is convex as a global program and subject to a set of convex constraints. These inequality and equality constraints, however, are known to the whole set of agents. The authors consider two main cases: one in which there is no equality constraint, and another in which all agents have identical local constraint sets. To solve these optimization problems, Zhu and Martínez [58] propose two algorithms that can be implemented over a network of agents with dynamically changing topologies. Both algorithms make use of distributed primal-dual sub-gradients to characterize the minimizers as points of the Lagrangian dual value functions. For the case whose equality constraint set is empty, the authors use a Lagrangian relaxation method, while for the case with identical local constraint sets, they adopted a relaxation of penalty type to do away with the extra constraint. The optimization problem was thus defined as:

$$Min_{q\in\mathbb{R}^n}\sum_{j=1}^{L}\xi_j(q),$$

Zhu and Martínez [58] focused on the optimization of multi-agent systems in topological spaces and not in L^p spaces. To achieve this, they used techniques such as primal-dual sub-gradients, rather than semi-continuous functions that we have employed. The main methods employed in their research were the use of Lagrangian relaxation and penalty relaxation approaches. This study focused on convex optimization in L^p-spaces employing the use of KKT optimality conditions to establish convex optimization conditions in L^p-spaces.

There has been a significant amount of research focused on optimization in Hilbert spaces, with notable contributions from studies such as [26], [4] and [44]. Alexanderian [1] focused on determining the real Hilbert space minimizers for convex functionals. The study employed use of lsc functions and convex functions in optimization. Some of the main findings that Alexanderian obtained from his study include:

Theorem 2.19 (1, Theorem 3.3)**.** *Let χ be a Hilbert space and $\varpi : \chi \to \mathbb{R}$ be a function. Then $\varpi \in ws - lsc(\chi) \iff epi(\varphi)$ is weakly sequentially closed.*

Theorem 2.19 establishes a connection between lower semi-continuity and convex optimization in Hilbert spaces, but not in L^p spaces. It states that the properties of lower semi-continuity and convex optimization in Hilbert spaces are related in some way to L^2 spaces when the 2-norm is defined in terms of inner products, but it does not necessarily apply to convex optimization in general L^p spaces where $p \neq 2$.

Theorem 2.20 ([1], Theorem 5.5)**.** *Let χ be a Hilbert space. If a convex subset $D \subseteq \chi$ is bounded and strongly closed in χ, and a function $\varpi : D \to \mathbb{R}$ is strongly lsc and convex, then ϖ is bounded below and has a minimum on D.*

Theorem 2.20 means that if ϖ on D satisfies the conditions of weak sequential lsc or strong lsc and if D bounded and is weakly or strongly sequentially closed, then a lower bound for ϖ exists and ϖ has a minimum value on the set D.

Alexandrian [1] conducted a study on convex optimization in Hilbert

spaces. The study used optimization tools that involve lower semi-continuous functions and convex functionals. The work focused on determining minimizers for convex programs in Hilbert spaces. The study applied the generalized Weierstrass Theorem to present conditions required for minimizers to be attained for Hilbert space convex problems. Since this assertion holds for a reflexive space such as L^2, which is a Hilbert space, it would be interesting to see whether it holds for general L^p-spaces, where $1 \leq p < \infty$, which was the aim of the current study. This study examined the applicability Alexandrian's findings to general L^p-spaces and sought to discuss properties of convex optimization in L^p-spaces.
In their research, Houska and Chachuat [26] developed the famous complete-search algorithm for Hilbert space optimization. Two theorems given below represent the main results obtained by Houska and Chachuat:

Theorem 2.21 ([26], Theorem 1)**.** *For any $\xi \in \chi$,*

$$\sup_{q \in D} |\langle \xi, q - P_i(q) \rangle| \leq R_D(i, \xi), \forall i \in \mathbb{N}$$

Theorem 2.22 ([26], Theorem 2)**.** *Let $B_i^0(S), D_i^0(S)$ be bounds and $\triangle_i(S)$ be computable for any feasible pair $(i, S) \in \mathbb{N} \times \mathbb{S}^{i+1}$. Then number of iterations before termination is at most $\overline{\sum}$ where*

$$\overline{\sum} \leq \sup_{0 \leq n \leq \overline{N}} \sum \left(\frac{\varepsilon P}{K(\xi + 1)}, n \right)$$

with $K = L \sup_{k \in N} \|\Phi_k\|$.

Houska and Chachuat [26] developed an algorithm that solved optimiza-

tion problems in Hilbert spaces. This study considered convex optimization in L^p-spaces in general.
In their research, Bay, Grammont and Maatouk [4] focused on optimization of general COPs. They developed an algorithm that approached a constrained interpolating function through the convergence of approximate solutions.

Theorem 2.23 ([4], Theorem 1). *Let* $(q, r)\chi_N = C_q^T \Gamma_N^{-1}(C_r)$, *with* $C_q = (q(t_N, 0), ..., q(t_N, N))^T$ *and* $C_r = (r(t_N, 0), ..., r(t_N, N))^T$. *Then*

$$\forall z', z \in [0,1], K_N(z', z) = \sum_{i,j=0}^{N} K(t_{N,i} t_{N,j})\varphi N, j(z)\varphi N, i(z').$$

Lemma 2.24 ([4], Lemma 2). *Let* $q_1 \in \chi \cap D \cap N$ *and* $q_0 \in \widehat{\chi \cap D} \cap N$. *Define* $q_r = (1-r)h_0 + rq_1 \in \chi, \forall r \in [0,1]$. *Then* $q_t \to q_1$ *as* $r \to 1$, $\forall r < 1, q_r \in \widehat{\chi \cap D} \cap N$.

Bay, Grammont, and Maatok [4] performed a study on convex optimization in Hilbert spaces. They formulated interpolating curves/surfaces as general convex optimization problems in a Hilbert space whose constraints were given in form of linear inequalities. However, the study was limited to inner product norms, whereas in the current research, the norms were defined as L^p-norms. This change in norm structure will provide a new perspective on the behavior of the functions representing the optimization problems and potentially offer new insights into how these problems can be solved.

Okelo [44] examined previously established principles of optimizing convex functionals within the context of Hilbert spaces. He utilized techniques from convex analysis and functions that are lower semi-continuous. Okelo [44] provided thorough demonstrations of classical theorems related to convex optimization. In addition, he presented a primary condition for optimal performance and provided a comprehensive example of how convex optimization can solve the problem posed by Dirichlet. The key outcomes of his research include:

Theorem 2.25 ([44], Theorem 3.1)**.** *Let χ be an infinite dimensional real separable Hilbert. Suppose D is a bounded weakly sequentially closed subset of χ. Let $\varpi : D \to \mathbb{R}$ be weakly sequentially lsc. Then ϖ has a lower bound and its minimizer belongs to D.*

Theorem 2.25 shows that a weakly sequentially *lsc* function in a Hilbert space achieves a minimizer on a weakly sequentially closed bounded set.

Theorem 2.26 ([44], Theorem 3.3)**.** *Let $\varpi : D \to \mathbb{R}$ be a strictly convex function on $D \subseteq \chi$. If ϖ is strictly convex on a Hilbert space χ then ϖ must achieve a unique minimizer on an infinite dimensional Hilbert space χ.*

Theorem 2.26 asserts that all strictly convex functions in infinite dimensional real Hilbert spaces have minimizers on a weakly sequentially bounded set. This assertion was proved to be true even for convex functions in L^p spaces.

Theorem 2.27 ([44], Theorem 3.5)**.** *If $\varpi : \mathbb{R}^n \to \mathbb{R}$ is differentiable at the point $q^* \in \mathbb{R}^n$. If q^* is a local minimizer of ϖ, then $\nabla\varpi(q^*) = 0$.*

Okelo's research in [44] focused on the study of local minima of differentiable functions in infinite dimensional real Hilbert spaces and how they minimize convex functions thereof, as outlined in Theorem 2.27. Okelo [44, examined the conditions for convex optimization in Hilbert spaces, however, his work did not cover the specific area of convex optimization in L^p spaces.

Okelo [44] conducted a study on optimization in Hilbert spaces posing the problems as convex functions. The results of the study showed that, if a function $\varpi : W \to \mathbb{R}$ is weakly sequentially lsc, then the function ϖ attains a minimizer on the convex set W. It was also established that if ϖ is closed, then the optimization problem $Inf_{q \in V} \psi(W)$ admits at least one global minimizer. In his study, in order to find minimizers, Okelo [44] employed the use of weak topologies. One of the key differences between Hilbert spaces and L^p-spaces is that given a sequence that is bounded it is easy to obtain a sub-sequence that converges weakly in a Hilbert space, which is not necessarily the case in L^p spaces. Therefore, this study aimed to establish optimization conditions for L^p-spaces using semi-boundedness imposed by semi-continuous functions.

Peypouquet [45] conducted research in the field of convex optimization in normed spaces. Conditions for general convex optimization in normed spaces were established, with a particular emphasis being put on linear functionals and the topological dual. The results of this research mainly focused on the compactness and closure of the weak topology. Conditions required for a convex function to attain minimizers in reflexive spaces were also presented. The following are some of the key findings from Peypouquet's research:

Proposition 2.28 ([45], Proposition 2.17). *Let* $(X, \|.\|)$ *be a normed linear space. Let* $\omega : X \to \mathbb{R} \cup \{+\infty\}$ *be a function, then the following statements hold:*

(i) If ω *is* lsc, *then* ω *is* $ws - lsc$.

(ii) If ω *is* $ws - lsc$, *then* ω *is* $s - lsc$.

(iii) If ω *is* lsc, *then* ω *is* $s - lsc$.

Peypouquet's [45] research in convex optimization in normed spaces led to the establishment of the relationship between lsc and $s - lsc$, as outlined in Proposition 2.28. This serves as a useful tool for convex optimization in normed spaces.

Proposition 2.29 ([45], Proposition 3.1). *Let* $\omega : X \to \mathbb{R} \cup \{+\infty\}$ *be a proper function. Then* ω *is convex and lsc, if and only if, a collection of affine functions* $(\omega_i)_{i \in \mathbb{I}}$ *that are continuous on* X *and satisfy* $\omega = \sup(\omega_i)$.

Corollary 2.30 (45, Corollary 3.57). *Let* $\omega : X \to \mathbb{R} \cup \{+\infty\}$ *be a proper function. Then,* ω^{**} *is the greatest lsc and convex function of* $(\omega_i)_{i \in \mathbb{I}}$.

Proposition 2.31 ([45], Proposition 3.6). *If* $(X, \|.\|)$ *is a complete normed space and suppose a convex mapping* $\omega : X \to R \cup \{+\infty\}$ *is lsc. Then* ω *is continuous on* $int(Dom(\omega))$.

Peypoquet [45] conducted a study on convex optimization in normed spaces. The study used optimization tools such as linear functionals, and topological duals, and the main methodology was weak topology and duality approach. The study developed optimality problems for constrained optimization in Hilbert spaces and general normed spaces. Peypoquet [45] noted that developing optimality conditions for constrained optimization in L^p spaces is made complex due to the intricate underlying L^p-norm structure and the infinite range of p i.e $1 \leq p < \infty$. Our current study sought to address this challenge by investigating convex optimization in L^p spaces, and aimed to find a solution to the complexities posed by Peypoquet in his research.

In [16] DeVore and Temlakov presented a variety of applications of E-greedy algorithms to convex optimization, including those that rely solely on evaluations of convex functions, as well as algorithms that utilize evaluations that give exact values and those that incorporate evaluations whose values are approximated. The study demonstrated that the upper bounds are dependent on the smoothness of the point in a convex space $\mathcal{X}$ which achieves a minimum point. It is worth noting that the research carried out by DeVore and Temlakov [16] on E-greedy algorithms and convex optimization has contributed valuable insights and tools for solving optimization problems and has the potential for further developments. Among the main results obtained by DeVore and Temlakov two key findings are presented below:

Theorem 2.32 ([16], Theorem 1.1)**.** *Let $Q^* = \inf_{q \in P_1(T)} Q(q)$ be a feasible set.*

(i) If the set Q is uniformly smooth on $P_1(T)$, then the Relaxed E-Greedy Algorithm $(REGA)$ converges such that:

$$\lim_{r \to \infty} Q(G_r) = Q^*.$$

(ii) Moreover, given $\{\rho(Q, P_1(T)) : a \leq \Gamma a^x, 1 < x \leq 2\}$, *one has*

$$Q(G_r) - Q^* \leq B(x, \gamma) r^{1-x},$$

where the constant $B(x, \Gamma)$ *is positive.*

Theorem 2.32 examines the rate at which the specific Relaxed E-Greedy Algorithm (REGA(co)) converges, utilizing the modulus of smoothness. It is noted that the point at which the minimum is achieved, Q^*, satisfies a sparsity constraint.

Theorem 2.33 ([16], Theorem 1.2)**.** *Let* T *be uniformly smooth on* $\mathcal{X}$ *and* $T^* := \inf_{q \in \mathcal{X}} T(q) = \inf_{q \in D_0} T(q)$.

