JARAMOGI OGINGA ODINGA UNIVERSITY OF SCIENCE AND TECHNOLOGY

SCHOOL OF MATHEMATICS AND ACTUARIAL SCIENCE

FIRST YEAR SECOND SEMESTER EXAMINATION FOR THE DEGREE OF MASTER OF SCIENCE IN PURE MATHEMATICS

SMA 813: FIELD THEORY

INSTRUCTIONS:

1. This paper consists of FIVE questions

2. Attempt any THREE questions.

3. Observe further instructions on the answer booklet.

QUESTION 1 **[20 Marks]**

(a) Construct a field of order 4. **[3 mks]**

(b) Let F be a field of prime characteristic p. Show that the Frobenius map of F is a 1-1 endomorphism.
[6 mks]

(c) Show that a finite subgroup of the multiplicative group of a field is cyclic
[6 mks]

(d) Show that for any field F, there is a unique F linear map $D : F[x] \to F[x]$ satisfying the following two conditions
i) $D(fg) = f.(Dg) + (Df).g$;
ii) $Dx = 1$. **[5 mks]**

QUESTION 2 **[20 Marks]**

(a) Distinguish between algebraic and transcendental numbers **[3 mks]**

(b) Suppose that E, F, G are fields with $E \subseteq F \subseteq G$. Show that $[G : E]$ is finite if and only if both $[G : F]$ and $[F : E]$ are finite, and that if this holds, then $[G : E] = [G : F].[F : E]$. **[9 mks]**

(c) Let E and F be fields with $E \subseteq F$. Show that the set of all elements of F which are algebraic over E is a field containing E. **[8 mks]**

QUESTION 3 **[20 Marks]**

(a) Let R be a commutative ring with identity. Show that R is a field if and only if the only ideals in R are $\{0\}$ and R itself. **[8 mks]**

(b) Let F be a field , and f a polynomial which is irreducible in $F[x]$. Show that there is a field K containing F and an element α satisfying $f(\alpha) = 0$.
[12 mks]

QUESTION 4 **[20 Marks]**

(a) State and prove Eisenstein's criterion of testing irreducibility of a polynomial over a principal ideal domain. Hence or otherwise, determine whether the polynomial $x^2 + y^2 - 1$ in $\mathbf{Q}[x, y]$ is irreducible. **[7 mks]**

(b) Let F be a field of characteristic $p \neq 0$, and let a be algebraic over F. Show that a is separable over F if and only if $F(a) = F(a^p)$. **[7 mks]**

(c) Let K/F be a Galois extension with Galois group G. Show that $\mid G \mid = [K : F]$. **[6 mks]**

QUESTION 5 **[20 Marks]**

(a) State the fundamental theorem of Galois theory. **[4 mks]**

(b) State and prove the theorem of the primitive element. **[5 mks]**

(c) Sketch the proofs of the following results

i) Any polynomial of odd degree over R has a root in R. **[4 mks]**

ii) Any positive real number has a real square root. **[3 mks]**

iii) Any complex number has a complex square root. **[4 mks]**