Properties of Local Automorphisms of commutative Banach Algebras

Thesis · December 2015

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ON NORM PRESERVING CONDITIONS FOR LOCAL AUTOMORPHISMS OF COMMUTATIVE BANACH ALGEBRAS

BY

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A THESIS SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE OF MASTER OF SCIENCE IN PURE MATHEMATICS

SCHOOL OF MATHEMATICS AND ACTUARIAL SCIENCE

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Declaration

This thesis is my own work and has not been presented for a degree award in any other institution.

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W251/4559/2014

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This thesis has been submitted for examination with our approval as the university supervisors.

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Acknowledgments

I thank God for enabling me to complete this work. I sincerely thank my supervisors Dr. Benard Okelo and Prof. Omolo Ongati for their thorough reading of this thesis, encouragements and guidance for this work to come to completion. Special thanks to Dr. Benard Okelo who mentored me during my undergraduate studies, being available all time and provided great suggestions towards this work. I would like to thank my fellow colleagues Nyaluke K. Wesley and Moses Ochola for their selfless support during this study. I wish to acknowledge JOOUST for accepting to offer me admission to the University. Am indebted to all SMAS members especially Dr. Aminer Titus for their encouragements and the support they gave me over this period. I thank my brothers, Cheruiyot, Kiprotich, Kiptoo and sisters, Margaret, Jepkosgei, Jemwetich, Jepchumba and Jerono for standing with me throughout my study period. Appreciation goes my Aunt Irene, Cousin Abitha Jepchumba and inlaw Evelyne for their continous encouragement, love and infinitely supporting me in many ways. I owe the successful completion of this thesis to you all.

Dedication

To my son Bourne Kiplagat Kipkemoi, wife Abigael and my parents.

May this work be a source of inspiration in your life.

Abstract

Many studies on preserver problems have been focusing on linear preserver problems in matrix theory. Kadison and Sourour showed that the local derivations of Von Neumann algebras are continous linear maps which coincide with some derivation at each point in the algebra over the field of complex numbers. Most of the studies have been focusing on the spectral norm preserver and rank preserver problems of linear maps on matrix algebras but not on norm preserver problems for local automorphism of commutative Banach algebras. In this study, we have investigated properties of local automorphism of commutative Banach algebras and determined their norms. The objectives of the study have been to: Investigate properties of local automorphisms of commutative Banach algebras; establish norm preserving conditions for local automorphisms of commutative Banach algebras and determine the norms of local automorphisms of commutative Banach algebras. The methodology involved the concept of local automorphisms and derivations introduced by Kadison and Sourour. We used Hahn-Banach extension theorem, Gelfand transform and the results developed by Molnar to investigate the preserver problems of local automorphisms. The results obtained show that local automorphisms are linear, inner, bounded, continous and their groups are algebraically reflexive. Moreover, the results on norms indicate that $\|\phi(x) + \phi(y)\| = \|x\| + \|y\|$ and $\|\phi_x(y)\| = 2\|y\|$. The results obtained in this study are useful in the applications of operator algebras and quantum mechanics.

Contents

Index of Notations

Chapter 1

INTRODUCTION

1.1 Mathematical background

The origin of Banach spaces dates back to 1920 when Banach submitted his thesis which was then followed by Hahn-Banach theorem and Banach monograph in 1932, while the roots of commutative algebra can be traced in the publication made by David [7] which has great applications in number theory. Moreover, detailed history of Banach algebras and linear operators have been outlined by Albert [3]. On the other hand, Kentaro [17] studied Banach Spaces in which he solved many problems from the theory of topological tensor products of locally convex spaces and the theory of nuclear spaces. At the same time, Kadison [15] introduced the study of local derivations of Von Nuemann algebra and some polynomial algebras in which he proved that each continous local derivation from a Von Neumann algebra M into a dual M-bimodule is a derivation. Finally, Sakai [31] established inner derivations of W*-algebra. Therefore, investigations of properties of local automorphisms of commutative of Banach algebra have not been exhausted. Several studies on norm preserver problems have been studied on Banach algebras and operators in; [1], [13], [20], [22], [24], [25], [27] and [28]. In this study, we endevered to investigate properties of local automorphisms of commutative Banach algebra and determine the norm preserver conditions for local automorphisms of commutative Banach algebra. In order to carry out our investigation successfully, we required some basic concepts which were useful in the sequel.

1.2 Basic concepts

In this section, we give a number of definitions of terms and examples which are encountered throughout this work.

Definition 1.1. A Group G is a non-empty set with a binary operation $* : G \times G \to G$ satisfying the following operation axioms:

(i). *Closure* $a * b \in G$ for all $a, b \in G$.

(ii). (*Associativity*) $(a * b) * c = a * (b * c)$, for all $a, b, c \in G$.

(iii). There exist an element $e \in G$ called an *identity* of G such that $a * e = e * a = a$, for all $a \in G$.

(iv). For each $a \in G$, there exist an element $a^{-1} \in G$ called *inverse* of a such that $a * a^{-1} = a^{-1} * a = e$.

Example 1.2. The set of all integers under addition is a group.

Example 1.3. The set of complex numbers $G = \{1, i, -1, -i\}$ under multiplication is a group.

Definition 1.4. Two groups G and E with elements $a, b, c \in G$ and $a', b', c' \in E$ are said to be isomorphic if there is a one-to-one correspondence between all their elements such that $ab = c \Rightarrow a'b' = c'$ and vice-versa that is a bijection $\varphi : G \to E$ such that $\varphi(ab) = \varphi(a)\varphi(b)$, for every $a, b \in G$.

Definition 1.5. A Ring R is a nonempty set with two binary operations associating each elements $a, b \in R$, a sum $a + b \in R$ and product $a.b$ or $ab \in R$ and satisfying the following laws:

(i). $a + b = b + a$ (Commutative law of addition).

(ii). $ab = ba$ (Commutative law of multiplication.

(iii). $(a + b) + c = a + (b + c)$ (Associative law of addition).

(iv). $a(bc) = (ab)c$ (Associative law of multiplication).

(v). There exist $0 \in R$ such that $a+0 = a$ (Neutral element of addition).

(vi). There exist $1 \in R$ such that $1a = a1 = a$ (Neutral element of multiplication).

(vii). For each $a \in R$, there exist $-a \in R$ such that $a + (-a) = 0$ (Additive inverse).

(viii). $(a + b)c = ac + bc$, $c(a + b) = ca + cb$ (Distributive law).

Definition 1.6. A Field is a set $\mathbb{K}$ together with two operations; multiplication $(\cdot)$ and addition $(+)$ for which the following conditions must hold:

(i). Closure; $\forall\, a, b \in \mathbb{K}$, then $a \cdot b \in \mathbb{K}$ and $a + b \in \mathbb{K}$.

(ii). Associativity; $\forall\ a, b, c \in \mathbb{K}$, then $(a.b).c = a.(b.c)$ and $(a+b)+c = a+(b+c)$.

(iii). Commutativity; $\forall\ a, b \in \mathbb{K}$, then $a.b = a.b$ and $a+b = a+b$.

(iv). Distributivity; $\forall\ a, b, c \in \mathbb{K}$, then $a.(b+c) = ab+ac$ and $(a+b).c = ac+bc$.

(v). Existence of additive identity; $\exists\ 0 \in \mathbb{K}$ for which $a+0 = a$ and $0+a = a$, $\forall\ a \in \mathbb{K}$.

(vi). Existence of multiplicative identity; $\exists\ 1 \in \mathbb{K}$ for which $a.1 = a$ and $1.a = a$; $\forall\ a \in \mathbb{K}$.

(vii). Existence of additive inverse; $\forall\ a \in \mathbb{K}\ \exists\ x \in K$ such that $a+x = 0$ and $x+a = 0$ hence $x = -a$ is the additive inverse.

(viii). Existence of multiplicative inverse; $\forall\ a \in \mathbb{K}$ with $a \neq 0$ then $a.x = 1$ and $x.a = 1$ then $x \in \mathbb{K}$ is called multiplicative inverse of a denoted by a^{-1}

Definition 1.7. A Vector Space over field $\mathbb{K}$ is a nonempty set V on which two operations are defined called scalar multiplication and addition denoted by $(\cdot)$ and $(+)$ respectively. For all $a, b \in \mathbb{K}$ and $u, v, w \in V$ the following must conditions hold:

(i). Closure; $a.v \in V$ and $u+v \in V$.

(ii). Commutativity of addition; $u+v = v+u \in V$.