(i) The $EGAFR(co)$ *converges*

$$\lim_{r \to \infty} T(G_r) = \inf_{q \in \mathcal{X}} T(q) = \inf_{q \in D_0} T(q) = T^*$$

(ii) If the inequality $\xi(T, a) \leq \Gamma a^x, 1 < x \leq 2$ *is satisfied by the modulus of smoothness of* T, *then the* $EGAFR(co)$ *satisfies the inequality*

$$T(G_r) - T^* \leq D(T, r, \Gamma) \varepsilon_r,$$

where $\varepsilon_r := \inf\{\varepsilon : P(\varepsilon)^x r^{1-x} \leq \varepsilon\}$.

Theorem 2.33 describes the convergence property of the E-Greedy Algorithm with Free Relaxation (EGAFR(co)) algorithm. It is used for the convex minimization of uniformly smooth sets. The current study, however, aims to expand this analysis by considering convex optimization on L^p-spaces from the perspective of semi-continuity. DeVore and Temlakov [16] proposed E-greedy algorithms, which utilize function evaluations, exact evaluations, and approximate evaluations, to solve general convex optimization problems. However, their research did not cover the application of these algorithms to convex optimization in L^p spaces. The current study utilized semi-continuous functions to identify the minimum and maximum points of a COP which was not the case in [16].

Based on this review, a clear research gap exists on the characterization of upper and lower semi-continuous functions in L^p spaces and the development of optimality conditions for convex optimization in these spaces. This gap in the current literature indicates that there is a great opportunity for further research in these areas to improve our understanding of the properties of lower and upper semi-continuous functions in L^p spaces and apply it to solve convex optimization problems in L^p spaces.

Chapter 3

RESEARCH METHODOLOGY

3.1 Introduction

For effective completion of this research, a strong understanding of functional analysis, operator theory, and norm structures was essential. In this chapter, we have summarized and restated important principles and results related to semi-continuity and convex optimization that were utilized in the research, such as the Dini's Theorem, Beer's Theorem and Fatou's Lemma. Additionally, classical inequalities, such as the Cauchy-Schwarz inequality and the Minkowski's inequality, were invoked in the findings. The technical methodology involved use of the KKT conditions for optimality.

3.2 Fundamental Principles

In this section we state classical results and important mathematical principles that we used in this research.

Theorem 3.1 (**Hahn-Banach Separation Theorem**)**.** *[[45], Theorem 1.10] Let t P and R be nonempty disjoint convex subsets of a normed space* $(\mathcal{Q}, \|.\|)$, *then*

(i) If the set $P \subseteq \mathcal{Q}$ *is open,* $\exists F \in \mathcal{Q}^* \setminus \{0\} : \langle F, q_1 \rangle < \langle F, q_2 \rangle, \forall q_1, q_2 \in P$.

(ii) If the set P *is compact and* R *is closed,* $\exists F \in \mathcal{Q}^* \setminus \{0\}$ *and* $\xi > 0$: $\langle F, q_1 \rangle + \xi \leq \langle F, q_2 \rangle$, $\forall q_1 \in P$ *and* $\forall q_2 \in .R$

Proposition 3.2 (45, Proposition 2.17)**.** *Let* $\mathcal{Q} \neq \emptyset$ *be a convex set and* $\vartheta : \mathcal{Q} \to \overline{\mathbb{R}}$ *be a function. If* ϑ *is convex then the following are equivalent:*

(i) ϑ *is* $w - lsc$.

(ii) ϑ *is* $ws - lsc$.

(iii) ϑ *is* $s - lsc$.

(iv) ϑ *is* lsc.

Proposition 3.3 ([45], Proposition 2.3)**.** *Let* $\vartheta : \mathcal{Q} \to [\mathbb{R} \cup \{+\infty\}$ *be a function, then the following statements are equivalent:*

(i) $\vartheta \in lsc(\mathcal{L})$.

(ii) the set $epi(\vartheta)$ *is closed in* $\mathcal{Q} \times \mathbb{R}$.

Theorem 3.4 (**Dini's Theorem**, [7], Theorem 2.1)**.** *Let* $\mathcal{Q}$ *be a compact set in a metric space* (X, d). *Let* ϑ_j *be a sequence of continuous functions that converge pointwise to a continuous function* ϑ. *If* $\vartheta_1(q) \geq \vartheta_2(q) ... \geq \vartheta_n(q) \geq ... \; \forall q \in \mathcal{Q}$, *then* $\{\vartheta_j\}$ *converges uniformly to* ϑ.

Theorem 3.5 (**Beer's Theorem**)**.** *[[5],Theorem 2] Given* $\eta > 0$ *define* $\vartheta_\eta^+(q) = \sup\{\lambda : (q, \lambda) \in B_\eta[hypo(\vartheta)]\}$. *Then* ϑ_η^+ *is a bounded usc function and* $|\vartheta_\eta^+ - \vartheta| = \eta$. *Moreover,* $|\vartheta - \varphi| \leq \eta \iff \vartheta \leq \varphi_\eta^+$ *and* $\varphi \leq \vartheta_\eta^+$.

Lemma 3.6 (**Fatou's Lemma**)**.** *[[52], Section 1 inequality 1.1] Let* $\{\vartheta_j\}_{j=1}^\infty$ *be a sequence of nonnegative measurable functions on a set* $\mathcal{L}$. *If* $\liminf_{j\to\infty} \vartheta_j(q) = \vartheta(q)$ *for each* $q \in \mathcal{L}$, *then,* $\liminf_{j\to\infty} \int_\mathcal{L} \vartheta_j(q) d\mu \geq \int_\mathcal{L} \vartheta(q) d\mu$.

3.3 Known Inequalities

Proposition 3.7 (**The Cauchy-Schwarz Inequality**)**.** *[45, Proposition 1.34] Let* $(\mathcal{Q}, \langle ., . \rangle)$ *be an inner product space and define the norm of* q *by* $\|q\| = (\langle q, q \rangle)^{\frac{1}{2}} \forall q \in \mathcal{Q}$ *then:*

$$|\langle q_1, q_2 \rangle| \leq \|q_1\| \|q_2\| \forall q_1, q_2 \in \mathcal{Q}. \tag{3.3.1}$$

Theorem 3.8 (**Minkowski's Inequality**)**.** *[51, Theorem 4.3] Let* $1 \leq p < \infty$, *then* $\forall q, w \in \mathbb{R}^n$ *we have,*

$$\left(\int_a^b |q_i + w_i|^p \right)^{\frac{1}{p}} \leq \left(\int_a^b |q_i|^p \right)^{\frac{1}{p}} + \left(\int_a^b |w_i|^p \right)^{\frac{1}{p}}$$

3.4 Technical Approach

The technical approach involved the use of Karush-Kuhn-Tucker Optimality Conditions to characterize conditions for convex optimization in L^p-spaces.

Karush-Kuhn-Tucker (KKT) **Optimality Conditions**

The KKT conditions for optimality are necessary for q^* to be a local optimal solution for the convex optimization problem below:

$$Min_{q\in\mathcal{Q}}\vartheta(q) \tag{3.4.1}$$

subject to

$$\begin{aligned} \varphi_i(q) &\leq 0, 1 \leq i \leq t & (3.4.2)\\ \varrho_j(q) &= 0, 1 \leq j \leq v, & (3.4.3) \end{aligned}$$

where $q \in \mathcal{Q}$ is the optimization variable and the set $\mathcal{Q}$ is convex; the COP 3.4.1 is convex; the inequality constraints 3.4.2 are convex functions and the equality constraints 3.4.3 are convex functions.

The KKT optimality conditions are given by:

$$\begin{aligned} \varphi_i(q^*) &\leq 0, \forall i = 1, ..., t & (3.4.4)\\ \varrho_j(q^*) &= 0, \forall j = 1, ..., v & (3.4.5)\\ \eta_i^* &\geq 0, \forall i = 1, ..., m & (3.4.6)\\ \eta_i^*\varphi_i(q^*) &= 0, \forall i = 1, ..., t & (3.4.7)\\ \nabla\varphi_0(q^*) + \sum_{i=1}^{p}\eta_i^*\varphi_1(q^*) + \sum_{j=1}^{p}\nabla\varrho_j(q^*) &= 0. & (3.4.8) \end{aligned}$$

The first two conditions (3.4.4 and 3.4.5) are primal feasibility conditions of q^*.

Condition 3.4.6 represents the dual feasibility condition for q^*.

Chapter 4

RESULTS AND DISCUSSION

4.1 Introduction

We give results on characterizations of lower semi-continuous functions and upper semi-continuous functions in L^p spaces. We also establish conditions for convex optimization in L^p spaces. We denote the collection of all *lsc* functions in an L^p-space $\mathcal{L}$ by $lsc(\mathcal{L})$ and $usc(\mathcal{L})$ denotes the collection of all *usc* functions in an L^p space $\mathcal{L}$.

4.2 Lower Semi-continuous functions in L^p Spaces

In this section, characterization of lower semi-continuous functions in L^p spaces is discussed. We begin with the following proposition in which lower semi-continuity is characterized using convex conjugates and bi-conjugates. We show that if a conjugate is convex in L^p spaces $\mathcal{L}$ then it is weak$^*-$ lower semi-continuous in an L^p space and a convex bi-conjugate is weak lower semi-continuous in an L^p space.

Proposition 4.1. *Suppose ϑ is a function in an L^p-space $\mathcal{L}$. Then ϑ^*, ϑ^{**} are convex functions, and*

(i) ϑ^* *is* $w^* - lsc$.

(ii) ϑ^{**} *is* $w - lsc$.

(iii) $\vartheta^{**} \leq \vartheta$.

Furthermore, if ϑ_1, ϑ_2 are convex functions satisfying $\vartheta_1 \leq \vartheta_2$, then $\vartheta_1^* \geq \vartheta_2^*$.

Proof. Let the convex function $\varrho \in \mathcal{L}^* : \varrho \to \langle q, \varrho\rangle, \forall q \in \text{dom}(\vartheta)$ be *weak** continuous function. Then, $\varrho \to \langle q, \varrho\rangle$ is $w^* - lsc$ for all finite or infinite function $\vartheta(q)$. Define ϑ^* as $\vartheta^* = \sup\{q : q \in lsc(\mathcal{L})\}$. Since the supremum of any collection of $lsc(\mathcal{L})$ is a lsc function, then ϑ^* supremum is a convex and $weak^* - lsc$ function. Now, the function $q \mapsto \langle q, \varrho\rangle$ is convex and weakly lsc. Define ϑ^{**} as $\vartheta^{**} = \sup\{q : q \mapsto \langle q, \varrho\rangle\}\forall \varrho \in \mathcal{L}^*$. Then ϑ^{**} is convex and weakly lsc. For each $q \in \text{dom}(\vartheta), \langle q, \varrho\rangle - \vartheta^*(\varrho) \leq \vartheta(q)$. Hence $\forall q \in \text{dom}(\vartheta), \vartheta^{**}(q) = \sup\{(\langle q, \varrho\rangle - \vartheta^*(\varrho))\} \leq \vartheta(q)$ this implies that $\vartheta^{**} \leq \vartheta$. Let $\vartheta_1 \leq \vartheta_2$, then, $\vartheta_2^*(\varrho) = \sup\{(\langle q, \varrho\rangle - \vartheta_2(q))\} \leq \sup\{(\langle q, \varrho\rangle - \vartheta_1(q))\} = \vartheta_1^*(\varrho)$ and this clearly shows that $\vartheta_1^* \geq \vartheta_2^*$. □

In the next lemma we have characterized lsc functions using Moore-Smith sequences. A function $\vartheta : \mathcal{Q} \to \overline{\mathbb{R}}$ where $\mathcal{Q} \subseteq \mathcal{L}$ is lower semi-continuous if given a Moore-Smith sequence $\{(q_j)\}_{j\in\mathbb{N}} \in \mathcal{Q}$, the inverse $\vartheta^{-1}(q,\infty), \forall q \in \overline{\mathbb{R}}$ is open.

