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On Compactness of Similarity Orbits of Norm-Attainable Operators

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Abstract: The notion of compactness plays an important role in analysis. It has been extensively discussed on both metric and topological spaces. Various properties of compactness have been proved under the underlying spaces. However, if we consider these sets to be from similarity orbits of norm-attainable operators, little has been done to investigate their compactness. In this paper, we introduce the concept of compactness of similarity orbits of norm-attainable operators in aspect of invariant topological spaces and investigate their properties.

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1. Introduction

The notion of similarity orbits of Hilbert space H operators was first initiated by Herrero [3] where he described the closure of similarity orbits of a normal operator with perfect spectrum. Since then, the norm closure of similarity orbits has been investigated extensively by many researchers such as Fialkow [1] and Hadwin et al. [2]. In this paper, we characterized similarity orbits in terms of compactness which is a generalization of the property of closedness and boundedness of subsets of the real line to topological space. Compactness can be characterized in many ways but the most fundamental ones are: sequential compactness which was developed by Bolzano and Weierstrass [4] grew out of functions developed on sequences of real numbers. The other characterization is in terms of open covers which was first introduced by Dirichlet and repeatedly introduced by Heine and continuously developed by Cousin, Lebesgue, Alexandroff and Uryson [4]. Compact sets enjoy a number of special properties not shared by other sets. For instance, compactness is preserved by continuous transformations. This implies that real-valued functions are bounded on compact sets and attain their maxima and minima. In this study, we regard similarity orbit as sets which consist of elements which are operators on which we assign a topology to become a topological space. In particular, invariant subsets of similarity orbits generate invariant topological space. Hence, we characterize similarity orbits of norm-attainable operators on invariant topological spaces in terms of compactness via the concept of open covers.

2. Preliminaries

In this section, we recall some key definitions and a result that are fundamental in the study:

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Definition 2.1 ([5]). *An operator $T \in B(H)$ is said to be norm-attainable if there exists a unit vector $x \in H$ such that $\|Tx\| = \|T\|$. The set of all norm-attainable operators on H is denoted by $NA(H)$.*

Definition 2.2 ([8]). *Let p be a point on a topological space (X, τ). A subset N of X is a neighborhood of p if and only if N is a superset of an open set G containing p. That is $p \subseteq G \subseteq N$.*

Definition 2.3 ([7]). *A subspace M of H is an invariant subspace of the operator T if for each $x \in M$, $Tx \in M$ i.e $TM \subseteq M$. M is also referred as $T-$ invariant or M is invariant under T.*

Definition 2.4. *A subset $S^o_{NA}(T)$ of an invariant topological space $(S_{NA}(T), \tau_{S_{NA}(T)})$ is compact if every open cover $\mathcal{O}_\alpha$, $\alpha \in \Lambda$ of $S_{NA}(T)$ is reducible to a finite subcover. i.e. $S^o_{NA}(T)$ is compact if for every open cover $\{\mathcal{O}_\alpha : \alpha \in \Lambda\}$ of $S_{NA}(T)$ such that $S^o_{NA}(T) \subseteq \cup_{\alpha=1}^{\infty}\mathcal{O}_\alpha$. For $\Gamma \subseteq \Lambda$, then there exist a finite subcover such that $\{\mathcal{O}_i : i \in \Gamma\}$ of $\mathcal{O}_\alpha$ such that $\{S^o_{NA}(T) \subseteq \cup_{i=1}^{n}\mathcal{O}_i\}$, (where Λ is an index set).*

Definition 2.5 ([12]). *A topological space denoted by (X, τ) is a non-empty set X together with a collection τ of subsets of X (referred to as open sets) that satisfies the following conditions:*

(1). the empty set $\emptyset$ and the whole space X are open sets.

(2). the union of any collection of open sets is itself an open set.

(3). the intersection of any finite collection of open sets is itself an open set.

Definition 2.6. *Let $S_{NA}(T)$ be a nonempty set and f be a map such that $f : S_{NA}(T) \to S_{NA}(T)$, then the set of all invariant subsets of $S_{NA}(T)$ related to f given by $\tau_f := \{S^o_{NA}(T) \subseteq S_{NA}(T) : f(S^o_{NA}(T)) \subseteq S^o_{NA}(T)\} \subseteq \mathcal{P}(S_{NA}(T))$ is a topology on $S_{NA}(T)$.*

Proposition 2.7. *Let $S_{NA}(T)$ be a nonempty set and τ_f be a collection of all invariant subsets of $S_{NA}(T)$ related to the map f, then $(S_{NA}(T), \tau_f)$ is a topological space.*

Proof. Let τ_f be defined by $\tau_f := \{S^o_{NA}(T) \subseteq S_{NA}(T) : f(S^o_{NA}(T)) \subseteq S^o_{NA}(T)\} \subseteq \mathcal{P}(S_{NA}(T))$, then we need to show that the three axioms of a topological space hold for τ_f.