(iii). Associativity: $u+(v+w) = (u+v)+w \in V$ and $(a.b)v = a.(b.v) \in V$.

(iv). Distributive law; $a.(u+v) = (au) + (av) \in V$ and $(a+b)v = (a.v) + (b.v) \in V$.

(v). Existence of additive identity: $\exists\ 0 \in V$ for which $v + 0 = v \in V$ and $0 + v = v \in V$.

(vi). Existence of additive inverse; $\exists\ x \in V$ such that $v + x = 0 = x + v$, $x = -v$.

(vii). Unitary law. $\forall\ v \in V$ then $1.v = v$.

Definition 1.8. Let A be a Banach algebra. An involution on Banach algebra A is a map $* : A \to A$ satisfying the following property.

(i) $(a^*)^* = a, \forall\ a \in A$.

(ii) $(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*, \forall\ a, b \in A \ \ and\ \alpha, \beta \in \mathbb{K}$.

(iii) $(ab)^* = b^* a^*, \forall\ a, b \in A$.

The pair $(A, *)$ is called an involutive Banach algebra.

Definition 1.9. A C^*-algebra is Banach algebra A with involution which satisfies $\|x^* x\| = \|x\|^2, \ \ \forall x \in A$.

Example 1.10. The conjugation map is an involution over the field complex numbers $\mathbb{C}$.

Definition 1.11. A subset $\mathcal{I}$ of an Banach algebra A denoted as $\mathcal{I} \subset A$ is an called an ideal if it is a linear subspace that satisfies $x \in \mathcal{I},\ y \in A$ then $xy \in \mathcal{I}$.

Definition 1.12. Let V be a vector space. A norm is a nonnegative real valued function $\|.\| : V \to \mathbb{K}$ such that $\forall\, x, y \in V$ and $\alpha \in \mathbb{K}$ the following properties are satisfied:

(i) $\|x\| \geq 0$ (nonnegativity).

(ii) $\|x\| = 0$ iff $x = 0$ (zero property).

(iii) $\|\alpha x\| = |\alpha|\|x\|$ (homogeneity).

(iv) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).

The ordered pair $(V, \|.\|)$ is called normed space.

Definition 1.13. A Banach algebra is a normed space $(A, \|.\|)$ over $\mathbb{K}$ that satisfies $\|ab\| \leq \|a\|\|b\|\ \forall\, a, b \in A$ (sub-multiplicative property).

Definition 1.14. Banach space is a complete normed vector space.

Definition 1.15. Let A be a Banach algebra. A is said to be *commutative* if $ab = ba$ for all $a, b \in A$.

Definition 1.16. Let X and Y be two normed vector spaces, a linear operator $T : X \to Y$ is bounded if for all $x \in X$ there exists a constant $c > 0$ such that $\|Tx\|_Y \leq c\|x\|_X$ holds.

Definition 1.17. Given a matrix $A = (a_{ij}) \in Mn(\mathbb{C})$, the trace $tr(A)$ of A is the sum of diagonal elements that is $tr(A) = a_{11}+a_{22}+...+a_{nn}$. Trace is a linear map so that $tr(\lambda A) = \lambda tr(A)$ and $tr(A + B) = tr(A) + tr(B)$.

Definition 1.18. A operator $\Phi : B(H) \to B(H)$ of the algebraic structure $B(H)$ is called a local automorphism if for every $x \in B(H)$ there is an automorphism $\Phi_x \in B(H)$ such that $\Phi_x = \Phi_x(x)$. Suppose $A \subset B(H)$ is a subalgebra. A linear operator $\phi : A \to B(H)$ is called an inner automorphism if $\phi_a(x) = xa - ax; \ \forall; \ a, x \in A$.

Definition 1.19. A subset $E \subset B(H)$ is called topologically reflexive if for any operator $T \in B(H)$ belongs to E whenever T has the property that $T(x) \in \overline{E(x)}, \ x \in H$.

Definition 1.20. Let A and B be Banach algebras. A monomorphism of A into B is an injective homomorphism of A into B.

Definition 1.21. Let A and B be Banach algebras. Isomorphism of A into B is a bijective homomorphism of A into B.

Definition 1.22. Let A be a Banach algebra. The spectrum of $x \in A$ is the set $\delta(x) = \{\lambda \in \mathbb{C} : x - \lambda I \ \ is \ \ not \ \ invertible\}$ while the spectral radius $r(x) = sup_{\lambda \in \delta(x)} |\lambda|$.

1.3 Statement of the problem

The basic question on the area of preserver problems is whether an operator between two spaces with the same structure (rings, groups and vector space) is a homomorphism and it preserves linearity. The preserver problem is concerned with a question of describing the general form of all transformation of a given structure of groups, rings and vector space which preserves the quantity attached to each element and distinguished set of elements and relations. Studies have been done concerning rank preserver problems in matrix theory, the commutativity preserver maps in full matrix algebras, derivations of commutative normed algebras, multiplicative norm preserver maps between invertible groups of Banach algebras, norm preserver inequalities of commutative unital Banach algebra and spectrum preserver problems of linear mapping in Banach algebras. However, studies on the norm preserver problem for local automorphisms of commutative Banach algebras have not been done. In this study, we investigated properties of local automorphism of commutative Banach algebras, established the norm preserver conditions for local automorphisms of commutative Banach algebras and determined the norms for local automorphisms of commutative Banach algebras.

1.4 Objectives of the study

The objectives of study have been to:

(i). Investigate the properties of local automorphisms of commutative Banach algebras.

(ii). Establish norm preserving condition for local automorphisms of commutative Banach algebras.

(iii). Determine the norms of local automorphisms of commutative Banach algebras.

1.5 Significance of the study

The results of this study are useful in analysis of local automorphism that preserve the norm in commutative Banach algebras. Moreover, the results are applicable in operators and function algebras in quantum symmetries.

1.6 Research methodology

Since the research was mainly dealing with establishing the norm preserver conditions for local automorphisms of commutative Banach algebra, a prerequisite knowledge was required in functional analysis, Operator theory, General topology and Commutative Banach algebra. We borrowed useful results from local spectral theory and Hilbert spaces.

1.6.1 Some Fundamental Results

We used known results in solving the stated objectives of the problem. The proofs were omitted.

Theorem 1.23 (4, Hahn-Banach Theorem D.1)**.** *Let X be a real linear space; $Y \subset X$ a linear subspace and let $p : X \to \mathbb{R}$ be a sublinear functional such that; $p(x+y) \leq p(x) + p(y)$ $\forall x, y \in X$ p is additive and $p(\lambda x) \leq \lambda p(x)$ $\forall$ $x \in X$ and $\lambda \geq 0$ p is non-negative subhomogenous. Then there is an $\mathbb{R}$ linear functional $f : A \to \mathbb{R}$ such that $\forall x \in E$ $f(X) \leq p(x)$.*

Theorem 1.24 (30, Theorem 2.35). *If B is a commutative Banach algebra, M is its maximal ideal space and $\Gamma : B \to C(M)$ is the Gelfand transform then:*

(i). *M is not empty.*

(ii). *Γ is an algebra homomorphism.*

(iii). *$\|\Gamma f\|_\infty \leq \|f\|$ for $f \in B$.*

(iv). *f is invertible in $C(M)$.*

Theorem 1.25 (18, Theorem 2.1). *Let X be an infinite-dimensional Banach space and let ϕ be a linear map from $B(X)$ onto itself such that ϕ is a local automorphism. Then ϕ is an automorphism.*

Theorem 1.26 (27, Theorem 4.2.7). *If a surjective mapping $T : A \to B$ between uniform algebras satisfies $\|Tf + Tg\| = \|f + g\|$ and $\||Tf| + |Tg|\| = \||f|+|g|\|$ for $f, g \in A$ there exist a homomorphism $\psi : \sigma B \to \sigma A$ such that $|(Tf)(y)| = |f(\psi(y))|$ for every $f \in A$ and $y \in \sigma(B)$.*

Corollary 1.27 (19, Proposition 4.1). *For $x, y \in H$ we have $|\langle x, y\rangle| \leq \|x\|\|y\|$ and equality holds if and if x and y are linearly dependent.*

Theorem 1.28 (19, Theorem 4.3). *Let H be Hilbert space and Y be closed subspace of H.Then for $x \in H$ there exist unique $y \in Y$ called a projection on Y and is denoted by $\Pi_y(x)$ such that $\|x - y\| = min_{z\in Y}\|x - z\|$. Moreover y is characterized by the property $\langle x - y, z\rangle = 0$ for $z \in Y$.*

1.6.2 Computational Techniques

Definition 1.29. Let U and V be two Vector Spaces over $\mathbb{K}$ and let T be the subspace of free vector space $\mathbb{K}_{U\times V}$ generated by all the vectors of the form
$r(u,v)+s(u',v)\ (ru+su',v)$ and $r(u,v)+s(u,v')\ (u,rv+sv')$
for all $r,s\in\mathbb{K}$ and $u,u',v,v'\in V$. Then the quotient space $\mathbb{K}_{U\times V}/T$ is called the **tensor product** of U and V and is denoted by $U\otimes V$.
An element of $U\otimes V$ has the form: $\Sigma r_i(u_i,v_i)+T$.