Lemma 4.2. *Let $\mathcal{L}$ be an L^p-space and $lsc(\mathcal{L})$ denote the collection of all lsc functions. Let ϑ be a measurable function and $\{(q_j)\}_{j\in\mathbb{N}} \in dom(\vartheta)$ be a Moore-Smith sequence. Then $\vartheta \in lsc(\mathcal{L})$, if and only if, $\int_{\mathcal{L}} \vartheta(q) \leq \liminf \int_{\mathcal{L}} \vartheta(q_j), \forall q \in dom(\vartheta)$ whenever $q_j \to q$.*

Proof. Let $\vartheta \in lsc(\mathcal{L})$. Assume $\{(q_j)_{j\in\mathbb{N}}\}$ is a Moore-Smith sequence converging to $q \in dom(\vartheta)$. If $r < \vartheta(q), \forall r \in \overline{\mathbb{R}}$, then $\vartheta^{-1}(r,\infty)$ is open since $\vartheta \in lsc(\mathcal{L})$. Now, $\forall q \in \vartheta^{-1}(r,\infty)$ and $q_j \to q$, $\exists j_r \leq j$ satisfying $q_j \in \vartheta^{-1}(r,\infty)$. So given $j \geq j_r$ we have $\vartheta(q_j) > r$ implying that $\liminf \vartheta(q_j) \geq r$. Now $\forall r < \vartheta(q)$ we obtain $\liminf \vartheta(q_j) \geq \vartheta(q)$. Hence, equivalently $\vartheta(q) \leq \liminf \vartheta(q_j)$. Given that ϑ is measurable over $\mathcal{L}$ and $q_j \to q$, then by Lemma 3.6 we have $\int_{\mathcal{L}} \vartheta(q) \leq \liminf \int_{\mathcal{L}} \vartheta(q_j)$.
Conversely, for each Moore-Smith sequence $q_j \to q$ let $\int_{\mathcal{L}} \vartheta(q) \leq \liminf \int_{\mathcal{L}} \vartheta(q_j)$. Then $\vartheta(q) \leq \liminf \vartheta(q_j)$. Let $\Omega = \vartheta^{-1}(-\infty, r], \forall r \in \overline{\mathbb{R}}$. If $q \in \overline{\Omega}$ and given $q_j \to q$, then $\{q_j\} \in \Omega$. Since $\Omega = \vartheta^{-1}(-\infty, r]$, we deduce that $\vartheta(q_j) \leq r$ for each j and so $\vartheta(q) \leq r$. Hence $q \in \Omega$, implying that Ω is closed and so the complement of Ω ($\Omega^c = \vartheta^{-1}(r,\infty)$) is open, showing that $\vartheta \in lsc(\mathcal{L})$. □

The next theorem shows that if two lower semi-continuous functions are such that neither takes the value $-\infty$, then their sum yields a lower semi-continuous function and the product of either of such functions with a positive scalar is also a lower semi-continuous function.

Theorem 4.3. *Let $\mathcal{L}$ be an L^p space and $\mathcal{Q}$ be a nonempty convex set . Suppose two functions $\vartheta : \mathcal{Q} \to \overline{\mathbb{R}}$ and $\varrho : \mathcal{Q} \to \overline{\mathbb{R}}$ are lsc in $\mathcal{L}$ such* *that* $\vartheta > -\infty, \varrho > -\infty$, *then their sum* $\vartheta + \varrho$ *is lsc in* $\mathcal{L}$ *and furthermore* $\forall \lambda > 0,\ \lambda\vartheta \in lsc(\mathcal{L}), \forall \vartheta \in lsc(\mathcal{L})$.

Proof. Let $\{(q_j)_{j\in\mathbb{N}}\} \subseteq \mathcal{Q}$ be a Moore-Smith sequence converging to $q \in \mathcal{Q}$, then by Lemma 4.2 we have,

$$\begin{aligned}(\varrho + \vartheta)(q) &\leq \liminf \varrho(q_j) + \liminf \vartheta(q_j) \\ &\leq \liminf(\varrho(q_j) + \vartheta(q_j)) \\ &= \liminf(\varrho + \vartheta)(q_j)\end{aligned}$$

Thus $(\varrho + \vartheta)(q) \leq \liminf(\varrho + \vartheta)(q_j)$, implying that $\int_{\mathcal{L}}(\varrho + \vartheta)(q) \leq \liminf \int_{\mathcal{L}}(\varrho + \vartheta)(q_j)$ by Lemma 3.6 meaning that $\varrho + \vartheta \in lsc(\mathcal{L})$.
It also follows that,

$$\begin{aligned}(\lambda\vartheta)(q) &= \lambda\vartheta(q) \\ &\leq \lambda \liminf \vartheta(q_j) \\ &= \liminf \lambda\vartheta(q_j) \\ &= \liminf(\lambda\vartheta)(q_j)\end{aligned}$$

Hence, $(\lambda\vartheta)(q) \leq \liminf(\lambda\vartheta)(q_j)$ yields $\int_{\mathcal{L}}(\lambda\vartheta)(q) \leq \liminf \int_{\mathcal{L}}(\lambda\vartheta)(q_j)$ showing that $\lambda\vartheta \in lsc(\mathcal{L})$. □

The following theorem shows that if a sequence of *lsc* functions $\{\vartheta_n\}$, with each of the functions being finite, converges uniformly to ϑ then, ϑ is also *lsc*.

Theorem 4.4. *Let* $\{\vartheta_n\}$ *be a finite sequence of lower semi-continuous functions in an* L^p*-space* $\mathcal{L}$*. If* $\{\vartheta_n\}_{n=1}^k$ *converges uniformly to* ϑ *then*

$\vartheta \in lsc(\mathcal{L})$.

Proof. Uniform convergence of $\{\vartheta_n\}$ implies that given a small positive real number ξ an integer N_ξ exists yielding $|\vartheta_n(q)-\vartheta(q)| < \xi, \forall n \geq N_\xi, q \in dom(\vartheta)$. So $|\vartheta_n(q) - \vartheta(q)| \leq \sup\{|\vartheta_n(q) - \vartheta(q)| < \xi : \forall q \in dom(\vartheta) \text{and} n \geq N_\xi\}$. Define η by $\eta = \sup\{|\vartheta_n(q) - \vartheta(q)| : q \in dom(\vartheta)\}$. Hence $-\eta \leq \vartheta_n(q) - \vartheta(q) \leq \eta$ implies that $\vartheta(q) \leq \eta + \vartheta_n(q)$. Now since $\{\vartheta_n\}$ are *lsc* functions, and given a Moore-Smith sequence $\{(q_j)\}_{j\in\mathbb{N}}$ converging strongly to q, we have $\vartheta_n(q) \leq \liminf \vartheta_n(q_j)$, thus

$$\vartheta(q) \quad \leq \quad \eta + \liminf \vartheta_n(q_j). \tag{4.2.1}$$

Since $\{\vartheta_n\}$ converges uniformly to ϑ, $\lim_{n\to\infty} (\liminf \vartheta_n(q_j)) = \liminf \vartheta(q_j)$ thus if we take limits as n tends to infinity inequality 4.2.1 becomes $\vartheta(q) \leq \eta + \liminf \vartheta(q_j)$. Choosing $\xi \geq \eta$ we have $\vartheta(q) \leq \xi + \liminf \vartheta(q_j)$. This holds for all $\xi > 0$. Hence, we get $\vartheta(q) \leq \liminf \vartheta(q_j)$ showing that $\vartheta \in lsc(\mathcal{L})$. □

In the following result, we express lower semi-continuity of functions using the concept of the epigraph of a function.

Theorem 4.5. *Let $\mathcal{Q}$ be a convex set. A function $\vartheta : \mathcal{Q} \to \overline{\mathbb{R}}$ in L^p space $\mathcal{L}$ is lsc if and only if $epi\vartheta \subseteq \mathcal{Q} \times \mathbb{R}$ is closed.*

Proof. Let $\vartheta \in lsc(\mathcal{L})$ and $\forall q \in \mathcal{Q}, \forall \kappa \in \mathbb{R}$ assume $(q_i, \kappa_i) \in epi\vartheta$, converges to $(q, \kappa) \in \mathcal{Q} \times \mathbb{R}$. Then $q_i \to q$ and $\kappa_i \to \kappa$. By Lemma 4.2 we have $\vartheta(q) \leq \liminf \vartheta(q_i) \leq \liminf \kappa_i = \lim \kappa_i = \kappa$. Thus, $\vartheta(q) \leq \kappa$ implies that $(q, \kappa) \in \mathrm{epi}(\vartheta)$. This shows that $\mathrm{epi}(\vartheta)$ is closed.

Conversely, let $\text{epi}(\vartheta)$ be closed. Then $\forall \kappa \in \mathbb{R}$, the set $(\mathcal{Q} \times \{\kappa\}) \cap \text{epi}(\vartheta) = \{(q, \kappa) : \kappa \geq \vartheta(q)\} \subset \mathcal{Q} \times \mathbb{R}$ is closed. This implies that $\vartheta^{-1}(-\infty, \kappa]$ as a subset of $\mathcal{L}$ is closed but $\vartheta^{-1}(\kappa, \infty), \forall q \in \mathcal{Q})$, is open. Since this holds true $\forall \kappa \in \mathbb{R}$, then $\vartheta \in lsc(\mathcal{L})$. □

A function taking a finite value at any point in its domain is termed as proper. The next proposition shows that if a convex *lsc* function can take $-\infty$ value at any point in its domain then it is infinite everywhere such that, should there exist a point at which the function is proper then it does not admit the value $-\infty$ at any other point in its domain.

Proposition 4.6. *Let $\mathcal{Q}$ be a convex set. Let a convex function $\vartheta : \mathcal{Q} \to \overline{\mathbb{R}}$ in L^p space $\mathcal{L}$ be lower semi-continuous. If there exists some $q_0 \in \mathcal{Q} : \vartheta(q_0) = -\infty$, then $\vartheta(q) = -\infty \ \forall q \in \mathcal{Q}$.*

Proof. Assume $q \in \mathcal{Q}$ implies $-\infty < \vartheta(q) < \infty$. Let $q_0 \in \mathcal{Q}$ satisfy $\vartheta(q_0) = -\infty$. Since ϑ is convex, we have $\vartheta((1-\eta)q + \eta q_0) \leq (1-\eta)\vartheta(q) + \eta\vartheta(q_0) = -\infty, \forall \eta \in (0, 1]$ hence $\vartheta((1-\eta)q + \eta q_0) = -\infty, \forall \eta \in (0, 1]$. Since $\vartheta \in lsc(\mathcal{L})$, we obtain $\vartheta(\lim_{\eta \to 0}(1-\eta)q + \eta q_0) \leq \liminf_{\eta \to 0} \vartheta((1-\eta)q + \eta q_0)$, and hence $\vartheta(q) \leq -\infty$, which is a contradiction. Thus, $q \in \mathcal{Q} : -\infty < \vartheta(q) < \infty$ does not exist, implying that $\forall q_0 \in \mathcal{Q} : \vartheta(q_0) = -\infty$ we have, $\vartheta(q) = \infty, \forall q \in \mathcal{Q}$ as desired. □

The next lemma shows that lower semi-continuity implies weak lower semi-continuity.

Lemma 4.7. *Let $\mathcal{Q}$ be a convex set. If a function $\vartheta : \mathcal{Q} \to \overline{\mathbb{R}}$ in an L^p-space $\mathcal{L}$ is lsc then, ϑ is weakly lower semi-continuous.*

Proof. The cartesian product $\mathcal{Q}\times\mathbb{R}$ is locally convex since $\mathcal{Q}$ is convex and $\mathbb{R}$ is locally convex. Therefore, $\mathcal{Q}_v\times\mathbb{R}$ forms a weak topology on $\mathcal{L}$. Hence, weak closure in $\mathcal{Q}\times\mathbb{R}$ implies closure in $\mathcal{Q}_v\times\mathbb{R}$. A convex subset of $\mathcal{Q}\times\mathbb{R}$ is always closed in $\mathcal{Q}_v\times\mathbb{R}$ as well as the whole domain of $\mathcal{Q}\times\mathbb{R}$. From the hypothesis $\vartheta \in lsc(\mathcal{L})$, so by Theorem 4.5 $\vartheta \in lsc(\mathcal{L}) \Leftrightarrow epi\vartheta \subseteq \mathcal{Q}\times\mathbb{R}$ is closed. Therefore, epi(ϑ) is closed implying that $\vartheta \subseteq \mathcal{Q}_v\times\mathbb{R}$. is $w-lsc$. □

Proposition 4.8 below shows that given a sequence of continuous functions that are Lebesgue integrable on $\mathcal{Q} \in [0,1]$, if ϑ is lsc then it is the least upper bound for the sequence.