(1). Let $\mathcal{O}_\alpha$ be a collection of open sets in τ_f for all $\alpha \in \Lambda$, then if $\mathcal{O}_\alpha = \emptyset \in \tau_f$ and similarly if $\mathcal{O}_\alpha = S_{NA}(T) \in \tau_f$.

(2). Let $\mathcal{O}_\alpha \in \tau_f$ for all $\alpha \in \Lambda$, then $f(\cup_\alpha \mathcal{O}_\alpha) = \cup_\alpha f(\mathcal{O}_\alpha) \subseteq \cup_\alpha \mathcal{O}_\alpha \in \tau_f$.

(3). Let $\mathcal{O}_i \in \tau_f$ for $i = 1, ..., n$, then $f(\cap_{i=1}^{n}\mathcal{O}_i) = \cap_{i=1}^{n} f(\mathcal{O}_i) \subseteq \cap_{i=1}^{n}\mathcal{O}_i \in \tau_f$.

Since the three axioms are satisfied, this implies that $(S_{NA}(T), \tau_f)$ is a topological space as required. □

Remark 2.8.

(1). In this study $(S_{NA}(T), \tau_f)$ is referred as an invariant topological space.

(2). The rest of this paper follow intuitively from Proposition 2.7.

3. Main results

We begin with basic facts about compact spaces.

Proposition 3.1. *Let $S_{NA}(T)$ be a non-empty set and $(S_{NA}(T), \tau_f)$ be an invariant topological space. If $S^o_{NA}(T)$ is a finite subset of $S_{NA}(T)$, then $S^o_{NA}(T)$ is a compact set.*

Proof. Suppose that $S^o_{NA}(T)$ is finite. This implies that either:

(1). $S^o_{NA}(T) = \emptyset$.

(2). $S^o_{NA}(T)$ is a finite non-empty set.

(1). If $S^o_{NA}(T) = \emptyset$, the proof is trivial since $S^o_{NA}(T)$ is vacuously compact. That is, the empty collection $\cup_{i\in\emptyset} O_i$ (where $O_i \cap O_j = \emptyset$, i.e. pairwise disjoint) is a finite subcover.
(2). Let $S^o_{NA}(T)$ be a non-empty finite set, then for $n \in \mathbb{N}$, $T_1, ..., T_n \in S_{NA}(T)$ is such that $S^o_{NA}(T) = \{T_1, T_2, ..., T_n\}$. Let $\mathcal{O}_{\alpha\in\Lambda}$ be a collection of open sets in $S^o_{NA}(T)$, then $\{T_1, T_2, ..., T_n\} = S^o_{NA}(T) \subseteq \cup_{O_i\in\mathcal{O}_{\alpha\in\Lambda}} O_i$. This implies that $\mathcal{O}_{\alpha\in\Lambda}$ is an open cover for $S^o_{NA}(T)$. It suffices to show that $S^o_{NA}(T)$ is compact. Thus invoking Definition 2.4, for each $O_i \in \mathcal{O}_{\alpha\in\Lambda}$ for some $\{T_i \in O_i : i = 1, ..., n\}$ and $\Gamma \subseteq \Lambda$, we have a finite sub-collection $\mathcal{O}_{\beta\in\Gamma}$ of $\mathcal{O}_{\alpha\in\Lambda}$ such that $S^o_{NA}(T) = \{T_1, T_2, ..., T_m\} \subseteq O_1 \cup O_2 \cup, ..., \cup O_m = \{\cup^m_{j=1} O_j : O_j \in \mathcal{O}_{\beta\in\Gamma}\}$ which also covers $S^o_{NA}(T)$. This shows that $S^o_{NA}(T)$ is a compact subset of $(S_{NA}(T), \tau_f)$. □

Proposition 3.1 shows that every finite subset of any arbitrary invariant topological space is compact.

Lemma 3.2. *Let $(S_{NA}(T), \tau_f)$ be an invariant topological space and $S^o_{NA}(T)$ be a subset of $S_{NA}(T)$, then $S^o_{NA}(T)$ is compact if and only if each of its open cover contains a finite subcover.*

Proof. Suppose that $S^o_{NA}(T)$ is compact, then invoking Definition 2.4, there exist a cover $\mathcal{O}_{\alpha\in\Lambda}$ of $S^o_{NA}(T)$ by open subsets of $S_{NA}(T)$. For each $T \in S^o_{NA}(T)$ there exist a neighborhood N_T of T and by Definition 2.2, we have that for each T, there is an open set O such that $T \subseteq O \subseteq N_T$. This shows that the collection $\mathcal{O}_{\alpha\in\Lambda} = \cup^n_{\alpha=1} O_\alpha$ is an open cover for $S^o_{NA}(T)$. Since $S^o_{NA}(T)$ is compact, then by Proposition 3.1 there exist a finite subcover of $\mathcal{O}_{\alpha\in\Lambda}$. Let $\{\mathcal{O}_{\beta\in\Gamma} : \Gamma \subseteq \Lambda\}$ be such a subcover, where $\{\mathcal{O}_\beta \subseteq N_{T_j} : j = 1, ..., m\}$ and hence $S^o_{NA}(T) \subseteq \bigcup^m_{j=1} O_{T_j} \subseteq \bigcup^m_{j=1} N_{T_j}$. This shows that the collection of $\{N_{T_j} : j = 1, 2, ..., m\}$ which is finite cover $S^o_{NA}(T)$ and hence compact.
Conversely, let $\mathcal{O}_{\alpha\in\Lambda}$ be an open cover of $S^o_{NA}(T)$, this means that each $T \in S^o_{NA}(T)$ belong to some member of $\mathcal{O}_{\alpha\in\Lambda}$. This implies that $\mathcal{O}_{\alpha\in\Lambda}$ is itself a collection of neighborhood which covers $S^o_{NA}(T)$. Now, suppose there exist a finite subcover $\{\mathcal{O}_{\beta\in\Gamma} : \Gamma \subseteq \Lambda\}$ of $\mathcal{O}_{\alpha\in\Lambda}$ such that $S^o_{NA}(T) \subseteq \cup^m_{\beta=1} \mathcal{O}_{\beta\in\Gamma}$. Then invoking Definition 2.4 completes the proof. □