Definition 1.30. Let V be vector space and let U,W be subspaces of V. The V is said to be **direct sum decomposition** of subspaces $U_1,...,U_k$, if it can expressed as $V=U_1\oplus...\oplus U_k$ if for all $v\in V$ there exist unique vectors $u_i\in U_i$ for $1\leq i\leq k$ such that $v=u_1+u_2+...+u_k$.

Chapter 2

LITERATURE REVIEW

2.1 Introduction

Many research studies have been on properties of commutative algebras and they have obtained interesting results. In this chapter, we discuss related literature which are useful to our study.

2.2 Automorphisms

Singer and Wermer [33] studied on derivations of commutative Banach algebras over the complex field and showed that derivations map commutative Banach algebra into its radical as shown in the theorem below:

Theorem 2.1 (33, Theorem 1)**.** *Let B be a commutative Banach algebra and ϕ be a bounded derivation on B. Then ϕ maps B into its radical. In particular if B is semi-simple then $\phi = 0$, where the radical of B is the intersection of all maximal ideals $M \in B$ and if the radical reduces to zero element, then B is called semi-simple.*

From the theorem above, we know that ϕ is bounded that is $\sup_{\|a\|=1} \|\phi(a)\| = \|\phi\| < \infty, \quad a \in B$. Singer and Wermer [33] studied the bounded derivation of commutative Banach algebra. However, they never investigated properties of local automorphism of commutative Banach algebras. In this study, we have showed that every local automorphism of commutative Banach algebras is bounded.

At the same time Sakai [31] worked on inner derivation of W^*-algebra and the main result is in the theorem below;

Theorem 2.2 (31, theorem 1)**.** *Every derivation of W^*-algebra is inner*

Sakai [31] did not investigate properties of local automorphism of commutative Banach algebra. In this thesis, we have showed that local automorphism of commutative Banach algebra is inner.

Sang and Seon [32], proved that every 2-local derivation on M_n, the $n \times n$ matrix algebra are derivations as shown in the theorem below.

Theorem 2.3 (32, Theorem 3)**.** *Let M_n be the $n \times n$ matrix algebra over $\mathbb{C}$ and $\phi : M_n \to M_n$ a 2-local derivation. Then ϕ is a derivation.*

In [32] they considered 2-local derivation over matrix algebras. However, they never considered local automorphisms of commutative Banach algebras. In our work, we have showed that every local automorphism is an automorphism.

Peter [26] proved that every 2-local automorphism $\phi : B(H) \to B(H)$ is an automorphism where $B(H)$ is the algebra of all bounded linear operators on H. The results are as follows:

Theorem 2.4 (26, Theorem 1)**.** *Let H be an infinite-dimensional separable Hilbert space and let $B(H)$ be the algebra of all bounded linear op-*

erators on H. Then every 2-local automorphism $\phi : B(H) \to B(H)$ (no linearity, surjectivity or continuity of ϕ is assumed) is an automorphism.

Theorem 2.5 (26, Theorem 2)**.** *Let H be an infinite-dimensional separable Hilbert space, and let $B(H)$ be the algebra of all linear bounded operators on H. Then every 2-local automorphism $\phi : B(H) \to B(H)$ (no linearity, surjectivity or continuity of ϕ is assumed) is a derivation.*

In Theorem 2.4 and Theorem 2.5, Peter established that every 2-local automorphism is an automorphism or derivation. However, investigation of properties of local automorphism of commutative Banach algebras has not been done. In the study, we have showed that every local automorphism is linear and continous.
The properties of group automorphism has been studied in [10]. Felix [10] studied the reflexivity of the automorphism group of the Banach algebras $L_\infty(\mu)$ for various measures μ. They proved that if μ is non-atomic σ-finite measure, then the automorphism group of $L_\infty(\mu)$ is algebraically reflexive if and only if $L_\infty(\mu)$ is $*$-isomorphic to $L_\infty[0, 1]$ but they did not consider group of local automorphism of commutative Banach algebras. In this work, we have showed that the group of every local automorphism of $B(H)$ is algebraically reflexive.
Further studies of locally inner derivation has been done in [21]. Molnar [21] proved that every locally inner derivation on symmetric norm ideal of operators is inner. A similar study of locally inner automorphisms has been done by David [8]. In [8] they established how locally inner automorphisms are related to diagonal sums and characterization of locally inner automorphisms of Von Nuemann algebras. Since Molnar [21] and David [8] did not investigate properties of local automorphisms of com-

mutative Banach algebras hence we have showed that local automorphism of commutative Banach algebras is inner.

2.3 Norm Preserver Problems

Osamu, Takeshi and Hiroyuki [24] established the multiplicative norm preserver maps between invertible groups of commutative Banach algebras and showed that C^*-algebras are isomorphic and they are norm preserving groups isomorphisms between group of algebras. In [24], it was established that the injective map T preserves the norm as illustrated by example below:

Example 2.6 (24, Example 3.6)**.** Let $C(X)$ be Banach algebra of complex valued continuous functions on a compact Hausdorff space X and $A = C(X) \oplus C(X)$. Let A be isometrically isomorphic to $C(X_1 \cup X_2)$ where X_1 and X_2 are two copies of X. Let T be a map from A^{-1} into itself defined by; $T(f \oplus g) = \frac{f^2 g}{|fg|} \oplus \frac{f^3 g^2}{|f^3 g|}$ for $f \oplus g \in A$. Then T is a norm preserving group automorphism on A^{-1} while T is not extended to linear map on A.

From Example 2.6, Osamu, Takeshi and Hiroyuki [24] used direct sum to determine the norm preserving group automorphism of A^{-1} but they did not consider if the map T preserves the norms of local automorphism of commutative Banach algebra. In our study, we have used tensor products to investigate properties of local automorphism of commutative Banach algebras and established its norm preserver conditions.

Takashi [34], also characterized the commutativity of unital Banach algebras A over a complex space $\mathbb{C}$ using the norm inequalities and gave the theorem below:

Theorem 2.7 (34, Theorem 1)**.** *Let A be a unital Banach algebra over a complex space $\mathbb{C}$ with the norm $\|.\|$. If there is a norm $\|.\|$ on S and positive constants γ, κ such that $\|S\| \leq \kappa\|S\|\|ST\| \leq \gamma\|TS\|$ for all $S, T \in A$, then A is commutative, that is $ST = TS$ for all $S, T \in A$.*

In Theorem 2.7, Takashi worked on commutativity using norm inequalities with similarity in transformation but did not established the norms of local automorphisms of commutative Banach algebras. We have determined the norms of local automorphisms of commutative Banach algebras.
A norm preserving condition has been done by Hosseini and Sady [13], that if we have two Hausdorff spaces say X and Y and two Banach spaces A and B, then a map $T : A \rightarrow B$ preserves multiplicatively norm on the range if $\|fg\|_X = \|TfTg\|_Y$ holds for all $f, g \in A$, hence the multiplicatively spectrum preserving map between two Banach algebras are defined. The results of the study are outlined in the theorem below.

Theorem 2.8 (13, Theorem 2.3)**.** *If $T : A \longrightarrow B$ is a surjective multiplicatively range preserving map, then there is a homeomorphism φ from X onto Y such that for each $x \in X$ and $f \in A$ then $\|f\|_{[x]} = \|Tf\|_{\varphi[x]}$. Moreover, if the points in $c(A)$ and $c(B)$ are all strong boundary points, then T has the following representation $Tf(y) = h(y)f(\varphi(y))$ $(f \in A, y \in c(B))$, where h is a continuous complex-valued function on $c(B)$ and is a homeomorphism from $c(B)$ onto $c(A)$.*

Hosseini and Sady [13], showed that T is surjective and preserves the norm however they did not explain norm preserver condition for local automorphism of commutative Banach algebras and so we established norm preserver conditions for local automorphisms of commutative Banach algebras.

