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On denseness of similarity orbits of norm-attainable operators

Research Article

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Abstract: The notion of dense sets has been extensively discussed on both metric and topological spaces. Various properties of the sets have been proved under the underlying spaces. However, if we consider these sets to be from similarity orbits where a topology has been developed on them, little has been done to investigate their denseness. In this paper, we introduce the concept of denseness of similarity orbits of norm-attainable operators in aspect of generalized sets in topological spaces and some of their properties.

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Keywords: Similarity orbits • Invariant subspace • dense sets • Topological space

1. Introduction

The notion of similarity orbits of Hilbert space operators was first initiated by Herrero [7] where he described the closure of similarity orbits of a normal with perfect spectrum. Since then, this concept has been investigated extensively by many researchers such as Fialkow [4], Rao[13] and Hadwin et al.[6]. In particular, [6] characterized the norm-closure of the similarity orbits $S(T)$ of bounded linear operator T on a Banach space X and established that if $T \in B(X)$ is not the sum of a scalar and a finite rank operator, then $S(T)$ is strongly dense in $B(X)$ which has useful application in showing that a transitive operator on X is an operator whose only invariant subspace are $\{0\}$ and X. The study of the denseness of norm-attaining operators was started by Lindenstrauss [9] where he showed that Bishop and Phelps theorem [2] is not true for operators. However, Lindenstrauss [9] gave isometric conditions known as property A and B on X and Y for which the set of norm-attainable operators from X to Y are dense in the space of all operators between these Banach spaces. In addition, M$\ddot{u}$ller [5] Characterized dense orbits where he established that given $T \in B(X)$, the set $\{T^n x : n = 0, 1, ...\}$ is dense in X if T is hypercyclic for all $x \in X$. The concept of dense sets has been extensively investigated on both metric and topological spaces. Sompong [5] investigated some fundamental properties of dense sets on bigeneralized topological spaces. Moreover, Al-shami [5] investigated somewhere dense sets as a new kind of generalized open sets and presented necessary and sufficient conditions under which the union/intersection of somewhere dense sets is also somewhere dense. In addition, Bourdon and Feldman [3], established that for a continuous linear operator T on locally convex topological vector space, if $x \in X$ has an orbit under T that is somewhere dense in X, then the orbit of x under T must be everywhere dense.

Since similarity orbit is a sequence of elements which are operators, then it forms a set on which we assign a topology to become a topological space. In this paper, we introduce the concept of dense sets of similarity orbits of norm-attainable operators on invariant topological spaces and study their properties.

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2. Preliminaries

In this section we start by defining some key terms and a result that are useful in the sequel.

Definition 2.1.
([5]) The similarity orbit of an operator $T \in B(H)$ is the set $S(T) = \{XTX^{-1} : X \text{ is invertible}\}$. For a particular case, similarity orbit denoted by, $S_{NA}(T)$ is the similarity orbits of norm-attainable operators.

Definition 2.2.
([5]) An operator $T \in B(H)$ is said to be norm-attainable if there exists a unit vector $x \in H$ such that $\|Tx\| = \|T\|$. The set of all norm-attainable operators on H is denoted by NA.

Definition 2.3.
A subset $S^o_{NA}(T)$ of an invariant topological space $(S_{NA}(T), \tau_{S_{NA}(T)})$ is dense if and only if for any nonempty open subset $\mathcal{O} \subseteq S_{NA}(T)$ we have $\mathcal{O} \cap S^o_{NA}(T) \neq \emptyset$.

Definition 2.4.
([5]) Let X be a topological space and let A be a subset of X. Then closure $\overline{A}$ of A in X is defined to be the intersection of all of the closed subsets of X that contain A.

Definition 2.5.
([5]) A topological space denoted by (X, τ) is a non-empty set X together with a collection of τ subsets X(referred to as open sets) that satisfies the following conditions:

i) the empty set and the whole space X are open sets.

ii) the union of any collection of open sets is itself an open set.

iii) the intersection of any finite collection of open sets is itself an open set.

Definition 2.6.
([5]) A subspace M of H is an invariant subspace of the operator T if for each $x \in M$, $Tx \in M$ i.e $T(M0 \subseteq M$. M is also referred as $T-$ invariant or M is invariant under T.

Definition 2.7.
([5]) Let (X, d_X) be a metric space. A set $D \subseteq X$ is dense in $E \subseteq X$ if $E \subseteq \overline{D}$ and D is dense if $\overline{D} = X$.

3. Main results

In this section, we characterize the denseness of similarity orbits. We derive various results concerning dense sets such as union, intersection and transitivity.

Proposition 3.1.
Let $S^o_{NA}(T)$ be a subset of an invariant topological space $(S_{NA}(T), \tau_{S_{NA}(T)})$, then $S^o_{NA}(T)$ is dense in $S_{NA}(T)$ if and only if the set of all limit points of $S^o_{NA}(T)$ coincides with $S_{NA}(T)$.