Example 3.3. *For an arbitrary set, let $S_{NA}(T) = \{T, T_1, T_2\}$, invoking Definition 2.6 we have $\tau_f = \{\{T\}, \{T_1\}, \{T_2\}, \{T, T_1\}, \{T, T_2\}, \{T_1, T_2\}, \emptyset, S_{NA}(T)\}$ and $\mathcal{O}_\alpha = \{\{T\}, \{T, T_1\}, \{T, T_2\}, \{T_2, T_3\}\}$, then $S_{NA}(T)$ is compact. Indeed, $\mathcal{O}_{\alpha\in\Lambda}$ is a cover since $\{T\} \cup \{T, T_1\} \cup \{T, T_2\} \cup \{T_2, T_3\} = \{T, T_1, T_2\} = S_{NA}(T)$. Moreover, $\mathcal{O}_{\alpha\in\Lambda}$ is an open cover since all elements of $\mathcal{O}_{\alpha\in\Lambda}$ are also in τ_f. In addition, if we let $\mathcal{O}_{\beta\in\Gamma} = \{\{T\}, \{T_2, T_3\}\}$ such that $\{\{T\} \cup \{T_2 \cup T_3\}\} = \{T, T_1, T_2\} = S_{NA}(T)$. Hence, $\mathcal{O}_{\beta\in\Gamma}$ is a subcover of $S_{NA}(T)$. Therefore this implies that $S_{NA}(T)$ is compact*

Example 3.4. *Let $S'_{NA}(T) = \mathbb{R}^1$ and $S_{NA}(T) = \mathbb{R}^2$, then the interval (a, b) is open in $S'_{NA}(T)$ but not in $S_{NA}(T)$ since none of the points in $S'_{NA}(T)$ is an interior point of $S_{NA}(T)$. This therefore implies that if $S^o_{NA}(T) \subseteq S'_{NA}(T) \subseteq S''_{NA}(T)$ and $S^o_{NA}(T)$ open relative to $S'_{NA}(T)$ and $S'_{NA}(T)$ open relative to $S''_{NA}(T)$ does not necessary imply that $S^o_{NA}(T)$ is open relative to $S''_{NA}(T)$.*

Example 3.4 implies that; the openness and closedness of a set depends on which space the set is embedded. But for compactness, this is not the case as shown by the following theorem.

Theorem 3.5. *Let $(S_{NA}(T), \tau_f)$ be an invariant compact topological space and $S^o_{NA}(T) \subseteq S'_{NA}(T) \subseteq S_{NA}(T)$, then $S^o_{NA}(T)$ is compact relative to $S_{NA}(T)$ if and only if $S^o_{NA}(T)$ is compact relative to $S'_{NA}(T)$.*

Proof. Suppose that $S^o_{NA}(T)$ is compact relative to $S_{NA}(T)$. This implies that if we take a collection of open subsets with respect to $S_{NA}(T)$ whose countable union also covers $S^o_{NA}(T)$, then there will be a finite subcover of these sets which also covers $S^o_{NA}(T)$. It suffices to show that $S^o_{NA}(T)$ is compact relative to $S_{NA}(T)$. Let us consider the open covers $\mathcal{O}_{\alpha\in\Lambda}$ of $S^o_{NA}(T)$ relative to $S'_{NA}(T)$. Since $S'_{NA}(T)$ has a subspace topology relative to $S_{NA}(T)$, then from [4], (if $A \subseteq X$, then $H \subseteq A$ is open relative to A if and only if $H = A \cap G$ for some open subset G of X). Using this result, let $\mathcal{V}_\alpha \subseteq S_{NA}(T)$ be a collection of open sets relative to $S_{NA}(T)$ such that $\mathcal{O}_{\alpha\in\Lambda} = S'_{NA}(T) \cap \mathcal{V}_{\alpha\in\Lambda}$. Then $\mathcal{V}_{\alpha\in\Lambda}$ forms an open cover for $S^o_{NA}(T)$ in $S_{NA}(T)$. Moreover, since $S^o_{NA}(T)$ is compact relative to $S_{NA}(T)$, so there exist a finite subcover $\mathcal{V}_{\beta\in\Gamma}$ of $\mathcal{V}_{\alpha\in\Lambda}$ such that $S^o_{NA}(T) \subseteq \cup^m_{\beta=1}\mathcal{V}_{\beta\in\Gamma}$. But, $S^o_{NA}(T) \subseteq S'_{NA}(T)$. This implies that $S^o_{NA}(T) = S^o_{NA}(T) \cap S'_{NA}(T) \subseteq (\cup^m_{\beta=1}\mathcal{V}_{\beta\in\Gamma}) \cap S'_{NA}(T) = \cup^m_{\beta=1}(\mathcal{V}_{\beta\in\Gamma} \cap S'_{NA}(T)) = \cup^m_{\beta=1}\mathcal{O}_{\beta\in\Gamma}$, where $\mathcal{O}_{\beta\in\Gamma}$ is a finite subcover of $\mathcal{O}_{\alpha\in\Lambda}$. This shows that $S^o_{NA}(T)$ is compact relative to $S'_{NA}(T)$.