Proposition 4.8. *Let $\mathcal{L}$ be an L^p-space. Let $\mathcal{Q}$ be a nonempty set. Denote a collection of continuous functions that are integrable on $\mathcal{Q} \in [0,1]$ by $\mathcal{C}(\vartheta)$. Given $\vartheta \in lsc(\mathcal{L}), \forall\vartheta \geq 0$ then $\vartheta = \sup\{\varphi : \varphi is\ continuous, \forall\varphi \leq \vartheta\}$.*

Proof. Suppose $\varphi \in \mathcal{C}(\vartheta) : \varphi \leq \vartheta$. Since $\mathcal{C}(\vartheta)$ is nonempty let $\mathfrak{L} = \vartheta^{-1}(-\infty, \vartheta(q)-\xi], \forall q \in \mathcal{Q}, \xi > 0$. Now, $\mathcal{Q}\setminus\mathfrak{L} = \vartheta^{-1}(\vartheta(q)-\xi, \infty)$ is open, so $\mathfrak{L}$ is closed. Therefore, q is not an element of $\mathfrak{L}$. Hence, a function $\rho : \mathcal{Q} \to [0,1]$ that is continuous exists, satisfying $\rho(\mathfrak{L}) = 0$ and $\rho(q) = 1$. Assuming $\vartheta(q)-\xi \leq 0$ then, given $\rho \geq 0$ and $\vartheta \geq 0$, $(\vartheta(q)-\xi)\varphi \leq \vartheta$. If we let $\vartheta(q)-\xi > 0$ and $w \in \mathcal{Q}$, then $(\vartheta(q)-\xi)\varphi(w) \leq (\vartheta(q)-\xi) \leq \varphi(w)$ when $w \notin \mathfrak{L}$ and $(\vartheta(q)-\xi)\varphi(w) = 0$ when $w \in \mathfrak{L}$. Thus $(\vartheta(q)-\xi)\varphi \leq \vartheta$ and continuity of $(\vartheta(q)-\xi)\varphi$ imply that $(\vartheta(q)-\xi)\varphi \in \mathcal{C}(\vartheta)$. Thus, we obtain $(\sup(\mathcal{C}(\vartheta))(q))) \geq (\vartheta(q)-\xi)\varphi(q) = \vartheta(q)-\xi$ where $\sup(\mathcal{C}(\vartheta)) = \sup\{\varphi : \varphi is\ continuous, \forall\varphi \leq \vartheta\}$. Since ξ was arbitrary we therefore have

$$(\sup(\mathcal{C}(\vartheta))\,(q)) \quad \geq \quad \vartheta(q) \tag{4.2.2}$$

and since q was also arbitrary we have

$$\sup(\mathcal{C}(\vartheta)) \quad \leq \quad \vartheta. \tag{4.2.3}$$

Therefore from 4.2.2 and 4.2.3 we have $\vartheta = \sup(\mathcal{C}(\vartheta))$. □

The next theorem shows that a *lsc* function in an L^p-space must attain an absolute minimum on a compact set.

Theorem 4.9. *Let ϑ be a function in an L^p-space $\mathcal{L}$. Let q be a member of a compact subset B of $\mathbb{R}^n$. If $\vartheta \in lsc(\mathcal{L})$ then, $\vartheta(q) \geq \vartheta(\overline{q}), \forall q \in B, \overline{q} = lim\{b_r\}$.*

Proof. By way of contradiction suppose that ϑ has no lower bound. Then there is $q_r \in B, \forall r \in \mathbb{N}$ satisfying $\vartheta(q_r) < -r$. Compactness of B implies that a sub sequence $\{q_{r_k}\}$ of $\{q_r\}$ exists which converges to $q_0 \in B$. Lower semi-continuity of ϑ means that $\vartheta \in lsc(\mathcal{L})$ at every point $\{q_0 \in B\}$ in the convergent sequence $\{q_r\}$ tending to q_0. Thus, by Lemma 4.2, $\liminf_{k\to\infty} \vartheta(q_{r_k}) \geq \vartheta(q_r)$. This shows a contradiction, since $\liminf \vartheta(q_{r_k}) \neq -\infty$. Hence, ϑ is bounded below. Suppose $\mathcal{F} = \inf\{\vartheta(q) : q \in B\}$. Then, $\mathcal{F} \in \mathbb{R}$ because $\vartheta(q)$ is not empty and is bounded below.
Let the sequence $\{b_r\} \in B$ be such that $\{\vartheta(b_r)\}$ converges to $\mathcal{F}$. Then since B is compact, there is $\{b_{r_k}\}$ a sub sequence of $\{b_r\}$ whose limit is $\overline{q} \in B$. Now, $\mathcal{F} = \lim_{k\to\infty} \vartheta(b_{r_k}) = \liminf_{k\to\infty} \vartheta(b_{r_k}) \geq \vartheta(\overline{q}) \geq \mathcal{F}$. This shows that $\mathcal{F} = \vartheta(\overline{q})$. Therefore, $\vartheta(q) \geq \vartheta(\overline{q}), \forall q \in B$. □

The next theorem proves that if a proper function is convex and *lsc* its lower bound is a functional that is continuous.

Theorem 4.10. *Suppose ϑ is a convex function in an L^p-space $\mathcal{L}$. If $\vartheta \in lsc(\mathcal{L})$ then, $\exists \rho \in \mathcal{L}^*, r \in \mathbb{R}$ satisfying $\vartheta \geq \rho + r$.*

Proof. The epi(ϑ) $\subseteq \mathcal{L} \times \mathbb{R}$ is convex given that ϑ is convex. Now, for each $r_0 \in$ dom ϑ we have $\vartheta(r_0) > -\infty$, by the fact that ϑ is proper. Given $\vartheta \in lsc(\mathcal{L})$, it follows from Theorem 4.5 that epi(ϑ) $\subseteq \mathcal{L} \times \mathbb{R}$ is closed. So, epi(ϑ) forms a closed convex set. Therefore, for each $r_0 \in$dom ϑ we have $(r_0, \vartheta(r) - 1) \notin$ epi(ϑ) guaranteeing convexity of the compact set $\{(r_0, \vartheta(r_0) - 1)\}$. Then, $\exists \sigma \in (\mathcal{L} \times \mathbb{R})^*$ and a function $\Gamma \in \mathbb{R}$ yielding $\sigma(\rho, r) < \Gamma < \sigma(r_0, \vartheta(r_0) - 1), \forall (r, \rho) \in epi\vartheta$. There exists $\rho \in \mathcal{L}^*, \sigma \in \mathbb{R}^*$ satisfying $\sigma(r, \rho) = \sigma\rho + \sigma r, \forall (\rho, r) \in \mathcal{L}^* \times \mathbb{R}$. Therefore by Theorem 3.1 $\sigma(\rho + r) < \Gamma < \sigma r_0 + \sigma(\vartheta(r_0) - 1), \forall (\rho, r) \in epi\vartheta$. For $(r_0, \vartheta(r_0)) \in epi\vartheta$, we have $\sigma(\rho + r) < \sigma r_0 + \sigma(\vartheta(r_0) - 1)$ implying that $\sigma < 0$. Now, for $r \in epi(\vartheta)$, $\vartheta(r) > -\frac{1}{\sigma}\rho r + \frac{1}{\sigma}\rho r_0 + \vartheta(r_0) - 1$. Since $\sigma < 0$ it follows that $\vartheta \geq \rho + r$. □

This theorem gives rise to the following two important corollaries:

Corollary 4.11. *Let the convex function ϑ in an L^p space $\mathcal{L}$ be lsc. Then $\vartheta < \infty$, if and only if, $\vartheta^* < \infty$.*

Proof. Let the convex function ϑ be proper. Thus, by Theorem 4.10, $\exists \rho \in \mathcal{L}^*, \exists r \in \mathbb{R}$ satisfying $\vartheta \geq \rho q + r, \forall q \in dom(\vartheta)$. Now, each $\eta \in \mathcal{L}^*$ satisfies $\vartheta^*(\eta) = \sup_{t \in dom(\vartheta)}(\eta q - \vartheta(q)) \leq \sup_{q \in \mathcal{L}}(\eta q - \rho q - r)$, hence $\vartheta^*(\rho) = -r < \infty$, implying that dom $\vartheta^* \neq \phi$. There also exists $q_0 \in dom(\vartheta)$ such that $\vartheta(q_0) \neq \infty$, resulting to $\sup_{q \in dom(\vartheta)}(\eta q - \vartheta(q)) \geq \eta q_0 - \vartheta(q_0) > -\infty$. It thus follows that $\vartheta^*(\eta) > -\infty, \forall \eta \in dom(\vartheta^*)$. Hence ϑ^* is proper. Conversely, let the convex function ϑ^* be proper. Let $q \in dom(\vartheta)$. Since

ϑ^* is proper, $\eta \in dom(\vartheta^*)$ satisfies $\vartheta^*(\eta) < \infty$. By Theorem 3.1 we get $\vartheta(q) \geq \langle q, \eta \rangle - \vartheta^*(\eta) > -\infty$. Therefore, $\forall q \in dom(\vartheta), \vartheta(q) > -\infty$, hence ϑ is proper. $\square$

Corollary 4.12. *Let ϑ be a convex function in an L^p-space $\mathcal{L}$. If $\vartheta \in lsc(\mathcal{L})$, and $\forall q_0 \in dom(\vartheta)$, $\exists r_0 > 0$ such that $\inf_{q \in B(q_0;r_0)} \vartheta(q) > -\infty$, then $\exists r \in \mathbb{R}$ satisfying $\vartheta(q) > \rho(q) + r$.*

Proof. Suppose $\inf_{q \in B(qz_0,r_0)} \vartheta(q) > -\infty$, then, there is $r \in \mathbb{R} : \vartheta(q) > r+1$ for each $q \in B(q_0, r_0)$. Clearly $(q_0, r) \notin \overline{epi(\vartheta)}$. Since $epi(\vartheta)$ is convex and closed, $\exists \rho \in dom(\vartheta^*) : \rho(q_0)+\alpha r < \rho(q)+\alpha\vartheta(q), \forall q \in dom(\vartheta), \alpha \in \mathbb{K}$. Therefore, $\vartheta(q) > -\frac{1}{\alpha}\rho(q) + \frac{1}{\alpha}(\rho(q_0) + \alpha r), \forall q \in dom(\vartheta), \alpha > 0$. Hence, $\vartheta(q) > \rho(q) + r \ \forall q \in dom(\vartheta)$. $\square$

Theorem 4.13. *Let $\mathcal{Q} \neq \emptyset$ be a set and $\mathcal{L}$ be an L^p-space. Let the functional $\Omega : \mathcal{Q} \to \mathbb{K}$ in $\mathcal{L}$ be convex. Then, a convex function ϑ exists to satisfy $\vartheta(r) = \vartheta_\varrho(r) = \inf_{\varrho(q)=r} \Omega(q), \forall r \in \mathbb{K}, q \in \mathcal{Q}$.*

Proof. Let $r_1, r_2 \in \mathbb{K}$ and $0 \leq \upsilon \leq 1$. Then we have,

$$
\begin{aligned}
\vartheta(\upsilon r_1 + (1-\upsilon)r_2) &= \inf_{\varrho(q)-\upsilon r_1+(1-\upsilon)r_2} \Omega(q), \forall q \in \mathcal{Q} \\
&= \inf_{\varrho(q_1)=r_1, \varrho(q_2)=r_2} \Omega(\upsilon q_1 + (1-\upsilon)q_2), \forall q_1, q_2 \in \mathcal{Q} \\
&\leq \inf_{\varrho(q_1)=r_1, \varrho(q_2)=r_2} \upsilon\Omega(q_1) + (1-\upsilon)\Omega(q_2) \\
&= (1-\upsilon) \inf_{\varrho(q)=r_2} \Omega(q) + \upsilon \inf_{\varrho(q)=r_1} \Omega(q) \\
&= \upsilon\vartheta(r_1) + (1-\upsilon)\varrho(r_2),
\end{aligned}
$$

so ϑ is convex on $\mathbb{K}$. Thus, since $\vartheta(r_0) = -\infty$ for some $r_0 \in \mathbb{K}$, we have, $\vartheta(r) < +\infty$. From property of convex functionals and $\mathbb{R} = [-\infty, +\infty]$ it follows that $\vartheta \equiv -\infty$. If $\vartheta > -\infty$ then, since dim $R = 1 < +\infty$, ϑ is continuous on $\mathbb{K}$ and $\vartheta(r) = \vartheta_{\varrho}(r) = \inf_{\varrho(q)=r} \Omega(q), \forall r \in \mathbb{K}, q \in \mathcal{Q}$. $\square$

Proposition 4.14. *Let a convex functional $\Omega : \mathcal{Q} \subset \mathbb{R}^n \to \mathbb{R}$ be continuous in an L^p-space $\mathcal{L}$. Suppose $Y \subseteq \mathcal{L}$ is bounded and satisfies* $\inf \Omega(q) < \sup \Omega(q), \forall q \in \mathcal{Q}, \forall q_0 \in Y$, *then we have $q_0 \in \vartheta(\Omega)$ if and only if* $\exists \varphi_0 \in \mathcal{L}', \varphi_0 \neq 0 : \varphi_0(q_0) = \sup_{q \in \mathcal{Q}} \varphi_0(q), \Omega(q) \leq \sup \Omega(Y)$.

