NORM INEQUALITIES OF NORM-ATTAINABLE OPERATORS AND THEIR ORTHOGONAL EXTENSIONS

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Abstract

We present new norm inequalities of matrices of norm-attainable operators and characterize the maps that act on matrices of these operators. Moreover, we characterize completely bounded norms, give their orthogonal extensions and extensions via norm-convergence in $NA(H)$-classes.

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1 Introduction

The algebra of norm-attainable operators, $NA(H)$, is one of the subclasses under consideration in this note. Consider an infinite dimensional complex Hilbert space H and $B(H)$ the algebra of all bounded linear operators on H. A lot of results have been obtained in the study of the properties of several classes and matrices of operators acting on Hilbert spaces. Considering norm-attainable operators, there are nice results on them especially on the necessary and sufficient conditions for norm-attainability [2]. Recently, characterizations on this subclass of operators has been done in [4]. For more details on norm-attainable operators see [4-18]. In this paper, we study some important properties of norm-attainable operators and characterize the maps that act on matrices of these operators. See details on completely bounded maps in [1 and 19] and the references therein. Lastly, we give the orthogonal extensions of these norms.

2 Preliminaries and Notations

In this section we give some basic definitions and notations that we shall use in the sequel.

Definition 2.1. *Let A, $S \in B(H)$, A is said to be positive if $\langle Ax, x\rangle \geq 0$, $\forall\ x \in H$ and normal if $AA^* = A^*A$. S is an isometry(co-isometry) if $S^*S = SS^* = I$ where I is an identity operator in $B(H)$.*

From this stage and in the sequel, we denote a bounded linear operator, a positive operator and a norm-attainable operator in $B(H)$ by A, A_P and A_N respectively. We also denote by $dimN(A_N)$, the dimension of the null space of A_N. We also denote the algebra of all norm-attainable operators by $NA(H)$.

Definition 2.2. *Let $\theta : B(H) \to B(H)$ be a bounded linear operator. θ is said to be completely bounded if $\sup \|\theta_n : \forall\ n \in \mathbb{N}\| < \infty$ and this supremum is called the completely bounded norm denoted by $||\cdot||_{CB}$.*

We denote by $CB[NA(H), NA(H)]$ the algebra of all completely bounded operators from $NA(H)$ to $NA(H)$. Clearly, this algebra is complete with respect to the completely bounded norm. HS-norm denotes the Hilbert-Schmidt norm.

Definition 2.3. *Let H be a Hilbert space, $B(H)$ the algebra of all bounded linear operators on H and let $N(H) \subseteq B(H)$ be a subalgebra. Let $M_{n,m}[B(H)]$ be a $n \times m$ matrix algebra with entries from $B(H)$. Then the inclusion, $M_{n,m}[NA(H)] \subseteq M_{n,m}[B(H)]$ endows this subalgebra with a collection of matrix norms and we call $N(AH)$ together with this collection of matrix norms on $M_{n,m}[NA(H)]$ an operator algebra. When $m = n$, we have*

$$M_{n,m}[NA(H)] = M_{n,n}[NA(H)] = M_n[NA(H)].$$

3 Norm-attainable Operators

In this section, we study norm-attainable operators and some of their properties.

Definition 3.1. *An operator $A_N \in NA(H)$ is said to be norm-attainable if there exists a unit vector $x \in H$, $\|A_N x\| = \|A_N\|$.*

Lemma 3.2. *If $A_N, B_N \in NA(H)$ are norm-attainable then $A_N + B_N$, $A_N - B_N$ and λA_N, $\lambda \in \mathbb{C}$ are norm-attainable.*

Proof. By direct summing of the operators A_N and B_N with a large enough rank-one projection, the sum and the difference of the two operators are norm-attainable. The case of coefficient λ is easy to see. □

Lemma 3.3. *For a norm-attainable operator $A_N \in NA(H)$, A_N is norm-attainable if its adjoint, A_N^*, is norm-attainable.*