Conversely, suppose that $S^o_{NA}(T)$ is compact relative to $S'_{NA}(T)$. Let $\mathcal{O}_{\alpha\in\Lambda}$ be open subsets of $S_{NA}(T)$ which covers $S^o_{NA}(T)$. Let $\mathcal{O}_{\alpha\in\Lambda} = S_{NA}(T) \cap \mathcal{V}_{\alpha\in\Lambda}$ where $\mathcal{V}_{\alpha\in\Lambda}$ is open in $S'_{NA}(T)$. Then $\mathcal{V}_{\alpha\in\Lambda}$ forms an open cover for $S^o_{NA}(T)$ in $S'_{NA}(T)$. Since $S^o_{NA}(T)$ is compact relative to $S'_{NA}(T)$, then this open cover $\mathcal{V}_{\alpha\in\Lambda}$ has a finite subcover $\mathcal{V}_{\beta\in\Gamma}$ of $S^o_{NA}(T)$ in $S'_{NA}(T)$ such that $S^o_{NA}(T) \subseteq \cup^m_{\beta=1}\mathcal{V}_{\beta\in\Gamma}$. Since $S^o_{NA}(T) \subseteq S_{NA}(T)$, this implies that $S^o_{NA}(T) = S^o_{NA}(T) \cap S_{NA}(T) \subseteq (\cup^m_{\beta=1}\mathcal{V}_{\beta\in\Gamma}) \cap S_{NA}(T) = \cup^m_{\beta=1}(\mathcal{V}_{\beta\in\Gamma} \cap S_{NA}(T)) = \cup^m_{\beta=1}\mathcal{O}_{\beta\in\Gamma}$, which is a finite subcover that covers $S^o_{NA}(T)$. This implies that $S^o_{NA}(T)$ is compact relative to $S_{NA}(T)$. □

The property of a space being compact and closed subset of an invariant topological space are related but are not equivalent in an arbitrary invariant topological space as shown in the sequel.

Theorem 3.6. *Let $(S_{NA}(T), \tau_f)$ be an invariant topological space and $S^o_{NA}(T) \subseteq S'_{NA}(T)$ such that $S'_{NA}(T)$ is a compact subset of $S_{NA}(T)$. In particular if $S^o_{NA}(T)$ is closed then it is also compact.*

Proof. Let $S^o_{NA}(T) \subseteq S'_{NA}(T) \subseteq S_{NA}(T)$ and let $S^o_{NA}(T)$ be a closed subset of a compact space $S'_{NA}(T)$. We want to prove that $S^o_{NA}(T)$ is compact. From Proposition 3.1, we can take $\mathcal{O}_{\alpha\in\Lambda}$ to be an open cover for $S^o_{NA}(T)$. Since $S'_{NA}(T)$ is compact we want to get an open cover of $S'_{NA}(T)$. Let $\mathcal{Q} = (S^o_{NA}(T))^c \cup \mathcal{O}_{\alpha\in\Lambda}$ be an open cover for $S'_{NA}(T)$, moreover since $S'_{NA}(T)$ is compact, then there exist some finite subcover

$$\{\mathcal{O}_{\beta\in\Gamma}\}^m_{\beta=1} \cup [S^o_{NA}(T)]^c \tag{1}$$

of $\mathcal{Q}$ which covers $S'_{NA}(T)$ and also covers $S^o_{NA}(T)$. But Expression 1 shows that we have extended to a bigger cover which has $[S^o_{NA}(T)]^c$ which was not in the original cover $\mathcal{O}_{\alpha\in\Lambda}$ and this implies that $\{\mathcal{O}_{\beta\in\Gamma}\}^m_{\beta=1} \cup [S^o_{NA}(T)]^c$ it is not a subcover of $\mathcal{O}_{\alpha\in\Lambda}$. Hence we have to remove $[S^o_{NA}(T)]^c$ to get a finite subcover of $\mathcal{O}_{\alpha\in\Lambda}$ which still covers $S^o_{NA}(T)$ and hence $S^o_{NA}(T)$ is compact. Therefore closed subset of compact set is also compact. □