Proof. Suppose that $q_0 \in \vartheta(\Omega)$. Let $\mathcal{M} = \{q \in \mathcal{Q} : \Omega(q) \leq \sup \Omega(Y)\}$ where $\mathcal{M} \subseteq \mathcal{L}$. Then $\mathcal{M}$ is convex and $Int\mathcal{M} = \{q \in \mathcal{Q} : \Omega(q) < \sup \Omega(Y) \neq \phi\}$. Now, since $q_0 \in \vartheta(\Omega)$ then $q_0 \notin Int\mathcal{M}$. Thus, there is $\varphi_0 \in \mathcal{L}'$ with $\varphi_0 \neq 0$, satisfying: $\varphi_0(q_0) \geq \sup \varphi_0(\mathcal{M})$. But $q_0 \in Y \subset \mathcal{M}$ implies that $\varphi_0(q_0) \leq \sup \varphi_0(\mathcal{M})$ also holds hence we have, $\varphi_0(q_0) = \sup_{q \in \mathcal{Q}} \varphi_0(q)$. Conversely, suppose that $\forall q_0 \in Y$, there is $\varphi_0 \in \mathcal{L}' : \varphi_0 \neq 0$ satisfying $\varphi_0(q_0) = \sup_{q \in \mathcal{Q}} \varphi_0(q), \Omega(q) \leq \sup \Omega(Y)$. Then, $q_0 \notin Int\mathcal{M}$ where $\mathcal{M}$ is a set defined by $\mathcal{M} = \{q \in \mathcal{Q} : \Omega(q) \leq \sup \Omega(Y)\}$. Therefore, since $Int\mathcal{M} = \{q \in \mathcal{Q} : \Omega(q) < \sup \Omega(Y) \neq \phi\}$ and $q_0 \in Y \subset \mathcal{M}$, we get $q_0 \in \vartheta(\Omega)$ as required. $\square$

Theorem 4.15. *Let $\mathcal{Q}$ be a proper subset of $\mathbb{R}^n$. Assume that a function $\vartheta : \mathcal{Q} \to \overline{\mathbb{R}}$ in an L^p-space $\mathcal{L}$ is convex. If ϑ is lsc then for each $q_0 \in \mathcal{Q}$, and convex function $\varphi : \mathcal{Q} \to \overline{\mathbb{R}}$,* $\vartheta(q_0) = \sup_{\varphi} \inf_{q \in \mathcal{Q}} \{\vartheta(q) - \varphi(q) + \varphi(q_0)\}$.

Proof. By definition of ϑ^* we deduce, $\varphi(q_0) - \vartheta^*(\varphi(q_0)) = \varphi(q_o) - \sup(\varphi - \vartheta)$ Thus $\varphi(q_0) - \vartheta^*(\varphi(q_0)) = \inf_{q \in \mathcal{Q}} \{\vartheta(q) - \varphi(q) + \varphi(q_0)\} (\forall \varphi \in \mathcal{Q}^*)$. Since $\vartheta \in \mathcal{L}$, we have $\vartheta^{**}(q_0) = \omega\vartheta(q_0)$. Therefore, $\vartheta(q_0) = \vartheta^{**}(q_0) =$

$\sup_{\varphi \in \mathcal{Q}^*}\{\varphi(q_0) - \vartheta^*(\varphi)\}$

$$\begin{aligned}\vartheta(q_0) = \vartheta^{**}(q_0) &= \sup_{\varphi \in \mathcal{Q}^*}\{\varphi(q_0) - \vartheta^*(\varphi(q_0))\} \\ &= \sup_{\varphi \in \mathcal{Q}^*} \inf_{q \in \mathcal{Q}}\{\vartheta(q) - \varphi(q) + \varphi(q_0)\}.\end{aligned}$$

□

The following corollary characterizes the notion of *lsc* to the almost *lsc* concept. A function $\vartheta : \mathcal{Q} \to \overline{\mathbb{R}}$ is said to be almost *lsc* if a sequence $\{q_n\} \subseteq \mathcal{Q}$ converging to $q_0 \in \mathcal{Q}$ one has $\vartheta(q_0) \leq \liminf_{n\to\infty} \vartheta(q_n)$.

Proposition 4.16. *Let $\mathcal{Q}$ be a convex and bounded set. Let $\vartheta : \mathcal{Q} \to \overline{\mathbb{R}}$ be a function in an L^p-space $\mathcal{L}$. Then ϑ is said to be almost lsc if and only if it is convex lsc.*

Proof. Let a convex set $\mathcal{Q} \subseteq \mathbb{R}$ be bounded. Suppose $\vartheta : \mathcal{Q} \to \overline{\mathbb{R}}$ is almost *lsc*. Let $\{q_n\}$ and $\{q'_n\}$ be two Moore-Smith sequences in $\mathcal{Q}$ converging to q and q' respectively such that $q, q' \in \mathcal{Q}$. Since ϑ is almost *lsc*, it is *lsc* at q and q'. Then $\vartheta(q) \leq \liminf_{n\to\infty} \vartheta(q_n)$ and $\vartheta(q') \leq \liminf_{n\to\infty} \vartheta(q'_n)$. So $\forall \eta \in [0, 1]$ we can obtain

$$\begin{aligned}\vartheta(\eta q + (1-\eta)q') &\leq \liminf_{n\to\infty}(\eta\vartheta(q_n) + (1-\eta)\vartheta(q'_n)) \\ &\leq \eta \liminf_{n\to\infty} \vartheta(q_n) + (1-\eta) \liminf_{n\to\infty} \vartheta(q'_n) \\ &\leq \eta\vartheta(q) + (1-\eta)\vartheta(q')\end{aligned}$$

This shows that ϑ is convex *lsc*.

Conversely, let ϑ be convex *lsc*. Then $\forall q_n \to q$ and $\forall q'_n \to q'$ we have

$$\vartheta(\eta q + (1-\eta)q') \leq \eta\vartheta(q) + (1-\eta)\vartheta(q').$$

Therefore, ϑ is *lsc* at q and q'. Since $\mathcal{Q}$ is bounded, it follows that, ϑ is almost *lsc*. □

4.3 Upper Semi-continuous functions in L^p Spaces

In this section, we have characterized upper semi-continuity in L^p-space. We let $\vartheta \in usc(\mathcal{L})$ to be an *usc* function and we define the δ-parallel function of ϑ as $\forall\delta > 0, \vartheta^+_\delta(q) = \sup\{\alpha : (q,\alpha) \in \beta_\delta|\overline{\vartheta}|\}$. We have used $B_{usc}(\mathcal{L})$ to denote a set containing all bounded *usc* functions on the L^p-space $\mathcal{L}$ while $C_{usc}(\mathcal{L})$ denotes a set comprising of all continuous *usc* functions on this space.

We start with the following proposition which characterizes *usc* functions of dense sets.

Proposition 4.17. *Let $\mathcal{L}$ be an L^p-space. Let $\{\vartheta_n\} \in \mathcal{L}$ be a convergent sequence of continuous functions whose limit is an usc function ϑ. Then there exists a dense set D such that $\forall q \in D$, $\vartheta(q)$ is a sub-sequential limit of $\{\vartheta_n(q)\}$.*

Proof. Let $m \in \mathbb{Z}^+$ and $k > 0$ then the set $S_{m,k} = \{q : \vartheta(q) - k\} \geq \{\vartheta_n(q), \forall n \geq m\}$ is closed and nowhere dense. Assume $q \in int(S_{m,k})$, then

$\exists\{q_n\} \to q$ as $n \to \infty$ satisfying $\vartheta(q) = \liminf_{n\to\infty}\{\vartheta_n(q_n)\}$. If $\exists M \in \mathbb{Z}^+$ then, $\forall n > M$, $\{\vartheta_n(q_n)\} > \{\vartheta(q) - \frac{k}{2}\}$ and $\{\vartheta(q_n)\} < \{\vartheta(q) + \frac{k}{2}\}$. This implies that $\{\vartheta_n(q_n)\} > \{\vartheta(q) - \frac{k}{2}\} > \{[\vartheta(q_n - \frac{k}{2}] - \frac{k}{2}\} = \{\vartheta(q_n) - k\}$. Clearly this is a contradiction to our assumption that $\{q_n\} \in S_{m,k}$. Now, for any $m, n \in \mathbb{Z}^+$ suppose $E_{m,n} = S_{m,\frac{1}{n}}$. Since $\mathcal{L}$ is an L^p-space, it is complete and so each Cauchy sequence existing in $\mathcal{L}$ must have its limit in $\mathcal{L}$. Thus $G = \cap_{m=1,n=1}^{\infty}(E_{m,n})^c$ is a dense G_δ set. Then a sub-sequence of G_n say $\{\vartheta_{n_m}\}$ exists satisfying $\forall q \in G, \vartheta_{n_m}(q) > \vartheta(q) - \frac{1}{k}$. Therefore, $\forall q \in \mathcal{L}, \limsup_{m\to\infty} \vartheta_{n_m}(q) \leq \vartheta(q)$. □

The next theorem shows that an *usc* function $\vartheta(\varphi)$ is equal to the Lebesgue integral of a convex function φ with respect to a Borel measure μ.

Theorem 4.18. *Let $\mathcal{L}$ be an L^p-space. Let ϑ be a function in $\mathcal{L}$ and let the function φ be integrable with respect to a Borel measure μ. If $\vartheta(\varphi) = \int_{\mathcal{L}} \varphi d\mu$ such that φ is measurable, then ϑ is usc.*

Proof. Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be sequence in $usc(\mathcal{L})$ whose limit is $\varphi \in usc(\mathcal{L})$. For any positive integer r we have $q_r = \varphi^+_{\frac{1}{r}}$. Clearly each $q_r \in usc(\mathcal{L})$. Now, since $\{q_r\}$ forms a monotonic sequence of limits that approach φ, $\vartheta(q_r) = \vartheta(\varphi)$. We see that $\vartheta\varphi) < \infty$ since φ is bounded above and $\mu(\mathcal{L}) < \infty$. Choose r such that $\vartheta(q_r) < \vartheta(\varphi) + \varepsilon, \forall\varepsilon > 0$. Because $\{\varphi_n\}$ converges to φ, then $\exists N \in \mathbb{Z}^+ : \forall n > N, |\varphi_n - \varphi| \leq \frac{1}{r}$. By Theorem 3.5 $\varphi_n \leq q_r$ so that $\vartheta(\varphi_n) \leq \vartheta(q_r) < \vartheta(\varphi) + \varepsilon$ showing that $\vartheta \in usc(\mathcal{L})$. □

The corollary below follows from Theorem 4.18.

Corollary 4.19. *Let $\mathcal{L}$ be an L^p-space and $\vartheta \in \mathcal{L}$ be a Lebesgue integral induced by a Borel measure μ. Suppose the sequence $\{\varphi_n\} \in usc(\mathcal{L})$*

converges to a measurable function $\varphi \in usc(\mathcal{L})$. *Define* $\vartheta_r(\varphi_n) = \vartheta[(\varphi_n)^+_{\frac{1}{r}}]$ *such that* $\vartheta_r : \{\varphi_n : n \in \mathbb{Z}^+\} \to \mathbb{R}, \forall r \in \mathbb{Z}^+$. *Then,* $lim_{n\to\infty}\vartheta(\varphi_n) = \vartheta(\varphi)$ $\iff$ $\{\vartheta_r\}$ *converges uniformly to* ϑ *on* $\{\varphi_n\}$.

Proof. Let $F = \{\varphi\} \cup \{\varphi_n\}$. Since $lim_{n\to\infty}\vartheta(\varphi_n) = \vartheta(\varphi)$, ϑ is continuous on F and φ is its unique limit point. Suppose $\vartheta_r(\varphi) = \vartheta(\varphi^+_{\frac{1}{r}})$. Then, ϑ is *usc* and forms a decreasing sequence. Therefore, for any $a \in F, \vartheta_1(a) > \vartheta_2(a) \geq ... \geq \vartheta_r(a)$ hence $\lim_{r\to\infty} \vartheta_r(a) = \vartheta(a)$. So by Theorem 3.4, it follows that, $\{\vartheta_r\}$ is uniformly convergent on $\{\varphi_n\}$. On the converse, let $\lim_{r\to\infty} \vartheta_r(a) \neq \vartheta(a)$. Because $\vartheta \in usc(\mathcal{L})$ at φ we know that, $|\varphi_n - \varphi| \leq \frac{1}{r}$ and $\vartheta(\varphi_n) < \vartheta(\varphi) - \varepsilon$.
Now, $(\varphi_n)^+_{\frac{1}{n}} > \varphi$ implies that

$$\vartheta_r(\varphi_n) \geq \vartheta(\varphi) > \vartheta(\varphi_n) + \varepsilon.$$

Thus $\{\vartheta_r\}$ cannot converge uniformly to ϑ on $\{\varphi_n\}$. Hence $\lim_{r\to\infty} \vartheta_r(a) = \vartheta(a)$. □

The next theorem characterizes upper semi-continuity in terms of the hypo-graph of a function.

Theorem 4.20. *If* φ *is a usc function in an* L^p*-space* $\mathcal{L}$, *then* $\forall\delta > 0$: $\beta_\delta|hypo(\varphi)| = hypo(\varphi)^+_\delta$.

Proof. If $(q, \upsilon) \in \eta_\delta|hypo\varphi|$ there is $(s, \eta) \in hypo\varphi$ yielding $\delta \geq d[(q, \upsilon), (s, \eta)]$. Now, $d[(s, \varphi(s)), (s, \upsilon + \varphi(s) - \eta] = d[(s, \eta), (q, \upsilon)]$ implying that $\upsilon + \varphi(s) - \eta \leq \varphi^+_\delta(q)$. From this inequality we see that $\upsilon \leq \upsilon + \varphi(s) - \eta$ showing that $(q, \upsilon) \in \text{hypo}(\varphi)^+_\delta$. On the converse assume that $(q, \upsilon) \in \text{hypo}(\varphi)^+_\delta$.