Emeka [16] tackled spectral preserver problems of linear mapping of Banach algebra. The results showed that a surjective spectrum preserving linear map between Von Neumann algebra is also a Jordan homomorphism and a surjective spectral isometry between finite dimensional semi-simple Banach algebra is a Jordan Isomorphism and the following two theorems gives explicit results:

Theorem 2.9 (16, Theorem 3.1)**.** *Let A and B be von Neumann algebras, and let $\Psi : A \longrightarrow B$ be a surjective, spectrum preserving linear map. Then Ψ is a Jordan isomorphism.*

Theorem 2.10 (16, Theorem 3.2)**.** *Let A and B be von semi-simple Banach algebras, and let $\Psi : A \longrightarrow B$ be a surjective, spectral isometry. Then $\Psi : A \longrightarrow B$ is a Banach Space isomorphism*

Emeka [16] focused on linear mapping between Banach algebra that preserve spectral properties but he did not looked into norm preserver conditions for local automorphism of commutative Banach algebra and hence we have determined norms of local automorphisms of commutative Banach algebra.

Studies on norms of inner derivations has been done in [5]. Baraa and Boumazgour [5] gave necessary and sufficient conditions when norm of the sum of two operators $A, B \in B(H)$ on Hilbert space H is equal to the sum of their norms that is $\|A+B\| = \|A\|+\|B\|$ but they did not characterized norms of local automorphisms of commutative Banach algebras. In this thesis, we have showed that the norms of local automorphisms of commutative Banach algebras the preserve additivity; $\|\phi(x)+\phi(y)\| = \|x\|+\|y\|$. Studies on norm preserver has been done by Tonev and Rebekah [28]. They provided sufficient conditions for norm-linear and norm-additive operators between uniform algebras which are isometric algebra isomorphisms. The results obtained are as follows:

Lemma 2.11 (28, Lemma 12)**.** *If $T : A \to B$ is norm additive operator and $f, g \in A$, then*

(i). T is norm preserving

(ii). T preserves the distances, i.e $\|Tf - Tg\| = \|f - g\|$

In [28], they considered norm preserver conditions for uniform algebras but not the norm preserver conditions for local automorphisms of commutative Banach algebras. Therefore, in this work we have established the norm preserver conditions for local automorphisms of commutative Banach algebras.
Bonyo and Agure [6], studied norms of inner derivations on norm ideals and established the inequalities between the diameter of numerical range and norms of inner derivations as seen in the theorem below:

Theorem 2.12 (6, Theorem 3.2)**.** . *Let $A \in B(H)$ be S-universal, then $diam(W(A)) = 2\|A\|$*

In the above study, they applied the concepts of S-universal to the theory of inner derivations. In our study we have applied the concept in Theorem 2.12 to determine the norms of local automorphisms of commutative

Banach algebras and $\|\phi_x(y)\| = 2\|y\|$ holds.

In conclusion, a lot has been done on preserver problems for example linear, spectral, order and rank preserver problems on uniform algebras, matrix algebras and Banach algebras but norm preserving condition for local automorphism of commutative Banach algebras has not been considered. Therefore, in this study we have investigated properties of local automorphisms of commutative Banach algebras and determined their norms . Moreover, we have determined their norm preserving conditions for local automorphism of commutative Banach algebras.

Chapter 3

LOCAL AUTOMORPHISMS

3.1 Introduction

Let H be an infinite dimensional complex Hilbert space and $B(H)$ the set of all bounded linear operators on H. In this chapter, we investigate properties of local automorphisms of commutative Banach algebras.

3.2 Properties of Local Automorphisms

Lemma 3.1. *Let H be an infinite dimensional complex Hilbert space and $B(H)$ the set of all bounded linear operators on H, then every local automorphism of $B(H)$ is an automorphism.*

Proof. : From [2], we have for non-zero $x, y \in H$ then $x \otimes y$ denote a rank one operator defined by $(x \otimes y)z = \langle z, y\rangle x,\ z \in H$. The spectrum of the operator $x \otimes y$ is equal to the set $\{0, \langle x, y\rangle\}$. Two operators M, N are said to be equal if there exist an invertible operator $A \in B(H)$ such that

$M = ANA^{-1}$. Since every automorphism of $B(H)$ is inner [18], a local automorphism can also be defined as a linear mapping with the property that the operators N and $\phi(N)$ are similar for every $N \in B(H)$. To complete the proof of Lemma 3.1, we outline the following propositions which are crucial.

Proposition 3.2 (2, Lemma 2.1)**.** *If X and Y are complex normed linear spaces and $A : X \to Y$ is a bijective linear operator such that A^{-1} carries closed hyperplanes to closed hyperplanes, then A is bounded. For the proof see [2].*

Proposition 3.3 (2, Lemma 2.2)**.** *Let H be an infinite dimensional separable Hilbert space and let ϕ be a local automorphism of $B(H)$. Then the restriction of ϕ to $F(H)$ is either a homomorphism or antihomomorphism. For the proof see [2].*

Proposition 3.4 (2, Lemma 2.3)**.** *Let H be an infinite dimensional separable Hilbert space and let ϕ be a local automorphism of $B(H)$. Then the restriction of ϕ to $F(H)$ is a homomorphism then ϕ is an automorphism. For the proof see [2].*

Now we complete the proof of Lemma 3.1. We know by Proposition 3.3, that the restriction of ϕ to $F(H)$ is either a homomorphism or an antihomorphism. In view of Proposition 3.4, it is sufficient enough to consider the situation when $\phi \mid F(H) = \Psi$ is an antihomomorphism. But then, as ψ maps $F(H)$ into itself $\phi^2 \mid F(H) = \psi^2$ is homomorphism and by definition of local automorphism ϕ is an automorphism. In particular, ϕ^2 is onto, which implies that ϕ is also an automorphism. Thus, ϕ satisfies the requirements of Larison and Sourour [18]. Hence ϕ is an automorphism. $\square$

Lemma 3.5. *Let H be an infinite dimensional complex Hilbert space and $B(H)$ the set of all bounded linear operators H, then every local automorphism of $B(H)$ is linear.*

Proof. Let $H^{'}$ denote the dual spaces of infinite dimensional separable complex Hilbert space H. For non-zero $x \in H$ and $y \in H^{'}$, we denote $x \otimes y \in F(H)$ a rank one operator defined by $(x \otimes y)z = y(z)x$ for all $z \in H$. Let $A \in B(H)$ be any bounded bijective linear operator. Since we know that two operators M and N are equal if $M = ANA^{-1}$ and from definition of local automorphism $\phi(x) = \phi_x(x)$, then $y(A^{-1})x = (My)(\phi(A)^{-1}Nx) = \phi(M(y)A^{-1}N(x)) = \phi(MN)(yA^{-1}x)$ $\phi(MN) = I$ where I is the identity, it follows that $y(A^{-1}x) = (yA^{-1}x)$ $\square$

Lemma 3.6. *Let H be an infinite dimensional complex Hilbert space and $B(H)$ the set of all bounded linear operators H, then every local automorphism of $B(H)$ is bounded.*

Proof. Let $\phi : B(H) \rightarrow B(H)$ be a local automorphism. From the definition of boundedness, ϕ is said to be bounded if there exist a scalar $k > 0$ such that $\|\phi(x)\| \leq k\|x\|, \forall x \in B(H)$. Recall that local automorphism $\phi(x)$ is defined by $\phi(x) = \phi_x(x)$. Let k be any constant, then it is easy to see that $\|\phi(kx)\| = \|\phi_x(kx)\| \leq \|k\phi_x(x)\| = k\|\phi_x(x)\|$. Let $\phi_x(x) = p, \forall\, p \in B(H)$, then $\|\phi_x(x)\| \leq k\|p\|$ $\square$

Lemma 3.7. *Let H be an infinite dimensional complex Hilbert space and $B(H)$ the set of all bounded linear operators H, then every local automorphism of $B(H)$ is continous.*

Proof. For the proof of Lemma 3.7, we give analogous proof of Molnar [22, Lemma 3.1.4]. To establish continuity, we show that there exist a projection (self-adjoint idempotents) $P \in B(H)$ with infinite rank and infinite corank for which the mapping $N \mapsto \phi(PNP)$ is continous. Suppose that $\phi \neq 0$, from [11, Theorem 3] we have every Jordan ideal of $B(H)$ is an associative ideal and from [9, Lemma 2.1], if we let ϕ be linear mapping from H into an H-bimodule then:

1. $(I-P)\phi(PRQ)(I-Q) = 0$ for every $R \in B(H)$ and any idempotents $P, Q \in B(H)$.
2. The mapping ϕ satisfies $\phi(PRQ) = \phi(PR)Q + P\phi(RQ) - P\phi(R)Q$ for every $R \in B(H)$ and idempotents $P, Q \in B(H)$.