Corollary 3.7. *Let $(S_{NA}(T), \tau_f)$ be an invariant topological space and $S^o_{NA}(T), S'_{NA}(T)$ be subsets of $S_{NA}(T)$. If $S^o_{NA}(T)$ is closed and $S'_{NA}(T)$ compact, then $S^o_{NA}(T) \cap S'_{NA}(T)$ is compact.*

Proof. Let $S^o_{NA}(T) \subseteq S'_{NA}(T) \subseteq S_{NA}(T)$ and since $S'_{NA}(T)$ is compact, then by Theorem 3.6 it is closed. Consequently, we can deduce that $S^o_{NA}(T)$ and $S'_{NA}(T)$ are closed relative to $S_{NA}(T)$ and hence from intersection property of closed sets it implies that $S^o_{NA}(T) \cap S'_{NA}(T)$ is also closed. Moreover, since

$$S^o_{NA}(T) \cap S'_{NA}(T) \subseteq S'_{NA}(T) \subseteq S_{NA}(T), \tag{2}$$

Expression 2 implies that $S'_{NA}(T) \cap S^o_{NA}(T)$ is a closed subset of a compact space $S'_{NA}(T)$ and invoking Theorem 3.6 this implies that $S'_{NA}(T) \cap S^o_{NA}(T)$ is compact. □

From Theorem 3.6, we can deduce that closedness implies compactness but compactness does not necessarily imply closedness. Therefore, for any arbitrary invariant topological space, the converse of Theorem 3.6 is not generally true as shown in the following counterexample.

Example 3.8. *A compact subset of an arbitrary an invariant topological space $(S_{NA}(T), \tau_f)$ is not closed since there exist topological spaces whose finite subsets are not all closed. Indeed, let $S_{NA}(T) = \mathbb{R}$ and $S^o_{NA}(T) \subseteq S_{NA}(T) = [0,2]$. The collection*

$$\{(n-1, n+1 : n = -\infty, ..., \infty)\} \tag{3}$$

which can be $\{(-1,1),(1,3)\}$ is an open cover of $[0,2]$ which is a collection of open sets and

$$[0,2] \subseteq (-1,1) \cup (1,3) = (-1,3). \tag{4}$$

Using Expression 3, it shows that Expression 4 is open in $\mathbb{R}$ but not in $[0,2]$. Moreover, consider a set $S_{NA}(T) = \{T, T_1\}$, its subset $S^o_{NA}(T) = \{T\}$ and Sierpinski topology $\{\emptyset, \{T\}, \{T, T_1\}\}$. The set $S^o_{NA}(T)$ is compact since it is finite but not closed since its complement $\{T_1\}$ is not open in the said topology.

Theorem 3.9. *Let $(S_{NA}(T), \tau_f)$ be an invariant topological space and $S'_{NA}(T)$ and $S''_{NA}(T)$ be compact subsets of $S_{NA}(T)$. Then the following are compact*

(1). $S'_{NA}(T) \cup S''_{NA}(T)$.

(2). $S'_{NA}(T) \cap S''_{NA}(T)$.

Proof.

(1). Let $S'_{NA}(T)$ and $S''_{NA}(T)$ be compact sets. Suppose $\mathcal{O}_{\alpha\in\Lambda}$ is an open covering of $S'_{NA}(T) \cup S''_{NA}(T)$, then by property of union, it is a covering of $S'_{NA}(T)$ and $S''_{NA}(T)$. Since these sets are compact, then from Proposition 3.1 we can choose a finite subcovers $\{\mathcal{O}'_{\beta\in\Gamma} : \beta = 1, ..., l\}$ of $\mathcal{O}_{\alpha\in\Lambda}$ to cover $S'_{NA}(T)$ and $\{\mathcal{O}''_{\beta\in\Gamma} : \beta = l+1, ..., m\}$ of $\mathcal{O}_{\alpha\in\Lambda}$ to cover $S''_{NA}(T)$. Hence, the finite subcover $\{\mathcal{O}_{\beta\in\Gamma} : \beta = 1, ..., m\}$ covers $S'_{NA}(T) \cup S''_{NA}(T)$. This therefore implies that $S'_{NA}(T) \cup S''_{NA}(T)$ is compact.