Then, $\exists\{s_n\} \in \mathcal{L} : d[(q, \varphi_\delta^+(q)), (s_n, \varphi(s_n))] \leq \delta + \frac{1}{n}, \forall n = 1, 2, ...$ Since $\{\varphi(s_n)\} \in B(\mathcal{L})$ we assume that the sub-sequence $\{(s_n, \varphi(s_n))\}$ converges to (s, η). Since $\varphi \in usc(\mathcal{L})$, then $hypo\varphi$ is closed and $(s, \eta) \in hypo\varphi$. Clearly $d[(q, \varphi_\delta^+(q)), (s, \eta)] \leq \delta$. Therefore, $(q, \varphi_\delta^+(q)) \in \eta_\delta[hypo\varphi]$. Moreover, $(q, \upsilon) \in \eta_\delta[\text{hypo}(\varphi)]$ since $\upsilon \leq \varphi_\delta^+(q)$. □

The corollary below follows from Theorem 4.20.

Corollary 4.21. *Let $\mathcal{L}$ be an L^p-space and suppose $\vartheta \in usc(\mathcal{L})$. Then $\forall\delta > 0 \in \mathbb{Z}^+$, a function ϑ_δ^+ is upper semi-continuous in $\mathcal{L}$ and bounded.*

Proof. We need to show that $\vartheta_\delta^+ \in usc(\mathcal{L})$ and is bounded below. Assume ϑ_δ^+ is not bounded below. Then $\exists\{q_n, h_n\} \in (\text{hypo}(\vartheta)_\delta^+)^c$ such that $h_n < -n, \forall n$. The distance between $\{q_n, h_n\}$ and each point of hypo(ϑ is greater or equal to δ. We assume that $a_n \to a$. Thus, (q_n, ϱ_n) is arbitrarily close to the half line $\{(q, \vartheta) : \vartheta(q) \geq q, \}$ as $n \to \infty$. Hence $\{(q, \vartheta) : q \in \vartheta(q)\}$ is a subset of the hypo-graph of ϑ which is a contradiction to our assumption. Therefore ϑ_δ^+ is bounded below. To show that $\vartheta_\delta^+ \in usc(\mathcal{L})$ we apply Theorem 4.20. Since ϑ_δ^+ closed and bounded, we have $\beta_\delta|\text{hypo}(\vartheta| = \text{hypo}(\vartheta_\delta^+$, because parallel bodies of closed sets are closed. Therefore, for every $\delta > 0$, we have, $\vartheta_\delta^+ \in usc(\mathcal{L})$. □

Proposition 4.22. *Let $\mathcal{Q}$ be a convex and bounded set. Let $d = \max\{d(r_1, r_2), |\alpha_1 - \alpha_2|\}$ define the distance between two points $(r_1, \alpha_1) \in \mathcal{Q}$ and $(r_2, \alpha_2) \in \mathcal{Q}$. If $\vartheta : \mathcal{Q} \to \overline{\mathbb{R}}$ and $\varpi : \mathcal{Q} \to \overline{\mathbb{R}}$ are usc functions in an L^p-space $\mathcal{L}$, then $d_1(\vartheta, \varpi) \geq d_2(\vartheta, \varpi) \geq d_3(\vartheta, \varpi)$.*

Proof. Assume $d_1(\vartheta, \varpi) = \delta$. Then, $\forall r \in \mathbb{R}^n$, $d[(r, \vartheta(r)), (r, \varpi(r))] \leq \delta$ implies $\vartheta \in \beta_\delta[\overline{\varpi}]$ and $\varpi \in \beta_\delta[\overline{\vartheta}]$. $\beta_\delta[\overline{\varpi}]$ is a closed set hence $\overline{\vartheta} \subset$

$\beta_\delta[\overline{\varpi}]$. Also $\beta_\delta[\overline{\vartheta}]$ is closed implying that $\overline{\varpi} \subset \beta_\delta[\overline{\vartheta}]$. Therefore, $d_2(\varphi, \varpi) \leq d_1(\vartheta, \varpi)$. Now, let $\beta_\delta[\overline{\vartheta}] \supset \overline{\varpi}$ and $\beta_\delta[\overline{\varpi}] \supset \overline{\vartheta}$. Then $\forall r \in \mathbb{R}^n$ we have, $\vartheta_\delta^+(r) \geq \varpi(r)$. Thus by Theorem 4.20, $\beta_\delta[hypo\vartheta] = hypo\vartheta_\delta^+ \supset hypo\varpi$. In the same manner $\beta_\delta[hypo\varpi] \supset hypo\vartheta$, hence, $d_2(\vartheta, \varpi) \geq d_3(\vartheta, \varpi)$. □

The theorem below presents a characterization of *usc* functions in terms of Moore-Smith sequences.

Theorem 4.23. *If* $\{\varphi_n\} \in usc(\mathcal{L})$ *in an* L^p*-space* $\mathcal{L}$, *then* $\{\varphi_n\}$ *converges to* φ *if*

(i) $\limsup_{n\to\infty} \varphi_n(q_n) \leq \varphi(q), \forall q \in \mathcal{Q} \subset \mathbb{R}^n$ *whenever* $\{q_n\} \to q$.

(ii) A Moore-Smith sequence $\{q_n\}$ *converging to* q *exists satisfying* $\forall q \in \mathcal{Q}$, $\limsup \varphi_n(q_n) \geq \varphi(q)$.

Proof. Let $\limsup_{n\to\infty} \varphi(q_n) \leq \varphi(q), \forall q \in \mathcal{Q}$ whenever $\{q_n\} \to q$. Then, $\forall \xi > 0$, there is an integer N_q and a function $g(q) \in (0, \xi)$ satisfying $\varphi_n(s) < \varphi(q) + \xi, \forall n \geq N_q$ whenever $d(q, s) < g(q)$. Pick $\{q_1, ..., q_k\} \subset \mathcal{Q}$ for which $\mathcal{Q} \subset \cup_{i=1}^k \{s : d(q_i, s) < g(q_i)\}$. Suppose $N = \max\{N_t, ..., N_{z_k}\}$ and for any arbitrary $q \in \mathcal{Q}$ choose q_i satisfying $d(q, q_i) < g(q_i) < \xi$. Then, $d[(q, \varphi_n(q)), (q_i, \varphi_n(q) - \xi)] = \xi$. Now if we let $n > N$, then $\varphi_n \subseteq \beta_\xi[hypo\varphi]$ because $(q_i, \varphi_n(q) - \xi) \in hypo\varphi$. Suppose now that a sequence $\{q_n\}$ exists converging to q satisfying $\forall q \in \mathcal{Q}, \liminf \varphi_n(q_n) \geq \varphi(q)$. Let $\xi > 0$ and for every $q \in \mathcal{Q}$ choose $g(q) < \frac{\xi}{2}$ such that $d(s, q) < g(q)$ implies $\varphi(s) < \varphi(q) + \frac{\xi}{2}$. Select $q_1, ..., q_k$ to satisfy $\mathcal{Q} \subset \cup_{i=1}^k \{s : d(s, q_i) < g(q_i)\}$. For $n \geq N$, a sub-sequence $\{q_n, ..., q_{n_k}\} \subset \mathcal{Q}$ satisfying $\forall i = 1, .., k, \varphi(q_i) - \varphi_n(q_{n_i}) < \frac{\xi}{2}$ and $d(q_{n_i}, q_i) < g(q)$. For arbitrary $q \in \mathcal{Q}$, pick q_i satisfying $d(q_i, q) < g(q_i)$. Clearly $d(q, q_{n_i}) < \xi$ and thus $\varphi(q) < \varphi_n(q_{n_i}) + \xi$. Hence

$(q, \varphi(q)) \in \beta_\xi[hypo\varphi_n], \forall n \geq N$. Thus $hypo\varphi \subset \beta_\xi[hypo\varphi_n]$. This shows that $\{\varphi_n\}$ converges to φ. Conversely, assume $\{\varphi_n\}$ converges to φ. If $\{[q_n\} \to q$, then, $\exists(s_n, \beta_n) \subset hypo\varphi : [(s_n, \beta_n), (q_n, \varphi_n(q_n))] \to 0$. Given that $\varphi \in usc(\mathcal{L})$ and $\{s_n\} \to q$, $\limsup_{n\to\infty} \varphi_n(q_n) = \limsup_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \varphi(s_n) \leq \varphi(q)$. Thus we have proved (i). In the same manner there exists $\{q_n, \alpha_n\} \in hypo\varphi_n$ such that $d[(q_n, \alpha_n), (q, \varphi(q))] \to 0$. Given that $\{q_n\} \to q$ and $\{\alpha_n\} \to \varphi(q)$, then $\forall n, \varphi_n(q_n) \geq \alpha_n$. It thus follows that

$$\varphi(q) = \lim_{n\to\infty} \alpha_n \leq \liminf_{n\to\infty} \varphi_n(q_n)$$

hence proving (ii). □

Theorem 4.24. *Let $\mathcal{L}$ be an L^p-space and let a nonempty convex set G form the domain of a function $\vartheta \in \mathcal{L}$. Then $\vartheta \in usc(\mathcal{L}) \iff \vartheta$ is convex.*

Proof. Let $\vartheta \in usc(\mathcal{L})$. Let $g, q \in G$. Given that G is an open set and from upper semi-continuity of ϑ an open set $H = \{g \in G : \vartheta(g) < \varphi(q) + 1\} = G \cap \vartheta^{-1}(-\infty, \vartheta(q) + 1)$ exists. Now $\forall g \in G$, $\vartheta(g) < \vartheta(g) + 1$ implying that ϑ is bounded on G. So G is a neighborhood of g. Boundedness from above property of ϑ implies that ϑ is continuous at g. Let ϑ at a point $g \in G$ be continuous and $q \in G$ be another point. Then $\vartheta \to g + \vartheta(q - g)$ is continuous. Since G is open, a line passing through g to q is contained in G for some length beyond q i.e $\exists \varrho > 1 : g + \varrho(q - 1) \in G$. Picking $\varrho = g + \vartheta(q - g)$ we deduce

$$\begin{aligned} \varrho &= (1 - \vartheta)g + gq \\ q &= (1 - \frac{1}{\vartheta})g + \frac{1}{\vartheta}\varrho \\ q &= \alpha g + (1 - \alpha)\varrho \end{aligned}$$

for all $0 < \alpha < 1$ with $\alpha = 1 - \frac{1}{q}$. Since ϑ is continuous at g, then $\forall \varepsilon > 0$ an open neighborhood S of 0 exists satisfying $|\vartheta(q) - \vartheta(g)| < \varepsilon$ whenever $q \in g + S$ and taking $g + S \subseteq G$. There is some M yielding $\vartheta(h) \leq M$ for $h \in g + S$.

$$\forall g \in S, \alpha h = \alpha g + (1-\alpha)\varrho + \alpha h = \alpha(g + S) + (1-\alpha)\varrho$$

where $g+S \in G$ and $\varrho \in G$. Since G is convex, then $z + \alpha h \in G$ implying that $q + \alpha h \subseteq G$. Since ϑ is convex,

$$\begin{aligned} \vartheta(q + \alpha h) &= \vartheta(\alpha(g+h) + (1-\alpha)\varrho) \\ &\leq \alpha\vartheta(g+h) + (1-\alpha)\vartheta(\varrho) \\ &\leq \alpha M + (1-\alpha)\vartheta(\varrho), \forall h \in S \end{aligned}$$

implying that ϑ is bounded above by $\alpha M + (1-\alpha)\vartheta(\varrho)$ on $g + \alpha h$. Hence ϑ is continuous at q and at each point in G. □

Corollary 4.25. *Let G be a compact convex subspace of $\mathcal{L}$. If $\vartheta \in usc(\mathcal{L})$, then there exists an extreme point of G which is a maximizer of ϑ.*

Proof. Let G be compact and ϑ be *usc* function. Then $H = \{y \in G : \vartheta(y) = \sup_{q \in G} \vartheta(q)\}$ where $H \subseteq G$ is nonempty. Now H is an extreme set of G given by the convexity of G and ϑ. Clearly, G being compact and $H \subseteq G$ is closed, H is also compact. Consequently $H \subseteq G$ being an extreme set that is compact in G, there exists an extreme point in $y \in H \subseteq G$ such that $\vartheta(y) \geq \vartheta(q), \forall q \in G$. □

4.4 Conditions for Convex Optimization in L^p-spaces

In this part we have developed and examined the requirements necessary for convex optimization in L^p-spaces. The first Propositions 4.26 and 4.27 below show that if a function in a strongly sequentially bounded convex sub-space of a convex L^p-space taking a convex closed set to the extended real line is Lipschitz continuous and *lsc* then it must attain minimizers in its domain.