For proof of above statements 1 and 2 see [9] and the same holds for the kernel $Ker\phi$.

Let P_k be an infinite dimensional projection, if $\phi(P) = 0$ then using the ideal property of $Ker\phi$ we obtain that $I \in Ker\phi$ yielding $Ker\phi = B(H)$ which contradicts $\phi(P) \neq 0$. Let P_k be a sequence of pairwise orthogonal infinite dimensional projections, we assert that there exist $k \in \mathbb{N}$ for which the linear operator $N \mapsto \phi(P_k N P_k)$ is bounded. Assume that for any $k \in \mathbb{N}$ there is an operator $N_k \in \mathbb{N}$ such that $\|N_k\| = 1$ and $\|\phi(P_k N P_k)\| \geq k2^k \|\phi(P_k)\|^2$. Define $N = \sum_k \frac{1}{2^k} P_k N P_k$ and we obtain

$N \in B(H)$ then,

$$\begin{aligned}
\|\phi(P_k)\|^2\|\phi(N)\| &\geq \|\phi(P_k)\phi(N)\phi(P_k)\| \\
&= \|\phi(P_kNP_k)\| \\
&= \frac{1}{2^k}\|\phi(P_kNP_k)\| \\
&= k\|\phi(P_k)\|^2
\end{aligned}$$

since $\|\phi(P_k)\| \neq 0$ and the inequality above holds for every $k \in \mathbb{N}$ hence we arrive at the contradiction. Now we obtain a projection $P \in B(H)$ with infinite rank and infinite corank for which the mapping $N \mapsto \phi(PNP)$ is continous.

Let $P = \Sigma_{k=1}^{\infty} a_k \otimes a_k$ where a_k is an orthonormal sequence. Let b_k be an orthonormal sequence in R which extends to a_k.

Consider

$$A = \Sigma_k b_k \otimes a_k \tag{3.2.1}$$

and

$$B = \Sigma_k a_k \otimes b_k \tag{3.2.2}$$

it follows that $APB = I$ and $BPA = Q$ which is a projection with an infinite dimensional range. Since

$$\phi(NQ) = \phi(APBNBPA) = \phi(A)\phi(P(BNB)P)\phi A$$

and the mapping $N \mapsto \phi(P(BNB)P)$ continous [22]. We obtain the continuity of the transformation $N \mapsto \phi(NQ)$ similarly the argument gives the same property for the mapping $N \mapsto \phi(QN)$. Therefore with the notation $Q^{\perp} = I - Q$ we have continuity of linear mapping

$$\begin{aligned} N \mapsto \phi((QN)Q) &= \phi(QNQ) \\ N \mapsto \phi((Q^{\perp}N)Q) &= \phi(Q^{\perp}NQ) \\ N \mapsto \phi(Q(NQ^{\perp})) &= \phi(QNQ^{\perp}) \end{aligned}$$

Let $Q = \Sigma_{k=1}^{\infty} a'_k \otimes a'_k$ be with orthonormal sequences (a'_k). Extend (b'_k) to a complete orthonormal sequences in H and define

$$R = \Sigma_k b'_k \otimes a'_k + \Sigma_k a'_k \otimes b'_k$$

Plainly $RQR = Q^{\perp}$ and the mapping

$$\begin{aligned} N \mapsto \phi(Q^{\perp}NQ^{\perp}) &= \phi(RQRNRQR) \\ &= \phi(R)\phi(Q(RNR)Q)\phi(R) \end{aligned}$$

is continous. Finally since

$$\phi(N) = \phi(QNQ) + \phi(QNQ^{\perp}) + \phi(Q^{\perp}NQ) + \phi(Q^{\perp}NQ^{\perp});\ \forall N \in B(H).$$

Hence we obtain a contradiction of ϕ. □

Theorem 3.8. *Let H be an infinite dimensional complex Hilbert space and $B(H)$ the set of all bounded linear operators H, then the group of every local automorphism of $B(H)$ is algebraically reflexive.*

Proof. To prove the above theorem we need to outline the following propositions.

Proposition 3.9 (23, Proposition 2.2)**.** *Let* $\mathbf{A}, \mathbf{B} \subset B(H)$ *be closed* $*$-*subalgebras and suppose for every self-adjoint elements* $A \in \mathbf{A}$ *the spectral measure of any any Borel subset of* $\sigma(A)$ *is bounded away from* 0 *belongs to* $\mathbf{A}$*. If* $\phi : \mathbf{A} \to \mathbf{B}$ *is continous linear map which sends projections to idempotents, then* ϕ *is a Jordan homomorphism. [For proof see 23]*

Proposition 3.10 (10, Lemma 1)**.** *Let* $\phi : \mathbf{A} \to \mathbf{B}$ *be a linear map where* $\mathbf{A}, \mathbf{B} \subset B(H)$*. Suppose* ϕ *is a local isomorphism in the sense that for every* $x \in \mathbf{A}$ *there exist an isomorphism* $\psi : \mathbf{A} \to \mathbf{B}$ *such that* $\phi(x) = \psi(x)$ *then* ϕ *is an injective homomorphism. For proof see [10].*

To complete the proof of Theorem 3.8, let ϕ be a local automorphism of $\mathbf{A}$ and ψ be an automorphism such that $\phi(x) = \psi(x)$ whenever $x \in \{0, 1\}$ is identity. We claim that $\phi = \psi$, since by Proposition 3.10, ϕ and ψ are weakly continous. It follows by Lemma 3.1, 3.5 and 3.7, that every local automorphism of $B(H)$ is an automorphism, linear and continous and $\phi(N) = N \circ \Pi = \psi(N)$ for any $N \in B(H)$ where $\Pi = \phi(x) = \psi(x)$. Finally by Proposition 3.9, the spectral measure of any borel subset of $\sigma(A)$ is bounded away from 0 belongs to $\mathbf{A}$ thus the group of local automorphism is algebraically reflexive. □

Theorem 3.11. *Let* H *be an infinite dimensional complex Hilbert space and* $B(H)$ *the set of all bounded linear operators* H*, then every local automorphism of* $B(H)$ *is inner.*

Proof. The following Proposition is helpful in our proof.

Proposition 3.12. *Every non-trivial ideal of $B(H)$ is selfadjoint, contains ideal $F(H)$ and is contained in the ideal of compact operators [21].*

In this proof we are considering subalgebras containing ideal of finite rank operators $F(H)$. We know that all commutative Banach algebras have the right and left ideal coincide. By [6], a symmetric norm ideal $(\mathcal{I}, \|.\|)$ in $B(H)$ consist of two-sided ideal $\mathcal{I}$ together with the norm $\|\mathcal{I}\| = \|F(H)\|$ for $F(H)$ a rank one operator.