(2). Since $S'_{NA}(T)$ and $S''_{NA}(T)$ are compact by (1), using Heine-Borel Theorem [4], we can deduce that $S'_{NA}(T)$ and $S''_{NA}(T)$ are closed sets. Let $\{\mathcal{C}_i : i = 1, ..., n\}$ be a finite collection of closed sets in $S'_{NA}(T)$, then $\cup_{i=1}^n \mathcal{C}_i$ is closed. Using De Morgan's law and the fact that for any collection of open sets its union is open [4], we have $[\cap_{\alpha\in\Lambda}\mathcal{C}_\alpha]^c = \cup_{\alpha\in\Lambda}\mathcal{C}^c_\alpha$ is open and this implies that

$$\cap_{\alpha\in\Lambda}\mathcal{C}_\alpha \tag{5}$$

is closed. Since $S'_{NA}(T) \cap S''_{NA}(T) \subseteq S'_{NA}(T)$ and from Expression 5, we can deduce that $S'_{NA}(T) \cap S''_{NA}(T)$ is a closed subset of the set $S'_{NA}(T)$. Thus, invoking Theorem 3.6, it follows that $S'_{NA}(T) \cap S''_{NA}(T)$ is compact as a closed subset of a compact space. □

Theorem 3.10. *Let $S^o_{NA}(T)$ be a subset of a T_2-space $S_{NA}(T)$. In particular if $S^o_{NA}(T)$ is compact, then it is closed.*

Proof. Let $S^o_{NA}(T)$ be a compact subset of a T_2-space $S_{NA}(T)$ and $T \in S^o_{NA}(T)$. It suffices to show that if $S^o_{NA}(T)$ is closed then its complement is open. Let $T_o \in (S^o_{NA}(T))^c = S_{NA}(T) \backslash S^o_{NA}(T)$, this implies that $T \neq T_o$. Since $S_{NA}(T)$ is a T_2-space, then we can separate T_o and $S^o_{NA}(T)$ by neighborhoods. Hence, by T_2 separation axiom there exist open sets $(S'_{NA}(T))_{T_o}$, $(S''_{NA}(T))_T$ such that $T_o \in (S'_{NA}(T))_{T_o} \subseteq (S^o_{NA}(T))^c$, $T \in (S''_{NA}(T))_T$ and $(S'_{NA}(T))_{T_o} \cap (S''_{NA}(T))_T = \emptyset$. This implies that $S^o_{NA}(T) \subseteq \cup\{(S''_{NA}(T))_T : T \in S_{NA}(T)\}$ is an open cover of $S^o_{NA}(T)$. But by the compactness of $S^o_{NA}(T)$, there exist a finite subcover $\{(S''_{NA}(T))_{T_1}, (S''_{NA}(T))_{T_2}, ..., (S''_{NA}(T))_{T_n}\}$ of $S^o_{NA}(T)$ such that $S^o_{NA}(T) \subseteq \{(S''_{NA}(T))_{T_1} \cup ... \cup (S''_{NA}(T))_{T_n}\}$. Thus, $S''_{NA}(T) = \cup_{i=1}^n (S''_{NA}(T))_{T_i}$ is the required neighborhood of $S^o_{NA}(T)$. Similarly it follows that $S'_{NA}(T) = \cap_{i=1}^n (S'_{NA}(T))_{T_{o_i}}$ forms the neighborhood of T_o which does not intersect $S^o_{NA}(T)$ since it does not intersect any of the elements of $\{(S''_{NA}(T))_{T_i} : i = 1, 2, .., n\}$. Then it follows that $S'_{NA}(T) \subseteq (S^o_{NA}(T))^c$ and $(S^o_{NA}(T))^c = \cup S'_{NA}(T)$, which implies that $(S^o_{NA}(T))^c$ is a neighborhood of each of its points and hence open. Therefore $S^o_{NA}(T) = ((S^o_{NA}(T))^c)^c$ is closed which completes the proof. □

Corollary 3.11. *Let $S^o_{NA}(T)$ be a subset of a T_2-space $S_{NA}(T)$, then $S^o_{NA}(T)$ is compact if and only if it is closed.*

Proof. The proof follows from Theorem 3.6 and Theorem 3.10. □

From Corollary 3.11, we can conclude that compact sets in T_2-space are always closed.

An important property of compactness as a topological concept is that it is invariant under continuous mapping as shown in the following theorem.

Lemma 3.12. *Let $(S'_{NA}(T), \tau_{f_1})$ and $(S''_{NA}(T), \tau_{f_2})$ be invariant topological spaces and $f : S'_{NA}(T) \to S''_{NA}(T)$ be a continuous map. If $S^o_{NA}(T) \subseteq S'_{NA}(T)$ is compact, then $f(S^o_{NA}(T))$ is also a compact subset of $S''_{NA}(T)$.*

Proof. Let $S^o_{NA}(T)$ be compact and $\mathcal{O}_{\alpha\in\Lambda}$ be an open cover of $f(S^o_{NA}(T))$. We need to get the inverse image of these open sets. Let $f^{-1}(\mathcal{O}_{\alpha\in\Lambda})$ be the open inverse of each $\mathcal{O}_{\alpha\in\Lambda}$ which is an open cover for $S^o_{NA}(T)$ since f is continuous this implies that the inverse image of an open set is open. By the compactness of $S^o_{NA}(T)$, any arbitrary open cover $\mathcal{O}_{\alpha\in\Lambda}$ of $f(S^o_{NA}(T))$ has a finite subcover. Hence, there exist say, for all $\beta \in \Gamma$;

$$f^{-1}(\mathcal{O}_{\beta_1}) \cup ... \cup f^{-1}(\mathcal{O}_{\beta_m}). \tag{6}$$