Proof. We show that a nonzero $A_N \in NA(H)$ is norm-attainable implies that A_N^* is norm-attainable. Let $A_N \in NA(H)$ be norm-attainable, then there exists a unit vector $x \in H$ such that $\|A_N x\| = \|A_N\|$. That is $A_N^* A_N x = \|A_N\|^2 x$. Let $\xi = \frac{A_N x}{\|A_N\|}$, then ξ is a unit vector and hence $\|A_N^* \xi\| = \|A_N\| = \|A_N^*\|$. □

Theorem 3.4. *Let S' and S'' be isometries or co-isometries in a unit ball $NA(H)_1$. For a norm-attainable operator $A_N \in NA(H)_1$ we have $A_N = \frac{S'+S''}{2}$.*

Proof. Let $A_N = S|T|$ be the polar decomposition of A_N. Now, since $A_N \in NA(H)_1$, T is a contraction (in fact, a positive contraction) in $NA(H)_1$, therefore, $I - T^2$ is also a positive contraction in $NA(H)_1$. Let K and K' respectively be defined by $K = T + i\sqrt{I - T^2}$ and $K' = T - i\sqrt{I - T^2}$. It is clear that $K^* = K'$ and hence, $KK^* = K^*K = I$ and $K'K'^* = K'^*K' = I$, so K and K' are unitaries and $T = \frac{K+K'}{2}$. If $dimN(A_N) < dimN(A_N^*)$, then S can be taken to be an isometry and therefore, putting $S' = S|K|$ and $S'' = S|K'|$, then S' and S'' are isometries and

$$A_N = S|T| = S|\frac{K+K'}{2}| = \frac{S'+S''}{2} \tag{1}$$

If $dimN(A_N) > dimN(A_N^*)$, then S can be taken to be co-isometry and therefore, S' and S'' in Equation (1) can be taken as co-isometries. □

Remark 3.5. *If $dimN(A_P) = dimN(A_P^*)$, then S can be taken to be a unitary and therefore, S' and S'' in Equation (1) can be taken as unitaries.*

Theorem 3.6. *An operator $A \in NA(H)$ is normal if it is norm-attainable.*

Proof. Assuming $A \in NA(H)$ is normal, we show that it is norm-attainable. Since A is normal, then $AA^* = A^*A$. By Lemma 3.3 its adjoint is norm-attainable. Consider a unit vector $x \in H$. Now $A(A^*x) = A^*(Ax)$. This implies that $\|Ax\|^2 = \langle Ax, Ax\rangle = \langle A^*Ax, x\rangle \le \|A(A^*x)\| = \|A^*(Ax)\| \le \|A^*\|\|Ax\| \le \|A^*\|\|A\|\|x\| = \|A\|^2\|x\| = \|A\|^2$. Taking a positive square root on both sides yields the required results. The reverse inequality is trivial and hence this completes the proof. □

4 Norm inequalities in $NA(H)$-Classes

Theorem 4.1. *Consider a C*-algebra $B(H)$, $NA(H)$ a subalgebra of $B(H)$ and a map θ, such that $\theta : NA(H) \to B(H)$. Let $T_{N:j,k} \in M_n[NA(H)]$ be a norm-attainable operator. For n-tuples of θ, whereby $\theta_n : M_n[NA(H)] \to M_n[B(H)]$, we define $\theta_n[T_{N:j,k}] = [\theta(T_{N:j,k})], \quad \forall\, T_{N:j,k} \in M_n[NA(H)]$. Moreover, $||\theta|| \leq ||\theta||_{CB}$ holds.*

Proof. For simplicity, we take $T_{N:j,k} = T_{j,k}$ throughout the proof. Now, when $n = 1$, then by definition of θ_n, θ_1 and θ are coincidental [1] hence, $||\theta|| = ||\theta_1||$. We therefore give proofs when $n = 2$ and when $n = 3$. We use an analogous technique to the one used in [1]. For $n = 2$, let $[T_{j,k}] \in M_2[NA(H)], \quad j,\ k = 1, 2$, then for $\theta_2 : M_2[NA(H)] \to M_2[B(H)]$, we have, $\theta_2\left[\begin{pmatrix} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{pmatrix}\right] = \begin{bmatrix} \theta(T_{1,1}) & \theta(T_{1,2}) \\ \theta(T_{2,1}) & \theta(T_{2,2}) \end{bmatrix}$ and