Proof. Suppose that for every open set $\{\mathcal{O} \in \tau_{S_{NA}(T)} : \mathcal{O} \neq \emptyset\}$, $S^o_{NA}(T) \cap \mathcal{O} \neq \emptyset$ holds. We consider two cases:

i) $S^o_{NA}(T) = S_{NA}(T)$.

The proof of this is trivial since its closure (i.e. $S^o_{NA}(T)$ contains all its limit points) is equal to $S_{NA}(T)$ and this implies that $S^o_{NA}(T)$ is dense in $S_{NA}(T)$.

ii) $S^o_{NA}(T) \neq S_{NA}(T)$.

Let $T \in (S^o_{NA}(T))^c$. Consider $\{\mathcal{O} \in \tau_{S_{NA}} : T \in \mathcal{O}\}$ with $\mathcal{O} \cap S^o_{NA}(T) \neq \emptyset$. This implies that T is a limit point of $S^o_{NA}(T)$. Since $\mathcal{O}$ intersect $S^o_{NA}(T)$, then $\mathcal{O}$ contains T and contains points of $S^o_{NA}(T)$. Hence,

$$\overline{S^o_{NA}(T)} = S^o_{NA}(T) \cup (S^o_{NA}(T))' = S_{NA}(T). \tag{1}$$

Equation 1 shows that all the points in $S_{NA}(T)$ are the limit points of $S^o_{NA}(T)$ and therefore $S^o_{NA}(T)$ is dense in $S_{NA}(T)$. Conversely, let $S^o_{NA}(T)$ be dense in $S_{NA}(T)$ and $\{\mathcal{O} \in \tau_{S_{NA}(T)} : \mathcal{O} \neq \emptyset\}$. Suppose

$$\mathcal{O} \cap S^o_{NA}(T) = \emptyset. \tag{2}$$

Let $T \in \mathcal{O}$ and $T \notin S^o_{NA}(T)$, this implies that T is not a limit point of $S^o_{NA}(T)$ because $T \in \mathcal{O} \cap S^o_{NA}(T) = \emptyset$ which is a contradiction since $S^o_{NA}(T)$ is dense in $S_{NA}(T)$ that is, $\overline{S^o_{NA}(T)} = S_{NA}(T)$ and this implies that each point not in $S^o_{NA}(T)$ must be a limit point of $S^o_{NA}(T)$. The conditions that $T \notin S^o_{NA}(T)$ and $\overline{S^o_{NA}(T)} = S_{NA}(T)$ contradict each other and this in turn implies that Equation 2 is false and therefore $\mathcal{O} \cap S^o_{NA}(T) \neq \emptyset$ and this implies that $S^o_{NA}(T)$ contains all its limit points as required. □

Lemma 3.1.
Let $(S_{NA}(T), \tau_{S_{NA}(T)})$ be an invariant topological space and $S^o_{NA}(T) \subseteq S_{NA}(T)$. If $S^o_{NA}(T)$ is dense, then $\overline{S^o_{NA}(T)} = S_{NA}(T)$.

Proof. Suppose that $S^o_{NA}(T)$ is dense, then we need to show that

$$\overline{S^o_{NA}(T)} = S_{NA}(T). \tag{3}$$

In order to prove Equation 3, we need to show that

i) $\overline{S^o_{NA}(T)} \subseteq S_{NA}(T)$

ii) $S_{NA}(T) \subseteq \overline{S^o_{NA}(T)}$

i) In general, if $S^o_{NA}(T) \subseteq S_{NA}(T)$, then $\overline{S^o_{NA}(T)} \subseteq S_{NA}(T)$. Hence, $S^o_{NA}(T) \subseteq S_{NA}(T)$ is generally true.
ii) Take every open set $\mathcal{O} \in \tau_{S_{NA}(T)} \setminus \emptyset$, since $S^o_{NA}(T)$ is dense, then from Definition 2.3 we have $S^o_{NA}(T) \cap \mathcal{O} \neq \emptyset$. In addition, since $S_{NA}(T)$ is also an open set, then replacing $\mathcal{O}$ by $S_{NA}(T)$ we have

$$S^o_{NA}(T) \cap S_{NA}(T) \neq \emptyset. \tag{4}$$

From Expression 4 there are two possibilities, i.e. $\forall\, T \in S_{NA}(T)$:

a) $T \in S^o_{NA}(T) \cap S_{NA}(T)$ or

b) $T \notin S^o_{NA}(T) \cap S_{NA}(T)$

a) From Proposition 3.1 there are also two possibilities, i.e. either $T \in S^o_{NA}(T)$ or $T \in \overline{S^o_{NA}(T)}$. By Definition 2.4 of the closure, $S^o_{NA}(T) \subseteq \overline{S^o_{NA}(T)}$, implying that

$$\text{if } T \in S^o_{NA}(T) \text{ then automatically } T \in \overline{S^o_{NA}(T)}. \tag{5}$$