Applying f to both Expression 6 and on $S^o_{NA}(T)$ we get $\mathcal{O}_{\beta_1} \cup ... \cup \mathcal{O}_{\beta_m}$ which is a finite subcover for $f(S^o_{NA}(T))$. Therefore, this implies that $f(S^o_{NA}(T))$ is compact. □

Theorem 3.13. *Let $(S'_{NA}(T), \tau_{f_1})$ and $(S''_{NA}(T), \tau_{f_2})$ be invariant topological spaces such that $f : S'_{NA}(T) \to S''_{NA}(T)$ is a continuous surjective map. If $S'_{NA}(T)$ is compact, then $S''_{NA}(T)$ is also compact.*

Proof. Let $S'_{NA}(T)$ be compact and since f is surjective; that is $f(S'_{NA}(T))$, we want to show that $S''_{NA}(T)$ is compact. Invoking Lemma 3.12, let $\mathcal{O}_{\alpha\in\Lambda}$ be an open cover of $S''_{NA}(T)$. It suffices to proof that this open cover has a finite subcover. By the continuity of f, $\mathcal{O}_{\alpha\in\Lambda}$ has an inverse image in $S'_{NA}(T)$ which must also be open. Let the inverse image be denoted by $\mathcal{V}_{\alpha\in\Lambda}$. Hence, $\mathcal{V}_\alpha : \mathcal{V}_\alpha = f^{-1}(\mathcal{O}_{\alpha\in\Lambda}) = S'_{NA}(T)$. By the compactness of $S'_{NA}(T)$, there exist a finite subcover $\mathcal{V}_{\beta\in\Gamma}$ of $S'_{NA}(T)$ such that

$$S'_{NA}(T) = \cup_{\beta=1}^{m}\mathcal{V}_{\beta\in\Gamma}. \tag{7}$$

Using the property of the image of a function and the union of them in [4], that is;

$$g(A \cup B) = g(A) \cup g(B). \tag{8}$$

Since $S'_{NA}(T) = \cup_{\beta=1}^{m}\mathcal{V}_{\beta\in\Gamma} = \mathcal{V}_{\beta_1} \cup \mathcal{V}_{\beta_2}\cup, ... \cup \mathcal{V}_{\beta_m}$, applying Equality 8 on Equality 7 we have $f(\mathcal{V}_{\beta_1} \cup \mathcal{V}_{\beta_2}\cup, ... \cup \mathcal{V}_{\beta_m}) = f(\mathcal{V}_{\beta_1}) \cup f(\mathcal{V}_{\beta_2})\cup, ... \cup f(\mathcal{V}_{\beta_n})$. Since $S'_{NA}(T) = \cup_{\beta=1}^{m}\mathcal{V}_{\beta\in\Gamma}$, then we can write $f(S'_{NA}(T)) = \cup_{\beta=1}^{m} f(\mathcal{V}_{\beta\in\Gamma})$. Moreover, since $\mathcal{V}_{\beta\in\Gamma}$ is the inverse of $\mathcal{O}_{\alpha\in\Lambda}$, then we have

$$\begin{aligned} \cup_{\beta=1}^{m} f(\mathcal{V}_{\beta\in\Gamma}) &= \cup_{i=1}^{m} f(f^{-1}(\mathcal{O}_{\beta\in\Gamma}) \\ &= \cup_{\beta=1}^{m}\mathcal{O}_{\beta\in\Gamma} \end{aligned}$$

But $\cup_{\beta=1}^{m}\mathcal{O}_{\beta\in\Gamma} \subseteq f(S'_{NA}(T)) = S''_{NA}(T)$. This implies that $S''_{NA}(T) = f(S'_{NA}(T)) \subseteq \cup_{\beta=1}^{m}\mathcal{O}_{\beta\in\Gamma} \subseteq f(S'_{NA}(T)) = S''_{NA}(T)$. Hence, $S''_{NA}(T) \subseteq \cup_{\beta=1}^{m}\mathcal{O}_{\beta\in\Gamma}$ which completes the proof. □

Lemma 3.14. *Let $(S'_{NA}(T), \tau_{f_1})$ and $(S''_{NA}(T), \tau_{f_2})$ be homeomorphic and invariant topological spaces such that $f : (S'_{NA}(T), \tau_{f_1}) \to (S''_{NA}(T), \tau_{f_2})$, then $S''_{NA}(T)$ is a T_2-space if and only if $S'_{NA}(T)$ is T_2-space.*