$$\begin{aligned}
\left\|\theta_2\left[\begin{pmatrix} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{pmatrix}\right]\right\| &= \left\|\begin{bmatrix} \theta(T_{1,1}) & \theta(T_{1,2}) \\ \theta(T_{2,1}) & \theta(T_{2,2}) \end{bmatrix}\right\| \\
&= \left[\sum_{j=1}^{2}\sum_{k=1}^{2} \|\theta(T_{j,k})\|^2\right]^{\frac{1}{2}} \qquad \text{by HS-norm} \\
&= \left(\|\theta(T_{1,1})\|^2 + \|\theta(T_{1,2})\|^2 + \|\theta(T_{2,1})\|^2 + \|\theta(T_{2,2})\|^2\right)^{\frac{1}{2}} \\
&\geq \left[\|\theta(T_{1,1})\|^2\right]^{\frac{1}{2}} \\
&= \|\theta(T_{1,1})\| \\
&= \|\theta_1(T_{1,1})\|.
\end{aligned}$$

Therefore,

$$||\theta_2|| = \sup\{||\theta_2([T_{j,k}]) : \quad [T_{j,k}] \in M_2[NA(H)]||\} \geq \sup\{||\theta_1(T_{1,1})||\} = ||\theta_1||.$$

and hence $||\theta_2|| \geq ||\theta_1||$.

When $n = 3$, $\theta_3\left[\begin{pmatrix} T_{1,1} & T_{1,2} & T_{1,3} \\ T_{2,1} & T_{2,2} & T_{2,3} \\ T_{3,1} & T_{3,2} & T_{3,3} \end{pmatrix}\right] = \begin{bmatrix} \theta(T_{1,1}) & \theta(T_{1,2}) & \theta(T_{1,3}) \\ \theta(T_{2,1}) & \theta(T_{2,2}) & \theta(T_{2,3}) \\ \theta(T_{3,1}) & \theta(T_{3,2}) & \theta(T_{3,3}) \end{bmatrix}$ which implies that

$$\begin{aligned}
\left\| \theta_3 \left[\begin{pmatrix} T_{1,1} & T_{1,2} & T_{1,3} \\ T_{2,1} & T_{2,2} & T_{2,3} \\ T_{3,1} & T_{3,2} & T_{3,3} \end{pmatrix} \right] \right\| &= \left\| \begin{bmatrix} \theta(T_{1,1}) & \theta(T_{1,2}) & \theta(T_{1,3}) \\ \theta(T_{2,1}) & \theta(T_{2,2}) & \theta(T_{2,3}) \\ \theta(T_{3,1}) & \theta(T_{3,2}) & \theta(T_{3,3}) \end{bmatrix} \right\| \\
&= \left[\sum_{j=1}^{3} \sum_{k=1}^{3} ||\theta(T_{j,k})||^2 \right]^{\frac{1}{2}} \\
&= [\|\theta(T_{1,1})\|^2 + \|\theta(T_{1,2})\|^2 + \|\theta(T_{1,3})\|^2 + \|\theta(T_{2,1})\|^2 + \\
&\quad \|\theta(T_{2,2})\|^2 + \|\theta(T_{2,3})\|^2 + \|\theta(T_{3,1})\|^2 + \|\theta(T_{3,2})\|^2 + \\
&\quad \|\theta(T_{3,3})\|^2]^{\frac{1}{2}} \\
&\geq \left[\|\theta(T_{1,1})\|^2 + \|\theta(T_{1,2})\|^2 + \|\theta(T_{2,1})\|^2 + \|\theta(T_{2,2})\|^2 \right]^{\frac{1}{2}} \\
&= \left[\sum_{j=1}^{2} \sum_{k=1}^{2} ||\theta(T_{j,k})||^2 \right]^{\frac{1}{2}} \\
&= ||\theta([T_{j,k}])||.
\end{aligned}$$