Similarly, in *b*)

$$\text{if } T \notin S^o_{NA}(T), \text{ then } T \in \overline{S^o_{NA}(T)}. \tag{6}$$

Both conditions in Expression 5 and Expression 6 show that eventually $T \in \overline{S^o_{NA}(T)}$. Moreover, since $T \in S_{NA}(T)$ and $T \in \overline{S^o_{NA}(T)}$ holds, this implies that $\overline{S^o_{NA}(T)} \subseteq S_{NA}(T)$. Therefore, combining the two expressions $S^o_{NA}(T) \subseteq S_{NA}(T)$ and $S_{NA}(T) \subseteq \overline{S^o_{NA}(T)}$ gives the desired result. □

Theorem 3.1.
If $S^o_{NA}(T)$ and $S'_{NA}(T)$ are dense subsets of an invariant topological space $(S_{NA}(T), \tau_{S_{NA}(T)})$, then $S^o_{NA}(T) \cap S'_{NA}(T)$ is dense.

Proof. Let $\mathcal{O}$ be any non-nonempty open set in $S_{NA}(T)$. Since the set $S^o_{NA}(T)$ is dense, then invoking Lemma 3.1 we have

$$\mathcal{O} \cap S^o_{NA}(T) \neq \emptyset. \tag{7}$$

Moreover, since $\mathcal{O}$ is an open set and for Equation 7 to hold, then it is sufficient for $S^o_{NA}(T)$ to be open but $S'_{NA}(T)$ can be any dense set. Hence, $\mathcal{O} \cap S^o_{NA}(T) \neq \emptyset$ is an open set as intersection of open sets. Now for the set $S'_{NA}(T)$, given a point $T \in S_{NA}(T)$ and open subset $\mathcal{O}$ such that $T \in \mathcal{O}$, we need to show that $S'_{NA}(T) \cap (S^o_{NA}(T) \cap \mathcal{O}) \neq \emptyset$. Since $S^o_{NA}(T)$ is dense and from Equation 7, let

$$T_1 \in (S'_{NA}(T) \cap (S^o_{NA}(T) \cap \mathcal{O})). \tag{8}$$

Expression 8, implies that $S'_{NA}(T) \cap (S^o_{NA}(T) \cap \mathcal{O})$ is not empty and since $S'_{NA}(T)$ is also dense, then by associativity we have:
$S'_{NA}(T) \cap (S^o_{NA}(T) \cap \mathcal{O}) = (S^o_{NA}(T) \cap S'_{NA}(T)) \cap \mathcal{O} \neq \emptyset$. Hence $S^o_{NA}(T) \cap S'_{NA}(T)$ is indeed dense in $S_{NA}(T)$. □

Theorem 3.1 gives a sufficient condition that for the intersection of two sets to be dense then one of the dense sets is open. Below we construct a counterexample showing that the intersection of two dense need not be dense.

Example 3.1.
Let $S^o_{NA}(T)$ and $S'_{NA}(T)$ be dense subsets of an invariant topological space $(S_{NA}(T), \tau)$, then $S^o_{NA}(T) \cap S'_{NA}(T) = \emptyset$. Indeed, if we let $S_{NA}(T) = \mathbb{R}$, $S'_{NA}(T) = \mathbb{Q}$ and $S^o_{NA}(T) = S_{NA}(T) \setminus S'_{NA}(T)$, then $\mathbb{Q}$ is closed which will imply that $S^o_{NA}(T)$ is open. Let $T_1 \in S^o_{NA}(T)$, then there exist a neighborhood N_{T_1} of T_1 such that given $\epsilon > 0$, $(T_1 - \epsilon, T_1 + \epsilon) \in S^o_{NA}(T)$. But this is not true since every interval of $S^o_{NA}(T)$ contains a rational number. Therefore this implies that $S^o_{NA}(T)$ is not open. Hence $S^o_{NA}(T) \cap S'_{NA}(T) = \emptyset$ is not dense in $S_{NA}(T)$.

For a more general phenomenon, in any an invariant topological space, the intersection of a finite collection of open dense sets is necessarily dense as proved in the following corollary.

Corollary 3.1.
If $(S_{NA}(T), \tau_{S_{NA}(T)})$ is an invariant topological space such that, for every finite subsets $(S^o_{NA}(T))_n$ and $(S'_{NA}(T))_n$ of $S_{NA}(T)$ are open and dense, then their intersection is also open and dense.