Proof. This Lemma show that the property of a space being T_2-space is preserved under homeomorphism. Let $S'_{NA}(T)$ and $S''_{NA}(T)$ be homeomorphic and $S'_{NA}(T)$ be a T_2-space. It suffices to show that $S''_{NA}(T)$ must be a T_2-space under homeomorphism. Let T_1, $T_2 \in S'_{NA}(T)$ be such that $T_1 \neq T_2$ and map these points to $S''_{NA}(T)$ using homeomorphism f, so that $f(f_1(x)) = A_1$ and $f(f_2(x)) = A_2$ for any $x \in H$. Since f is one−to−one, this implies that $A_1 \neq A_2$. Next, lets take two arbitrary open sets around T_1 and T_2, O_1 and O_2 respectively such that $O_1 \cap O_2 = \emptyset$ since $T_1 \neq T_2$ and again map these open sets to $S''_{NA}(T)$ using f, that is $f(O_1)$ and $f(O_2)$. These sets are also open in $S''_{NA}(T)$, $A_1 \in O_1$, $A_2 \in O_2$ and $f(O_1) \neq A_2$ by homeomorphism. Moreover, f is an onto mapping and $f(O_1) \cap f(O_2) = \emptyset$ which implies that, A_1, $A_2 \in S''_{NA}(T)$ are such that $A_1 \neq A_2$ and hence $S''_{NA}(T)$ is a T_2-space. □

Theorem 3.15. *If $(S'_{NA}(T), \tau_{f_1})$ and $(S''_{NA}(T), \tau_{f_2})$ are homeomorphic and invariant topological spaces, then $S'_{NA}(T)$ is compact if and only if $S''_{NA}(T)$ is compact.*

Proof. Suppose that $S'_{NA}(T)$ is compact and from Lemma 3.14, let $(S'_{NA}(T), \tau_{f_1})$ and $(S''_{NA}(T), \tau_{f_2})$ be homeomorphic topological spaces, then there exist an operator T which induces a one-to-one and onto correspondence between the open sets of $S'_{NA}(T)$ and the open sets of $S''_{NA}(T)$, such that for all open set $O_1 \in \tau_{f_1}$, $O_2 \in \tau_{f_2}$, $f(O_1) = f(O_2), \Rightarrow O_1 = O_2$ and $f(O_1) = O_2$. Since T is bijective, it implies that

$$(f^{-1})^{-1}(O_1)^c = f(O_1)^c. \tag{9}$$

Where $(O_1)^c$ is a closed subset of the compact space $S'_{NA}(T)$. From Theorem 3.10, this implies that $(O_1)^c$ is a compact set. Hence $f(O_1)^c$ is a compact subset of the $S''_{NA}(T)$ that is, $f(O_1)^c$ is a closed set in $S''_{NA}(T)$. Next we need to show that O_1 is an open set in $(S'_{NA}(T), \tau)$. From Equation 9, $(f^{-1})^{-1}(O_1)^c$ is a closed set which implies that $f(O_1) = (f^{-1})^{-1}(O_1) = S''_{NA}(T) \backslash (f^{-1})^{-1}(O_1)^c$ is an open set. Hence $f^{-1} : (S''_{NA}(T), \tau_{f_2}) \to (S'_{NA}(T), \tau_{f_1})$ is continuous.

The converse follows from Theorem 3.13, that $S'_{NA}(T)$ is compact if and only if $S''_{NA}(T)$ is compact. □

Corollary 3.16. *Suppose that $(S'_{NA}(T), \tau_{f_1})$ is an invariant compact topological space and $(S''_{NA}(T), \tau_{f_1})$ be a T_2-space such that $f : S'_{NA}(T) \to S''_{NA}(T)$ is a continuous map, if $S'_{NA}(T)$ is compact then $f(S'_{NA}(T))$ is closed in $S''_{NA}(T)$.*

Proof. The proof follows from Theorem 3.13 that guarantees that $f(S'_{NA}(T))$ is compact. Similarly by Theorem 3.6, $f(S'_{NA}(T))$ is closed. □

Theorem 3.13 is very important in the sense that continuous real-valued functions are compact sets and attain their maximum and minimum values as demonstrated by the following theorem.

Theorem 3.17. *Let $f : S'_{NA}(T) \to \mathbb{R}$ be a continuous function where $S'_{NA}(T)$ is an invariant topological space and $S^o_{NA}(T) \subseteq S'_{NA}(T)$ be compact. Then there are $T_1, T_2 \in S^o_{NA}(T)$ such that $f(T_1) \le f(T) \le f(T_2), \forall\, T \in S^o_{NA}(T)$.*

Proof. Since f is continuous then by Theorem 3.13, $f(S^o_{NA}(T))$ is compact in $\mathbb{R}$. Consequently, by Heine-Borel theorem, $f(S^o_{NA}(T))$ is closed and bounded. This implies that we have $m = \inf(f(S^o_{NA}(T)))$ and $M = \sup(f(S^o_{NA}(T)))$ both in $f(S^o_{NA}(T))$. Hence, there exist $T_1 \in S^o_{NA}(T)$ with $f(T_1) = m$ and $T_2 \in S^o_{NA}(T)$ with $f(T_2) = M$ such that for any $T \in S^o_{NA}(T)$ we have $f(T_1) \le f(T) \le f(T_2)$. □

4. Conclusion

In this paper, we introduced the concept of compactness of similarity orbits of norm-attainable operators on an invariant topological spaces via the notion of open covers and characterized their properties. The results obtained are useful in mathematical physics and quantum information theory.

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