Now we complete the proof of Theorem 3.11. Suppose that $\mathcal{I}$ is non-trivial ideal in $B(H)$, then by Lemma 3.1 every local automorphism is an automorphism. Let $A \subset B(H)$ be such that for every $M \in \mathcal{I}$ and there is $N_M \in \mathcal{I}$ for which $MA - AM = MN_M - N_M M$. Considering this relation for M^* and by Definition 1.18, we take the adjoints hence $MA^* - A^*M = MN^*_{M^*} - N^*_{M^*}M$. Consequently the equations of self adjoint operators $Re(A)$ and $Im(A)$ induces a local automorphism on ideal $\mathcal{I}$. It follows from Proposition 3.12 that A is self-adjoint. By the Weyl-Von Neumann theorem there are complete orthonormal system $\{a_n\}n \in \mathbb{N}$, a bounded sequence $\{b_n\}n \in \mathbb{N}$ of real numbers and compact self adjoint operators K such that $A = \Sigma_{n=1}^{\infty} b_n a_n \otimes a_n + K$.
Since $\mathcal{I} \neq F(H)$, $\mathcal{I}$ must contain an operator with infinite dimensional range. Let $M \mapsto MA - AM$ for $A \in B(H)$. Since there is compact N for which $M(A_K) - (A - K)M = MN - NM$ then it implies that $(b_n)n \in \mathbb{N}$ converges to real b.
Let $A^{'} = A - bI$ which is compact self-adjoint operators inducing inner automorphism as A does. Suppose that A is compact, from equation 3.2.1, A can be written in the form $A = \Sigma_{n=1}^{\infty} \lambda_n a_n \otimes a_n$, where $\lambda \to 0$ and

$\{a_n\}n \in \mathbb{N}$ is a complete orthonormal system. Let $M \in \mathcal{I}$ and $N \in \mathcal{I}$ such that $MA - AM = MN - NM$. From Lemma 3.1 $x \otimes y = \langle x, y\rangle$ then we can also define $\lambda_n = \langle Na_n, a_n\rangle$; $(n \in \mathbb{N})$

If $N = \Sigma_{k=1} a_n c_n \otimes c_n$ is a complete orthonormal system of N then, $\langle Na_n, a_n\rangle = \Sigma_k d_k \langle a_n, c_k\rangle\langle c_k, a_n\rangle$ since

$$\begin{aligned} \Sigma_k |\langle a_n, c_k\rangle||\langle c_k, a_n\rangle| &\leq \|a_n\|^2 = 1 \\ \Sigma_n |\langle a_n, c_k\rangle||\langle c_k, a_n\rangle| &\leq \|c_k\|c_k\| \leq 1 \end{aligned}$$

Furthermore since A is also an ideal it follows that $A = \mathcal{I}$. □

Theorem 3.13. *Let $\phi : B(H) \to B(H)$ be a local automorphism, then there exist a projection $Q \in B(H)$ such that for any family X_n of pairwise orthogonal rank one projections the sequence $\Sigma_{i=1}^n \phi(X_n)$ converges strongly to $Q(x)$.*

Proof. Two idempotents $P, Q \in B(H)$ are algebraically orthogonal if $PQ = QP = 0$ holds true. We know that ϕ is a local automorphism that maps idempotents to idempotents and preserve orthogonality between them . If P and Q are orthogonal idempotents then we have $0 = \phi(PQ + QP) = \phi(P)\phi(Q) + \phi(Q)\phi(P)$ which implies that there is orthogonality between $\phi(P)$ and $\phi(Q)$.

Let X_n be a maximal family of pairwise orthogonal rank one projection in $B(H)$ which is necessarily closed maximal proper two sided proper ideal. Let $F_n = \Sigma_1^n \phi(X_n)$ and define E as the idempotent having $R = \overline{Span}\{RanF_n : n \in \mathbb{N}\}$ and $KerK = \cap KerF_n$. We need to show that $R \cap K = \{0\}$. Let (f_n) be a sequence in $Span\{RanF_n : n \in \mathbb{N}\}$ which converges to $r \in K$. By Lemma 3.6, we know ϕ is bounded. Let M

denote the norm of ϕ. For every $\epsilon > 0$ there exist an index $n_0 \in \mathbb{N}$ such that $\|r - f_n\| < \frac{\epsilon}{M}$ $(n \geq n_0)$. We can see that $RanF_n$ is monotone increasing sequence of subspaces of $B(H)$. Therefore (f_n) is in the range of an idempotent F_k while r is in its $KerF_k$ which implies that

$$\begin{aligned}
\|0 - f_n\| &= \|F_k r - F_k f_n\| \\
&\leq \|F_k\|\|r - f_n\| \\
&= \|\phi(\Sigma_{i=1}^{k})\|\|r - f_n\| \\
&= M\|r - f_n\| < \epsilon
\end{aligned}$$

for every $n \in \mathbb{N}$. Thus $f_n \to 0$ and we have $r = 0$. This gives us that $R \cap K = \{0\}$.

We also see that $F_n h \to E_n$ whenever $h \in Span\{RanF_n : n \in \mathbb{N}\}$ or $h \in KerK$. So as F_n is bounded using Banach-Steinhaus theorem, we get that F_n converges strongly to E. Furthermore, the involved operators $\phi(P), \phi(Q)$ are idempotents and algebraic arguments proves that $\phi(P)\phi(Q) = \phi(Q)\phi(P)$ holds for all $n \in \mathbb{N}$. By spectral theorem of self-adjoint operators and continuity of ϕ, E commutes with the range and hence $Q = E$. Hence $s - limX_n \to Q$. $\square$

Chapter 4

NORM PRESERVING CONDITIONS

4.1 Introduction

Most of the preserver problems have been on linear preserver problems in matrix algebras, spectrum preserver or subset of spectra algebra preserver and norms preservers on uniform algebras and Banach algebras. In this chapter, we present results on norm preserver conditions for local automorphisms of commutative Banach algebras B.

4.2 Conditions preserving norms

Since our work is in commutative Banach algebras. We give conditions that preserve norm with respect to additivity, multiplicativity and identity.

Theorem 4.1 (27, Theorem 5.0.8)**.** *A mapping $T : A \to B$ between uniform algebras preserve the peaking functions of algebra that is $T(P(A)) = P(B)$ satisfying the equation $\|TfTg\| = \|fg\|$ for every $f, g \in A$ if and only if there exist a homomorphism $\psi : \sigma(B) \to \sigma(A)$ such that $|Tf(y)| = |f\psi(g)|$ for every $f \in A$ and $y \in \sigma(B)$.*

Lemma 4.2. *Let B be a commutative Banach algebra and $\phi : B \to B$ be a local automorphism. Then $\|\phi(x) + \phi(y)\| = \|x\| + \|y\|$ for all $x, y \in B$.*

Proof. Let A be a commutative Banach sub-algebra B. Let $\phi : B \to B$ be a local automorphism. By Hahn-Banach extension Theorem, there is a continued extension of $\phi : A \to A$ so that for all $x, y \in A$ there exist a peaking function $\phi(x)$ with supremum norm defined by:
$\|\phi(x)\| = \sup_{\|x\|=1} \left\{ \frac{\|\phi(x)\|}{\|x\|}, x \in A \right\}.$
Now, we show that $\|\phi(x)\| + \|\phi(y)\| = \|x\| + \|y\|$.
Indeed,

$$\begin{aligned}
\|\phi(x) + \phi(y)\| &= \sup_{\|x\|=1, \|y\|=1} \{\|\phi(x) + \phi(y)\|;\ x, y \in A\} \\
&= \sup_{\|x\|=1, \|y\|=1} \{\|\phi(x)\| + \|\phi(y)\|;\ x, y \in A\} \\
&\leq \sup_{\|x\|=1} \{\|\phi(x)\|;\ x \in A\} + \sup_{\|y\|=1} \{\|\phi(y)\|;\ y \in A\} \\
&= \|\phi(x)\| + \|\phi(y)\|.
\end{aligned}$$

The reverse inequality is trivial since norm is nonnegative. But by [27, Proposition 4.0.9] $\|\phi(x) + \phi(y)\| = |\phi(x)\| + \|\phi(y)\| = \|x\| + \|y\|$,
iff $\|\phi\| = 1$. □

Corollary 4.3. *Let B be a commutative Banach algebra and $\phi : B \to B$ be a local automorphism, then $\phi : B \to B$ is norm-additive if $\|\alpha\phi(x) +$*

$\beta\phi(y)\| = |\alpha\|x\| + |\beta|\|y\|$ *for all* $x, y \in A$ *and* $\alpha, \beta \in \mathbb{K}$ *with* $|\alpha| = |\beta| = 1$.

Proof. We show that linearity is preserved in norm additive of local automorphisms. For all $x, y \in A$ and $\alpha, \beta \in \mathbb{K}$ with $|\alpha| = |\beta| = 1$.
Then we have,

$$\begin{aligned} \|\alpha\phi(x) + \beta\phi(y)\| &= \sup_{\|x\|=1, \|y\|=1} \left\{ \| \frac{\phi\alpha(x)}{\|x\|} + \frac{\phi\beta(y)}{\|y\|} \|, \ x, y \in A \right\} \\ &\leq |\alpha| \sup_{\|x\|=1} \left\{ \frac{\|\phi(x)\|}{\|x\|}, x \in A \right\} \\ &+ |\beta| \sup_{\|y\|=1} \left\{ \frac{|\beta|\|\phi(y)\|}{\|y\|}, \ y \in A \right\} \\ &= |\alpha|\|\phi(x)\| + |\beta|\|\phi(y)\| \ \alpha, \beta \in \mathbb{K}, \ x, y \in A \end{aligned}$$

iff $\forall \alpha, \beta \in \mathbb{K}, |\alpha| = |\beta| = 1$ *and* $x, y \in A$.