Proposition 4.26. *Let the sub-space $\mathcal{Q} \subset \mathcal{L}$ of a convex L^p-space $\mathcal{L}$ be strongly sequentially bounded. If a function $\vartheta : G \to \overline{\mathbb{R}}$, where G is a convex closed set, is Lipschitz continuous in G, then ϑ is lsc and attains a minimizer on G.*

Proof. Let $\{q_n\}_{n\in\mathbb{N}} \in G$ be a sequence converging strongly to $\overline{q}$. Since $\mathcal{Q}$ is bounded (from hypothesis), a subsequence $\{q_{n_k}\}$ of $\{q_n\}$ exists converging strongly to $\overline{q}$. Closure of G implies that $\overline{q} \in G$. Now, since ϑ is given to be Lipschitz continuous and $\{q_{n_k}\}$ converges to $\overline{q}$, we have $\vartheta(\overline{q}) \leq \liminf \vartheta(q_{n_k})$. Clearly $\vartheta \in lsc(\mathcal{L})$. We proceed to show that a minimizer exists in $\mathcal{Q}$. Given that the sequence $\{q_n\}$ is convergent, we have $\vartheta(q_n) \to \inf \vartheta(q)$ for each $q \in G$. This shows that $\{q_n\}$ is minimized on G by ϑ. Since $\mathcal{Q}$ is strongly sequentially bounded and closed, there exists a subsequence $\{q_{n_k}\}$ of $\{q_n\} \in G$ converging strongly to $\overline{q} \in G$. Furthermore, if α is a minimizer on G, since $\vartheta \in lsc(\mathcal{L})$ we obtain $\vartheta(\overline{q}) \leq \liminf \vartheta(q_{n_k}) = \lim \vartheta(q_{n_k}) = \alpha$. Therefore, $\vartheta(\overline{q}) = \alpha$ is the required minimizer on G. □

Proposition 4.27. *Let $\mathcal{Q} \subset \mathcal{L}$ be a strongly sequentially bounded subspace* *of a convex L^p-space $\mathcal{L}$. Let G be a a nonempty convex closed set. If a function $\vartheta : G \to \overline{\mathbb{R}}$ is Lipschitz continuous in $\mathcal{Q}$ then, ϑ is weakly lsc in $\mathcal{Q}$ and attains a minimizer on G.*

Proof. Take a convergent sequence $\{e_n\}, \forall e \in G$ to approach $\overline{e}$ strongly. Since G is strongly sequentially bounded and closed, $\overline{e} \in \vartheta(\overline{e})$. So $\langle \overline{e} - \vartheta(\overline{e}), e_n - \vartheta(\overline{e}), r - \vartheta(\overline{e}) \rangle \leq 0, \forall r \in G, n \in \mathbb{N}$. Therefore,

$$\begin{aligned} \langle \overline{e} - \vartheta(\overline{e}), e_n - \vartheta(\overline{e}) \rangle &\leq 0 \\ \|\overline{e} - \vartheta(\overline{e})\|^2 &= \langle \overline{e} - \vartheta(\overline{e}), \overline{q} - \vartheta(\overline{e}) \rangle \end{aligned}$$

Since $e_n \to \overline{e}$ we obtain, $\lim_{n\to\infty} \langle \overline{e} - \vartheta(\overline{e}), e_n - \vartheta(\overline{e}) \rangle = \langle \overline{e} - \vartheta(\overline{e}), e_n - \vartheta(\overline{e}) \rangle = 0$. Hence, $\|\overline{e} - \vartheta(\overline{e})\| = 0$, thus $\overline{e} = \vartheta(\overline{e})$, showing that G is weakly sequentially closed. Since $\vartheta : G \to \overline{\mathbb{R}}$ is convex and Lipschitz continuous, then by Proposition 4.26 $\vartheta \in lsc(\mathcal{L})$ and has a minimizer in G. We now need to show that ϑ is weakly lsc function in $\mathcal{Q}$. Clearly epi(ϑ) is convex since ϑ was given as convex. Strong lower semi-continuity of ϑ implies strong closure of epi(ϑ). Therefore,epi(ϑ) is wsc indicating that ϑ is $w - lsc$. □

The following lemma characterizes solvability property for unconstrained convex program with a $w - lsc$ coercive objective.

Lemma 4.28. *Let $(\mathcal{L}, \|.\|_p)$ be an L^p-space and G be a convex set. Assume a function $\vartheta : G \to \overline{\mathbb{R}}$ in $\mathcal{L}$ satisfying $\vartheta \neq +\infty$ is $w - lsc$ and coercive. If $\vartheta(q) = \inf_{q\in G} \vartheta(q)$ is a convex optimization problem, then, $\vartheta(q)$ attains a solution $q^* \in G$.*

Proof. Suppose $q^* = \inf_{q \in G} \vartheta(q_n)$ and assume ϑ minimizes a convergent sequence $\{q_n\}_{n \in \mathbb{N}}$ in G to $\vartheta(q_n) \to q^*$ as $n \to \infty$. Since ϑ is coercive and $q^* < +\infty$, then $\{q_n\}$ is a bounded sequence. Therefore, $\exists \{q_{n_k}\} : q_{n_k} \to q^* \in G$ strongly. Furthermore, since ϑ is $w-lsc$, $\vartheta(q) \leq \inf_{q \in G} \vartheta(q_n) = q^*$. Hence $\vartheta(q) = q^*$. □

Theorem 4.29. *Suppose a finite function $\vartheta : G \to \overline{\mathbb{R}}$ is convex and coercive in an L^p-space $\mathcal{L}$. Assume that ϑ is $w-lsc$. If $\vartheta(q) = \inf_{q \in G} \vartheta(q)$ is a convex optimization problem, then, $\vartheta(q)$ attains a solution $q^* \in G$. Furthermore, strict convexity in ϑ guarantees a unique solution q^*.*

Proof. Let $\vartheta(q) = \inf_{q \in G} \vartheta(q)$ be a convex optimization problem. Since $\vartheta \in lsc(\mathcal{L})$ is convex and coercive (from hypothesis), then $\vartheta(q_n) \to \vartheta(q)$ as $\|q_n\| \to q \ \forall n \in \mathbb{N}$ (by Lemma 4.28). Thus $\vartheta(q)$ attains an optimal solution $q^* \in G$. To prove uniqueness of this solution assume $q_1^* \neq q_2^*$ two optimal solutions for the unconstrained convex optimization problem $\vartheta(q) = \inf_{q \in G} \vartheta(q)$. Then we have, $\vartheta(\frac{1}{2}(q_1^* + q_2^*)) < \frac{1}{2}\vartheta(q_1^*) + \frac{1}{2}\vartheta(q_2^*) = \inf_{q \in G} \vartheta(q^*)$. This is a contradiction. Thus $q_1^* = q_2^* = q^*$. □

The next theorem establishes that if a function $\vartheta : G \to \overline{\mathbb{R}}$ is Gateaux-differentiable over a convex set G of constraints, then for $q \in G$ to minimize ϑ, the Gateaux-derivative of ϑ with respect to q is a necessary condition.

Theorem 4.30. *Let a function $\vartheta < +\infty$ in an L^p-space $\mathcal{L}$ be Gateaux-differentiable over a convex set G. If the Gateaux-derivative of ϑ is given by $\vartheta'(q) : q \in G$ then, $\langle \vartheta'(q), r \rangle = 0, \forall r \in G$ is necessary for $q \in G$ to minimize ϑ.*

Proof. Let $q \in G$ minimize ϑ then, $\vartheta(q \pm \kappa r) \geq \vartheta(q), \forall \kappa \geq 0, r \in G$, hence, $\langle \vartheta'(q), r \rangle \geq 0$. Take $\langle \vartheta'(q), r \rangle = 0$. Now, since G is convex we have,

$$\begin{aligned} \vartheta(q + \kappa(r - q)) &= \vartheta(\kappa r + (1 - \kappa)q) \\ &\leq \kappa\vartheta(q) + (1 - \kappa)\vartheta(q) \end{aligned}$$

So

$$\begin{aligned} 0 &= \langle \vartheta'(q), r - q \rangle \\ w &= lim_{\kappa \to 0} \frac{\vartheta(q + \kappa(r - q)) - \vartheta(q)}{\kappa} \\ &\leq \vartheta(r) - \vartheta(q). \end{aligned}$$

□

The findings in the following propositions and theorem show the conditions for existence of minimizers in sequentially bounded and compact regions.

Proposition 4.31. *Let G be a sequentially bounded and compact set. Let $\vartheta : G \to [-\infty, +\infty]$ be a lsc function in an L^p-space $\mathcal{L}$. If the convex set $\{\vartheta(q) \leq Q, \forall q \in G, \forall Q \in \mathbb{R}\}$ is compact then, there is a local minimizer $\overline{q}$ of $min_{q \in G} \vartheta(q)$*

Proof. Suppose $\eta = \inf_{q \in R} \vartheta(q)$. Since $\vartheta(q)$ is bounded, then $\exists \{q_n\}$: $\vartheta(q_n) \to \eta$. As n becomes sufficiently large, we have $\vartheta(q_n) \leq Q$, implying that $\{q_n\}$ is in a compact set. Since the bounded sequence $\{q_n\}$ is compact, $\exists \{q_{n_k}\}$ tending to $\overline{q}$ for some $\overline{q} \in G$. Since $\vartheta \in usc(\mathcal{L})$,

$\eta \leq \vartheta(\overline{q}) \leq \liminf_{n\to\infty} \vartheta(q_n) = \eta$ showing that the global minimizer of $\vartheta(q)$ is $\overline{q}$. Hence, $\overline{q}$ is also the local minimizer because $\vartheta(q)$ is convex. □

Theorem 4.32. *Let G be a sequentially bounded and compact set. Let $\vartheta \neq \emptyset : G \to [-\infty, +\infty]$ be a lsc function in an L^p-space $\mathcal{L}$. If ϑ meets the compactness and convexity conditions, then the set of all local minimizers of $min_{q\in G}\vartheta(q)$ is compact.*

Proof. Given that the set of constraints G is sequentially bounded and compact, then by Proposition 4.31 and convexity of ϑ local minimizers of $\min_{q\in G}\vartheta(q)$ exist and they lie in the level set $\{\vartheta(q) \leq Q, \forall Q \in \mathbb{R}.\}$ This shows pre-compactness. It now suffices to prove the closedness property. Since $\vartheta \in lsc(\mathcal{L})$, then $\forall q$ in the closure of G we obtain,

$$\eta \leq \vartheta(q) \leq \inf_{q\in R} \vartheta(q) \leq \eta.$$

□

Proposition 4.33. *Let the constraint set G be sequentially bounded and compact. Let $\vartheta \neq \emptyset : G \to [-\infty, +\infty]$ in an L^p-space $\mathcal{L}$ be a Fréchet-differentiable linear function. Supposing that the local minimizer of $min_{q\in G}\vartheta(q)$ is $\overline{q}$ then, $\vartheta'(\overline{q}) = 0$.*

Proof. Let $\vartheta(q) \to \min_{q\in G}$ be a convex optimization problem whose local minimizer is $\overline{q}$. For each $r \in G$ and $\eta \in \mathbb{R}^+$, $\vartheta(\overline{q}+\eta r)-\vartheta(\overline{q}) \geq 0$ holds true as η approaches zero. Therefore, $\vartheta(\overline{q})r = \lim_{q\downarrow 0} \frac{\vartheta(\overline{q}+\eta r)-\vartheta(\overline{q})}{\eta} \geq 0, \forall r \in G$. Linearity of $\vartheta(\overline{q})$ gives us $0 \leq \vartheta(\overline{q})(-r) = -\vartheta(\overline{q})r \leq 0$ implying that $\vartheta'(\overline{q})r = 0, \forall r \in G$. □

The next proposition shows a condition for optimality in convex optimization using the notion of Fréchet-differentiability.

Proposition 4.34. *Let $\mathcal{L}$ be an L^p-space and $\vartheta \in \mathcal{L}$ be a convex function from a convex set G to the extended real line $\overline{\mathbb{R}}$. If ϑ is Fréchet-differentiable then it satisfies $\partial\vartheta(q) = \{\vartheta'(q)\}$ at $q \in G$.*

Proof. If $g \in \partial\vartheta(q)$ define linear functionals in $\mathcal{L}^*$ then, $\forall\kappa > 0$ and $\forall r \in G$,

$$\begin{aligned} \frac{\vartheta(q+\kappa r)-\vartheta(q)}{\kappa} &\geq \langle g, r\rangle \\ \frac{\vartheta(q-\kappa r)-\vartheta(q)}{\kappa} &\geq -\langle g, r\rangle. \end{aligned}$$

Thus, as $\kappa \to 0$, $\langle g, r\rangle \leq \vartheta'(q)r \leq \langle g, r\rangle, \forall r \in G$. This shows that $g = \vartheta'(q)$ in $\mathcal{L}^*$ hence $\partial\vartheta(q) = \{\vartheta'(q)\}$. □

The following results presented by Lemma 4.35, Theorem 4.36 and Theorem 4.37 establish convex optimality conditions for twice continuous Fréchet-differentiable functions.