This implies that

$$\begin{aligned}
||\theta_3|| &= \sup\{||\theta_3[(T_{j,k})] : \ [T_{j,k}] \in M_3[NA(H)]||\} \\
&\geq \sup\{||\theta_2[(T_{j,k})] : \ [T_{j,k}] \in M_2[NA(H)]||\} \\
&= ||\theta_2||
\end{aligned}$$

and therefore, $||\theta_3|| \geq ||\theta_2||$. Lastly, consider $\theta_{n+1} : M_{n+1}[NA(H)] \to M_{n+1}[B(H)]$ defined by $\theta_{n+1}[(T_{j,k})] = [\theta(T_{j,k})]$ for all $j, k = 1, \ldots, n+1$. We obtain,

$$\begin{aligned}
\|\theta_{n+1}[(T_{j,k})]\| &= \|[\theta(T_{j,k})]\| \\
&= \left[\sum_{j=1}^{n+1} \sum_{k=1}^{n+1} \|\theta(T_{j,k})\|^2 \right]^{\frac{1}{2}} \\
&\geq \left[\sum_{j=1}^{n} \sum_{k=1}^{n} \|\theta(T_{j,k})\|^2 \right]^{\frac{1}{2}} \\
&= ||\theta_n([T_{j,k}])||
\end{aligned}$$

So, $\|\theta_{n+1}\| \geq \|\theta_n\|$ by taking supremum on both sides of the inequality above. By complete boundedness of the norm of θ, $||\theta||_{CB} = \sup\{||\theta_n|| : n \in \mathbb{N}\}$ which implies that $\|\theta\|_{CB} \geq \|\theta_n\| \ \ \forall \ n \in \mathbb{N}$. Therefore, $||\theta|| \leq ||\theta||_{CB}$ which completes the proof. $\square$

5 Orthogonal Extensions

In this section, we give norm-attainable operator-valued orthogonal extensions of matrix inequalities. Orthogonal extensions are known in different settings, but we include simple proofs, since more elaborate ones have appeared in literature.

Definition 5.1. *Two operators T and P in $NA(H)$ are said to be orthogonal if $\langle T, P\rangle = 0$. Operators $T_j, T_k,\ (j, k = 0, 1, ...)$ are said to have orthogonal extensions if $\langle T_j, T_k\rangle = 0$.*

Proposition 5.2. *Let $(M_{j,k})$ be a positive definite $n \times n$ matrix and $T_j,\ (1 \leq j \leq n)$ are elements of $NA(H)$, then $\Sigma_{j=1}^{n}\Sigma_{k=1}^{n}\langle T_j, T_k\rangle \geq 0$.*

Proof. Consider an orthonormal basis $\{e_1, e_2, ..., e_m$ of the subspace of $NA(H)$ generated by the elements T_j, and we write $T_j = \Sigma_{r=1}^{m} T_j(r)e_r$. From Theorem 4.1 for each r we have positivity attained and taking summation over r completes the proof. □

Theorem 5.3. *The matrix $[T]_B^{i,j}$ of an orthogonal operator $T \in NA(H)$ in an arbitrary orthonormal basis B is orthogonal. Conversely, if in an orthonormal basis B the matrix of a linear operator $T \in NA(H)$ is orthogonal, then T is orthogonal.*

Proof. Consider an orthonormal basis $B = e_1, ..., e_n$ in $NA(H)$. Let T be a linear operator. Then $[T]_B^{i,j} = ([T(e_1)]_B^{i,j}, ..., [T(e_n)]_B^{i,j})$. Recall that for an orthonormal basis, the inner product in $NA(H)$ is equal to the standard scalar product in $\mathbb{R}^n$ of their respective coordinate column (or row) operators. Suppose that T is orthogonal, the set of Operators $T(e_1), ..., [T(e_n)$ is orthonormal (since $e_1, ..., e_n$ is orthonormal). Hence the columns $[T(e_1)]_B^{i,j}, ..., [T(e_n)]_B^{i,j}$ are orthonormal, hence $[T]_B^{i,j}$ is an orthogonal matrix. Conversely, assume that the matrix $[T]_B^{i,j}$ is orthogonal. $\Longrightarrow T(e_1), ..., [T(e_n)$ is an orthonormal set, i.e., T preserves all pairwise scalar products of the elements of the basis B. It follows then that T preserves all inner products of vectors of $NA(H)$, i.e., T is an orthogonal operator. □

Now it is easy to see that without loss of generality, fixing an orthonormal basis in $NA(H)$, gives a $1-1$ correspondence between orthogonal matrices and orthogonal operators. At this point we consider extensions via norm convergence.