The reverse inequality holds since norm is nonnegative. □

Example 4.4 (27, Example 23)**.** An operator $\phi : B \to B$ for which $\phi(x) = ix$ is norm additive in modulus because $\||\phi(x)| + |\phi(y)|\| = \||ix| + |iy|\| = |i|\||x| + |y|\| = \||x| + |y|\|$. The operator $\phi : B \to B$ for which $\phi(x) = -x$ is similarly norm additive. In fact all operators $\phi : B \to B$ such that $\phi(x) = \alpha x$ with $\alpha \in \mathbb{K}$ and $|\alpha(x)| = 1$ for every $x \in A$ and norm additive since $\||\phi(x)| + |\phi(y)|\| = \||\alpha x| + |\alpha y|\| = |\alpha\|x| + |y|\| = \||x| + |y|\|$.

Example 4.5 (27, Example 24)**.** The operator $\phi : B \to B$ defined by $\phi(x) = \|x\|, \ \forall x \in A$ is also norm additive in modulus i.e
$\||\phi(x)| + |\phi(y)|\| = \||x|\| + \||y|\| = \||x| + |y|\|$.
We note that the operator ϕ does not preserve $|x|$ unless it is a constant function.

Lemma 4.6. *Let B be a commutative Banach algebra and $\phi : B \to B$ be a local automorphism, then $\|\phi(xy)\| = |x|\|\phi(y)\|$ if x is a scalar operator.*

Proof. Let k be a peaking constant function on commutative Banach algebra B such that $\phi(x) = k$ then ϕ is a norm preserver i.e
$\|\phi(x)\| = \|x\| = |k|$ it follows that when $|k| > 0$ we have

$$\begin{aligned}
\|\phi(xy)\| &= \sup_{\|x\|=1,\ \|y\|=1} \left\{ \frac{\|\phi(x)\|}{\|x\|} \frac{\|\phi(y)\|}{\|y\|},\ x, y \in A \right\} \\
&= \sup_{\|x\|=1} \left\{ \frac{\|\phi(x)\|}{\|x\|},\ x \in A \right\} \sup_{\|y\|=1} \left\{ \frac{\|\phi(y)\|}{\|y\|},\ y \in A \right\} \\
&= |k| \sup_{\|x\|=1} \left\{ \frac{\|\phi(y)\|}{\|y\|},\ y \in A \right\} \\
&= \|x\|\|\phi(y)\|,\ \text{iff } \|x\| = |k| = |x| \\
&= |x|\|\phi(y)\|.
\end{aligned}$$

Hence $\|\phi(xy)\| = |x|\|\phi(y)\|$ if and only if x is a scalar operator □

Lemma 4.7. *Let $\phi : B \to B$ be a local automorphism, then for all $\alpha, \beta \in \mathbb{K}$ and $x, y \in B$ the following hold.*

(i) $\|\alpha\phi(x)\| = |\alpha|$.

(ii) $\|\phi(I)^2\| = 1$.

Proof. Part (i) we use supremum norm

$$\begin{aligned}
\|\alpha\phi(x)\| &= \sup_{\|x\|=1} \left\{ \frac{\|\alpha\phi(x)\|}{\|x\|}, x \in A \right\} \\
&= |\alpha| \sup_{\|x\|=1} \left\{ \frac{\|\phi(x)\|}{\|x\|}, x \in A \right\} \\
&= |\alpha|\|\phi(x)\|.
\end{aligned}$$

For peaking functions [28, Lemma 12] we have $\|\phi(x)\| = \|x\|$ and [30, Proposition 2.22] $\|\phi(x)\| = 1$ holds hence $\|\phi(x)\| = \|x\| = 1$. Therefore it follows that $\|\alpha\phi(x)\| = |\alpha|$

For the second part (ii) $\|\phi(I)^2\| = 1$. We know that $\phi(I) = 1$ and by Gelfand-transform [30, Theorem 2.25], ϕ preserves multiplication that is $\phi(a)\phi(b) = \phi(a)\phi(b)$. Suppose that $a = b$ then $\phi(ab) = \phi(aa) = \phi(a)^2$ but when $a = I$ then $\phi(I)^2 = \phi(I.I) = \phi(I)\phi(I) = 1$ but $\|\phi(I)\| = 1$ then $\|\phi(I)^2\| = 1.1 = 1$. □

4.3 Norms of local Automorphisms

In this section, we determine norms of local automorphisms. In particular, we determine norms of special cases of local automorphisms and generalized automorphisms.

Lemma 4.8. *Let $\phi : B \to B$ be defined by $\phi_x(y) = xy - yx$ then $\|\phi_x(y)\| = 2\|y\|$.*

Proof. Let $A \subset B$. The inner automorphism induced by x fixed in A is the operator $\phi_x(y)$ defined by $\phi_x(y) = xy - yx, \ \ \forall y \in A$. By Stampfli [14], the norm of inner derivation is computed as $\|\phi_x(y)\| = 2\|y\|$. Consider $x \otimes I \in A \otimes A$ where $I \in A$ and $\|x \otimes I\|_{CB} = \|x\|\|I\| = \|x\|$. Define $\|\phi(x)\| = \sup_{x\otimes I \in A\otimes A,\ \|x\otimes I\| = \|x\|} \{\|x \otimes I - I \otimes x\|.\}$

Now,

$$\begin{aligned}
\|\phi_x(y)\| &= \sup_{x\otimes y\in A\otimes A,\ \|x\otimes y\|=\|x\|\|y\|} \{\|x\otimes y - y\otimes x\|\} \\
&= \sup_{x\otimes y\in A\otimes A,\ \|x\otimes y\|=\|x\|\|y\|} \{\|(x-\alpha I)\otimes y - y\otimes (x-\alpha I)\|,\ \alpha\in\mathbb{K}\} \\
&= \sup_{x\otimes y\in A\otimes A,\ \|x\otimes y\|=\|x\|\|y\|} \{\|(x-\alpha I)\otimes y + [-1(y\otimes (x-\alpha I))]\|\ \alpha\in\mathbb{K}\} \\
&\leq \sup_{x\otimes y\in A\otimes A,\ \|x\otimes y\|=\|x\|\|y\|} \{\|(x-\alpha I)\otimes y\|\ \alpha\in\mathbb{K}\} + \\
&\quad |-1| \sup_{x\otimes y\in A\otimes A,\ \|x\otimes y\|=\|x\|\|y\|} \{\|y\otimes (x-\alpha I)\|.\ \alpha\in\mathbb{K}\}
\end{aligned}$$

By [29, Theorem 2], we know that a rank one operator $x\in B$ satifies the following condition:

For $\alpha\in\mathbb{C}$, then $\|x-\alpha I\| \geq \sup_{0\neq x\in A,\ \|x\|=1} \left\{\frac{\|\phi(x)\|}{\|x\|}\right\} = \|x\|$ holds. Hence,

$$\sup_{x\otimes y\in A\otimes A,\ \|x\otimes y\|=\|x\|\|y\|} \{\|(x-\alpha I)\otimes y\|\ \alpha\in\mathbb{K}\} = \|x\|\|y\|$$

and

$$|-1| \sup_{x\otimes y\in A\otimes A,\ \|x\otimes y\|=\|x\|\|y\|} \{\|y\otimes (x-\alpha I)\|\ \alpha\in\mathbb{K}\} = \|y\|\|x\|. (4.3.1)$$

$\|\phi_x(y)\| = \|x\|\|y\|+\|y\|\|x\|$ and by Lemma 4.7 we have $\|\phi(x)\| = \|x\| = 1$. It follows that $\|\phi_x(y)\| = \|x\|\|y\| + \|y\|\|x\| = \|y\| + \|y\| = 2\|y\|$. $\square$

Theorem 4.9. *Let $\phi : B \to B$ be defined by $\phi_{x,z}(y) = xy - yz$ then $\|\phi_{x,z}(y)\| = \|x\| + \|z\|$, iff $\|y\| = 1$.*