Lemma 4.35. *Let G be a convex constraint set for a convex optimization problem $min_{q\in G}\vartheta(q)$. Let $\vartheta : G \to [-\infty, +\infty]$ n an L^p-space $\mathcal{L}$ be twice continuous Fréchet-differentiable. If $\vartheta'(\overline{q}) = 0, \forall\overline{q} \in G$ and $\vartheta''(\overline{q})\langle r, r\rangle \geq \kappa\|r\|^2$ for each $r \in G$ and $\kappa \neq r \in \mathbb{R}^+$ then, $\overline{q}$ forms a local minimizer of ϑ.*

Proof. The Taylor expansion of $\vartheta(r)$ yields

$$\vartheta(r) = \vartheta(\overline{q}) + \vartheta'(\overline{q})\langle r - \overline{q}\rangle + \frac{1}{2}\vartheta''(\overline{q})\langle r - \overline{q}, r - \overline{q}\rangle + 0(\|r - \overline{q}\|^2).$$

This is true in the whole neighborhood of $\overline{q}$. Assume there exists a $\delta > 0$ satisfying

$$\vartheta(r) \geq \vartheta(\overline{q}) + \vartheta'(\overline{q})\langle r - \overline{q}\rangle + \frac{1}{2}\vartheta''(\overline{q})\langle r - \overline{q}, r - \overline{q} - \frac{\kappa}{4}\|r - \overline{q}\|^2$$

such that $\|r - \overline{q}\| < \delta$.

Substituting $\vartheta(\overline{q}) = 0$ and applying the positive definiteness property, $\vartheta''(\overline{q})\langle r, r\rangle \geq \kappa\|r\|^2$, we obtain $\vartheta(r) \geq \vartheta(\overline{q}) + \frac{\kappa}{4}\|r - \overline{q}\|^2$ leading to $\|r - \overline{q}\| < \delta$. Therefore, $\vartheta(r) > \vartheta(\overline{q})$ if $r \neq \overline{q}$. Hence $\overline{q}$ is a local minimizer. □

Theorem 4.36. *Let the set G be sequentially bounded. Let a set $S \subseteq G$ be closed. Define a tangential cone at $q \in G$ by $T_S(q) = \{g \in G : \exists\xi > 0, \forall\kappa \in [0, \xi], \exists g(\kappa) \in G : \|q + \kappa s - g(\kappa)\| = O(\kappa)\}$. Let $\vartheta : G \to [-\infty, +\infty]$ in an L^p-space $\mathcal{L}$ be twice continuous Fréchet-differentiable. If $\overline{q}$ minimizes $\vartheta(q)$ locally in S then, $\vartheta'(\overline{q})s \geq 0, \forall s \in T_S(\overline{q})$.*

Proof. For each $s \in T_S(q)$ and $0 \leq \kappa \leq \xi$ we get,

$$\begin{aligned}\vartheta(g(\kappa)) &= \vartheta(\overline{q} + \kappa s) + O(\kappa) \\ &= \vartheta(\overline{q}) + \kappa\vartheta'(\overline{q})s + O(\kappa).\end{aligned}$$

Suppose $\vartheta'(\overline{q})s < 0$ then, as κ approaches zero we have

$$\vartheta(y(\kappa)) \leq \vartheta(\overline{q}) + w\vartheta'(\overline{q})s - \frac{\kappa}{2}\vartheta'(\overline{q})s < \vartheta(\overline{q}).$$

This contradicts our assumption that $\overline{q}$ is a local minimizer. Therefore, $\vartheta'(\overline{q})s \geq 0, \forall s \in T_S(\overline{q})$ as required. □

Theorem 4.37. *Let G be a sequentially bounded set and $S \subseteq G$ be closed* *and convex. Let $\vartheta : G \to [-\infty, +\infty]$ in an L^p-space $\mathcal{L}$ be twice continuous Fréchet-differentiable. Then $\overline{q} \in S$ forms a local minimizer of $\vartheta(q)$ in S if it satisfies $\vartheta'(\overline{q})s \geq 0, \forall s \in T_S(\overline{q})$ and $\vartheta''(\overline{q})\langle s, s\rangle \geq \eta \|s\|^2, \forall s \in T_S(\overline{q}), \eta >$* 0.

Proof. Let $\|y - \overline{q}\| \to 0$, for each $y \in S$. Since S is convex, we have $\overline{q} + ws \in S, \forall 0 \leq w \leq 1 : s = y - \overline{q}$. If $s = T_S(\overline{q})$ we get

$$\vartheta(y) \geq \vartheta(\overline{q}) + \vartheta'(\overline{q})s + \frac{1}{2}\vartheta''(\overline{q})\langle s, s\rangle - \frac{\kappa}{4}\|s\|^2$$

for $\|s\| \to 0$. Hence, $\vartheta(y) \geq \vartheta(\overline{q}) + \frac{\kappa}{4}\|s\|^2 > \vartheta(\overline{q}), \forall y \in S$ such that $\|y - q\| \to 0$. Thus, $\overline{q}$ forms a local minimizer of $\vartheta(q)$ in S. □

Theorem 4.38. *Let the set G be a convex constraint set for the convex optimization problem $min_{q \in G}\vartheta(q)$. If a function $\vartheta : G \to [-\infty, +\infty]$ in an L^p-space $\mathcal{L}$ is convex, then, every single local minimum forms a global minimum and moreover, $\overline{q} \in G$ forms a minimizer for $\vartheta(q) \iff$ $0 \in \partial\vartheta(q)$.*

Proof. Assume that $\overline{q} \in G$ minimizes $\vartheta(q)$ locally and not globally. So $\exists y \in G : \vartheta(y) < \vartheta(\overline{q})$. Convexity of G yields $q_\kappa = \kappa y + (1 - \kappa)\overline{q} \in G, \forall \kappa \in [0, 1]$. Also, claiming convexity of ϑ we deduce $\vartheta(q_\kappa) = \kappa\vartheta(y) + (1 - \kappa)\vartheta(\overline{q}) < \vartheta(\overline{q})$ for $q_\kappa \to \overline{q}$ as $\kappa \to 0$. This contradicts the assumption of $\overline{q}$ minimizes $\vartheta(q)$ strictly locally. Hence $\overline{q}$ is a global minimizer.
Now, suppose $0 \in \partial\vartheta(\overline{q})$. Then, $\forall h \in G$,

$$\begin{aligned} \vartheta(h) &\geq \vartheta(\overline{q}) + \langle 0, h - \overline{q}\rangle \\ &= \vartheta(\overline{q}) \end{aligned}$$

showing that $\overline{q}$ is a global minimizer.

Conversely, assume $\overline{q}$ minimizes $\vartheta(q)$ globally and let $0 \notin \partial\vartheta(\overline{q})$. Then $\vartheta(h) < \vartheta(\overline{q}) + \langle 0, h - \overline{q} \rangle = \vartheta(\overline{q})$ implying that $\overline{q}$ is not a minimizer. This is a contradiction. Hence, $0 \in \partial\vartheta(\overline{q})$. $\square$

Chapter 5

CONCLUSION AND RECOMMENDATIONS

5.1 Introduction

In the final chapter of this study, a conclusion is drawn based on the three specific objectives that were established at the beginning of the research and the findings that were obtained during the course of the investigation. Additionally, we propose a number of directions for future research that may be useful for scholars and practitioners in this field. These recommendations are based on the gaps that were identified in the current literature and the areas where further investigation is needed in order to deepen our understanding of the topic.

5.2 Conclusion

It is gratifying to report that all of the results of this research met our three specific objectives. This suggests that the methodology used was effective in achieving the goals of the study. It is important to note, that the findings of this research should be considered within the context of the L^p-space, however, befitting generalizations can be made appropriately. Future research may seek to replicate or extend these findings in order to build upon the knowledge that has been gained through this study.

The first objective of this study was to characterize *lsc* functions in the context of L^p spaces. Specifically, to establish a characterization of functions that map elements of an L^p space $\mathcal{L}$ to real numbers in the range of $[-\infty, \infty]$. Lemma 4.2, Theorems 4.3 and 4.4 analyze *lsc* functions using Moore-Smith sequences. Theorem 4.3 showed that L^p-spaces preserve lower semi-continuity property under point-wise operations and multiplication by scalars; so if two functions ϑ and φ are *lsc*, then their sum, $\vartheta + \varphi$ is *lsc* and so is $t\vartheta : t \in \mathbb{R}^+$. If a finite sequence of *lsc* functions $\{\vartheta_n\}$ converges uniformly to a function $\vartheta \in \mathcal{L}$, then ϑ is lower semi-continuous. In Theorem 4.5, the known fact that a function ϑ is lower semi-continuous if and only if its epigraph is closed was verified to hold in L^p-spaces. Convexity property of *lsc* functions is characterized and found to be key as it guarantees exactly one minimum point and ensures stability when we take a point-wise minimum. In Lemma 4.7 we have shown that if ϑ is a convex function then ϑ is *lsc* if and only if it is $w - lsc$. It has been proved in Theorem 4.10 that a *lsc* function in an L^p-space attains an absolute minimum on a compact set. In Proposition 4.16 we proved that a function is almost lsc on a convex and bounded set if and only if it is convex lsc.

The second objective of this study was to characterize usc in L^p spaces. We have shown by Theorem 4.18 that if an integrable function satisfies $\vartheta(\varphi) = \int_{\mathcal{L}} \varphi d\mu$ then ϑ is usc. In Theorem 4.20 upper semi-continuous functions in L^p spaces have been characterized in terms of the hypograph of a function and proved that $\forall \delta > 0$ we have $\beta_\delta |\text{hypo}(\varphi)| = \text{hypo}(\varphi)^+_\delta$. If a function ϑ is usc, then a bounded function ϑ^+_δ is also usc whenever a positive integer δ exists. We further proved in Theorem 4.24 that a function ϑ from a convex L^p-space to the extended real line is convex if it satisfies the condition for upper semi-continuity. Additionally, it was shown by Corollary 4.25, that any extreme point of a compact convex L^p-space, if it exists, forms a maximizer for any usc function.

The third objective was to establish the optimality conditions necessary for convex optimization in L^p-spaces. Propositions 4.26 and 4.27 establish that objective functions that are both Lipschitz-continuous and lsc or even weakly lsc attain minimizers on an L^p-space: If a function ϑ is Lipschitz-continuous, then, $\vartheta \in usc(\mathcal{L})$ achieves a minimizer on $R \subseteq \mathcal{L}$; Also, if a convex Lipschitz-continuous function ϑ is weakly lsc its minimizer is always attained on $R \subseteq \mathcal{L}$. Theorem 4.30 links Gateaux-differentiability with inner products of convex functions to the existence minimizers. We further illustrated how compactness and convexity concepts impact on existence of minimizers: If $\{\vartheta(q) \leq T, \forall q \in R, T \in \mathbb{R}^+\}$ is convex and compact then, $\overline{q}$ minimizes $\vartheta(q) \to \min_{q \in R}$ locally; If a non empty compact and convex function ϑ is lsc then, $\vartheta(q) \to \min_{q \in R}$ is compact. In Proposition 4.34, Lemma 4.35 and Theorem 4.36 we have discussed the conditions of Frechet-differentiability in convex optimization and shown that these conditions are necessary for L^p-space lsc functions to attain minimizers. Additionally, in Theorem 4.36 and Theorem 4.37 we illustrated the condition of sequential boundedness together with double Frechet-differentiabilty in locating optimum points that meet the KKT optimality conditions.

5.3 Recommendations

The results obtained in this study are specific to L^p-spaces. In this section we recommend areas of further research relating to this study.

The main focus of the study was to understand how these functions behave in L^p-spaces and how they can be utilized in optimization. However, the study suggests that it would be interesting to investigate the properties of lsc functions in other spaces such as Sobolev spaces. Additionally, the study suggests that it would be fascinating to investigate whether convex lsc functions can attain minimizers under smooth function spaces with compact Riemann manifolds.

Our findings in this study were focused on characterizing upper semi-continuous (usc) functions in L^p spaces, specifically, convex and sequentially bounded usc functions. However, there is potential for further research to be done on non-convex usc functions or cases where the functions are unbounded. Additionally, it would be interesting to investigate the characterization of *usc* functions in other spaces such as Sobolev spaces or function spaces on manifolds.

This study obtained important results that focused on conditions for convex optimization in L^p spaces. However, there is potential for further research to be done on convex optimization conditions in L^p spaces to ensure that they also satisfy the second-order conditions for optimality. In this study, we applied the use of lower semi-continuous (lsc) functions that were compact, bounded, and coercive to guarantee the attainment of minimizers for convex problems in L^p spaces. We recommend exploring other methods such as gradient descent method, conjugate gradient methods, or interior-point methods. Our study focused on convex optimization in L^p spaces. However, it would be interesting to investigate convex optimization in other spaces like Sobolev spaces and function spaces on manifolds. Additionally, it would be valuable to study non-convex optimization in L^p spaces and other spaces such as Banach spaces and Hilbert spaces. These spaces have different properties and it is expected that non-convex optimization would have different characteristics in these spaces. This research would provide a deeper understanding of the properties of non-convex optimization problems and how they can be solved in different settings.

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