6 Extensions via Norm-convergence

Norm inequalities can be extended via norm-convergence. We note that a sequence $\{T_j\}$ of operators converges strongly to T if $\lim_{j\to\infty} T_j x = T$, for all

$x \in H$. Norm-convergence implies strong convergence while boundedness of $\Sigma_{j=1}^{n} T_j^* T_j$ is actually equivalent to strong convergence of $\Sigma_{j=1}^{n} T_j^* T_j$ since any norm-bounded, increasing sequence of self-adjoint operators converges strongly. It is interesting to show that either norm-convergence or strong convergence is inherited by the transformed sequence.
Considering weaker conditions, the norm of the limit of a weakly convergent sequence of operators in $NA(H)$ may be strictly less than the norms of the terms in the sequence, corresponding to a loss of energy in oscillations, at a singularity, or by escape to infinity in the weak limit. Therefore, the expansion of any positive functional ϕ in any orthonormal basis contains coefficients that wander off to infinity. Hence, we note that if the norms of a weakly convergent sequence converge to the norm of the weak limit, then the sequence is strongly convergent.

It is known that the boundedness of the pointwise values of a family of linear functional ϕ implies the boundedness of their norms. We prove that a weakly convergent sequence is bounded, hence gives a necessary and sufficient condition for weak convergence of operators in $NA(H)$.

Theorem 6.1. *Let T_j be a sequence of operators in $NA(H)$ and $\mathcal{G}$ a dense subset of $NA(H)$. Then T_j converges weakly to T if and only if $\|T_j\| \leq \lambda$ for some constant λ. Moreover, $\langle T_j, P\rangle \to \langle T, P\rangle$ as $j \to \infty$ for all $P \in \mathcal{G}$.*

Proof. Suppose that T_j is a weakly convergent sequence. By the definition of bounded linear functionals ϕ_n given by $\phi_n(T) = \langle T_j, T\rangle$ Then $\|\phi_n\| = \|T_j\|$. But $\phi_n(T)$ is convergent for each $T \in NA(H)$, it is a bounded sequence, and by uniform boundedness theorem $\{\|\phi_n\|\}$ is bounded. Hence both conditions are necessary and satisfied. Next, we prove the reverse inclusion. Suppose that T_j satisfies the two conditions. If $A \in NA(H)$, then for any $\beta > 0$ there is a $P \in \mathcal{G}$ such that $\|A - P\| < \beta$, and there is an M such that $|\langle T_j - T, P\rangle| < \beta$, for $j \geq M$. From the first condition in the theorem,and by Cauchy-Schwarz inequality it follows that for $j \geq M$

$$
\begin{aligned}
|\langle T_j - T, A\rangle| &\leq |\langle T_j - T, P\rangle| + |\langle T_j - T, A - P\rangle| \\
&\leq \beta + \|T_j - T\| + \|A - P\| \\
&= (1 + \lambda + \|T\|)\beta.
\end{aligned}
$$

Thus, it follows that $T_j \rightharpoonup T$which completes the proof □

Example 6.2. *Suppose that $e_i, (i = 1, 2, ...)$ is an orthonormal basis of $NA(H)$. Then a sequence T_j is weakly convergent to T if and only if it is bounded and its coordinates converge, i.e. $\langle T_j, e_i\rangle \to \langle T, e_i\rangle$ for each $i = 1, 2, ...$ Thus, the boundedness of the sequence is sufficient to ensure weak convergence in $NA(H)$.*

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