Proof. Let A be a commutative Banach sub-algebra of B and $x\otimes I, y\otimes I$

and $z \otimes I \in A \otimes A$. For $\phi_{x,z}(y) = xy - yz$, we have

$$
\begin{aligned}
\|\phi_{x,z}(y)\| &= \sup_{y\otimes I \in A\otimes A,\ \|y\otimes I\|=\|y\|} \{\|x \otimes y - y \otimes z\|\} \\
&= \sup_{y\otimes I \in A\otimes A,\ \|y\otimes I\|=\|y\|} \{\|x \otimes (y - \alpha I) - (y - \alpha I) \otimes z\|,\ \alpha \in \mathbb{K}\} \\
&= \sup_{y\otimes I \in A\otimes A,\ \|y\otimes I\|=\|y\|} \{\|x \otimes (y - \alpha I) + [-1((y - \alpha I) \otimes z)]\|,\ \alpha \in \mathbb{K}\} \\
&\leq \sup_{y\otimes I \in A\otimes A,\ \|y\otimes I\|=\|y\|} \{\|x \otimes (y - \alpha I)\|,\ \alpha \in \mathbb{K}\} \\
&+ \ |-1| \sup_{y\otimes I \in A\otimes A,\ \|y\otimes I\|=\|y\|} \{\|(y - \alpha I) \otimes z\|,\ \alpha \in \mathbb{K}\}
\end{aligned}
$$

By Lemma 4.8, Equation 4.3.1 and nonnegativity of the norm, it follows that

$$
\begin{aligned}
\|\phi_{x,z}(y)\| &= \|y\|\|x\| + \|y\|\|z\| \\
&= \|x\| + \|z\|, \text{ iff } \|y\| = 1
\end{aligned}
$$

□

Theorem 4.10. *Let A be sub-algebra of commutative Banach algebra B. Let $\phi : A \otimes A \to B$ be an inner automorphism defined by $\phi_a(b) = ab - ba$. Let $c \in A$ be unique orthogonal projection and $\phi_c(b) = cb - bc$ then $\|\phi_a(b)\| = \|\phi_c(b)\| = 2\|b\|$.*

Proof. Consider $a \otimes I, b \otimes I \in B \otimes B$ and $c \otimes I \in A \otimes A$. Let $\|a \otimes I\|_{CB} = \|a\|\|I\| = \|a\|$, $\|b \otimes I\|_{CB} = \|b\|\|I\| = \|b\|$ and $\|c \otimes I\|_{CB} = \|c\|\|I\| = \|c\|$. Then by [19, Theorem 4.3] and the parallelogram identity we show the uniqueness of $c \in A$ in which it is sufficient to see that

$$\|a \otimes I + b \otimes I\|^2 + \|a \otimes I - b \otimes I\|^2 = 2\|a\|^2 + 2\|b\|^2 \quad (4.3.2)$$

for all $a \otimes I, b \otimes I \in B \otimes B$ holds.

Now, we show that for all $c \otimes I \in A \otimes A$ and the following statements hold.

(i). $\|a \otimes I - b \otimes I\| = \|a \otimes I - c \otimes I\|$

(ii). $\langle a \otimes I - b \otimes I, c \otimes I \rangle = 0$

For part (i) let $\alpha = \sup_{c\otimes I \in A \otimes A,} \{\|a \otimes I - c \otimes I\|,\ \|c \otimes I\| = \|c\|\}$ and from [12, Lemma 1] and [12, Corollary 1], we choose a family of pairwise orthogonal projections $P_n \in A$ so that $\|a \otimes I - P_n\| \to \alpha$. We show that P_n is a cauchy sequence. Hence for $n, m \in \mathbb{K}$ we have the parallelogram identity (4.3.2)

$$\begin{aligned}\|(a \otimes I - P_n) + (a \otimes I - P_m)\|^2 + \|(a \otimes I - P_n) - (a \otimes I - P_m)\|^2 &= \\ 2\|(a \otimes I - P_n)\|^2 + 2\|(a \otimes I - P_m)\|^2.&\end{aligned}$$

Consequently,

$$\|P_n - P_m\|^2 = 2\|a \otimes I - P_n\|^2 + 2\|a \otimes I - P_m\|^2 - 4\|a \otimes I - \frac{P_n + P_m}{2}\|^2$$

Taking into account $\frac{P_n + P_m}{2} \in A$, we have

$$\|P_n - P_m\|^2 = 2\|a \otimes I - P_n\|^2 + 2\|a \otimes I - P_m\|^2 - 4\alpha^2$$

so that $\|P_n - P_m\| \to 0$ as $n, m \to \infty$ thus P_n is cauchy sequence and since B is complete and $A \otimes A \subset B$ its convergent to $b \otimes I \in A \otimes A$. Since $\|a \otimes I - P_n\| \to \|a \otimes I - b \otimes I\|$, we have that $\|a \otimes I - b \otimes I\| = \alpha$.

Indeed uniqueness also follows by parallelogram identity,

$$\begin{aligned}
\|b \otimes I - d \otimes I\|^2 &\leq 2\|a \otimes I - b \otimes I\|^2 + 2\|a \otimes I - d \otimes I\|^2 \\
&- 4\|a \otimes I - \frac{a \otimes I + d \otimes I}{2}\|^2 \\
&\leq 2\alpha^2 + 2\alpha^2 - 4\alpha^2 = 0
\end{aligned}$$

where $b \otimes I, d \otimes I \in A \otimes A$.

For part (ii) we adopt analogous proof of [19, Proposition 4.1]. We define

$$\begin{aligned}
F(\beta) &= \|a \otimes I - b \otimes I, \beta c \otimes I\|^2 \\
&= |\beta|\|c \otimes I\|^2 - 2\beta\langle a \otimes I - b \otimes I, c \otimes I\rangle \\
&+ \|a \otimes I - b \otimes I\|, \ \forall \beta \in \mathbb{K}
\end{aligned}$$

but when $\beta = 0$ then $F(\beta) = \langle a \otimes I - b \otimes I, c \otimes I\rangle = 0$ holds for all $c \otimes I \in A \otimes A$.

Hence

$$\begin{aligned}
\|a \otimes I - b \otimes I - c \otimes I\|^2 &= \|c \otimes I\|^2 + \|a \otimes I - b \otimes I\|^2 \\
&\geq \|a \otimes I - b \otimes I\|^2
\end{aligned}$$

Since $c \in A$. By Lemma 4.8, then $\|\phi_a(b)\| = 2\|b\|$ and $\|\phi_c(b)\| = 2\|b\|$ holds. Therefore $\|\phi_a(b)\| = \|\phi_c(b)\| = 2\|b\|$. □

Chapter 5

CONCLUSION AND RECOMMENDATIONS

5.1 Introduction

In this last chapter, we draw conclusions and make recommendations based on the stated objectives of the study and the results.

5.2 Conclusion

The study of norm preserver conditions is an extensive area in operator algebras. In the study, we have outlined the basic concepts and some fundamental techniques used to solve the stated objectives in chapter one. In chapter two, we have outlined related literature on automorphisms and norm preserver problems on Banach algebras. In chapter three, we have investigated properties of local automorphisms of commutative Banach algebras with respect to the first objective. We have showed that every local automorphism on $B(H)$ is an automorphism, linear, bounded, inner and continous. Moreover, the group of local automorphisms are algebraically reflexive. In chapter four, we have established norm preserver conditions for local automorphism of commutative Banach algebras which includes; $\|\phi(x)+\phi(y)\| = \|x\|+\|y\|$, $\|\alpha\phi(x)+\beta\phi(y)\| = \|\alpha(x)+\beta(y)\|$ for all $x, y \in A$ and $\alpha, \beta \in \mathbb{K}$ with $|\alpha| = |\beta| = 1$, $\|\phi(xy)\| = |x|\|\phi(y)\|$ if x is a scalar operator, $\|\alpha\phi(x)\| = |\alpha|$ and $\|\phi(I)^2\| = 1$. In our third objective we have determined the norms of local automorphism and the results are: $\|\phi_x(y)\| = 2\|y\|$, $\|\phi_{x,z}(y)\| = \|x\| + \|z\|$ iff $\|y\| = 1$ and $\|\phi_a(b)\| = \|\phi_c(b)\| = 2\|b\|$ iff c is an orthogonal projection on A.

5.3 Recommendation

Local automorphisms of commutative Banach algebras plays a key role in operator algebras in describing two observable physical quantities in quantum mechanics especially for normal operators. We have established the norm preserver conditions for local automorphisms of commutative Banach algebra. However, we recommend further research on norm preserving conditions for local automorphisms with denseness, limit convergence and separability in commutative Banach algebras may pursued by other researchers, to establish whether our results holds in commutative Banach algebras with these properties.

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