Proof. For $n = 1$, we have two open dense sets $S^o_{NA}(T)$ and $S'_{NA}(T)$ where the proof follows from Theorem 3.1 and by the fact that the intersection of two open set is always open. For the general case of an arbitrary non-empty set $S_{NA}(T)$, let $n > 1$, and $(S^o_{NA}(T))_n$ be open dense sets. Since the intersection $\cap_{n=1}^{k}(S^o_{NA}(T))_n$ is open dense, then by induction we need to show that it is also true for $(S^o_{NA}(T))_{k+1}$. Hence $\cap_{n=1}^{k}(S^o_{NA}(T))_n \cap (S^o_{NA}(T))_{k+1} = \cap_{n=1}^{k+1}(S^o_{NA}(T))_n$ which is open and dense. □

Remark 3.1.
For infinite intersection of dense open sets, Corollary 3.1 is not necessarily true. Particulary, if we consider $(S_{NA}(T), \tau_{S_{NA}(T)})$ to be a cofinite topology for $S_{NA}(T)$ being countably infinite, then for every $T \in S_{NA}(T)$, the set $S_{NA}(T) \setminus \{T\}$ is open and dense. The countable intersection $\cap_{n\in\mathbb{N}}(S_{NA}(T) \setminus \{T\}) = \emptyset$ and hence not dense.

Corollary 3.2.
Let $S_{NA}(T)$ be a Banach space, then the intersection of any collection of dense open subset of $S_{NA}(T)$ is dense in $S_{NA}(T)$.

Proof. Let $(S^o_{NA}(T))_n$ for all $n \in \mathbb{N}$ be an arbitrary sequence of dense open subsets of a Banach space $S_{NA}(T)$ and let $S^o_{NA}(T) = \cap_{n\in\mathbb{N}}(S^o_{NA}(T))_n$. We need to show that $(S^o_{NA}(T))_n$ is dense in $S_{NA}(T)$. Let $\mathcal{O}$ be any nonempty open subset of $S_{NA}(T)$. Since $(S^o_{NA}(T))_1$ is dense in $S_{NA}(T)$, then by Theorem 3.1 the open subset $\mathcal{O} \cap (S^o_{NA}(T))_1 \neq \emptyset$. Let $T_1 \in \mathcal{O} \cap (S^o_{NA}(T))_1$. Then there exist a positive number $r_1 < 1$ such that

$$\overline{B_{r_1}(T_1)} \in S_{NA}(T) \subseteq \mathcal{O} \cap (S^o_{NA}(T))_1. \tag{9}$$

Similarly, Since $(S^o_{NA}(T))_2$ is dense in $S_{NA}(T)$ the open set $B_{r_1}(T_1) \cap (S^o_{NA}(T))_2 \neq \emptyset$. Let $T_2 \in B_{r_1}(T_1) \cap (S^o_{NA}(T))_2$, taking a positive partitioning of r_1 then there exist a positive real number $r_2 < \frac{1}{2}$ such that

$$\overline{B_{r_2}(T_2)} \in B_{r_1}(T_1) \cap (S^o_{NA}(T))_2. \tag{10}$$

From Expression 9 and Equation 10 we deduce that $\overline{B_{r_2}(T_2)} \subseteq \overline{B_{r_1}(T_1)}$. Inductively, we obtain for every $n \in \mathbb{N}$ a point T_n and a positive real number $r_n < \frac{1}{n}$ such that $\overline{B_{r_n}(T_n)} \in S_{NA}(T) \subseteq B_{r_{n-1}}(T_{n-1}) \cap (S^o_{NA}(T))_n$. Since $\overline{B_{r_n}(T_n)} \subseteq \overline{B_{r_{n-1}}(T_{n-1})}$, we obtain a decreasing sequence $\overline{B_{r_n}(T_n)}$ of nonempty closed sets such that $\text{diam}(\overline{B_{r_n}(T_n)}) \to 0$. This shows that the sequence converges and since a Banach space is complete, then the sequence converges to a point. Let $T \in S_{NA}(T)$ such that $\cap_{n\in\mathbb{N}}\overline{B_{r_n}(T_n)} = T$, then

$$T \in \overline{B_{r_n}(T_n)} \subseteq B_{r_{n-1}}(T_{n-1}) \cap_{n\in\mathbb{N}} (S^o_{NA}(T))_{n-1} \subseteq \mathcal{O} \cap_{n\in\mathbb{N}} (S^o_{NA}(T))_n.$$

Hence $T \in \mathcal{O} \cap (S^o_{NA}(T))_n$. This implies that $T \in \mathcal{O}$ and $T \in \cap_{n\in\mathbb{N}}(S^o_{NA}(T))_n$, and therefore $\cap_{n\in\mathbb{N}}(S^o_{NA}(T))_n$ is dense in $S_{NA}(T)$. □

For a countably collection of dense open sets in $S_{NA}(T)$, the Banach space in Corollary 3.2 becomes a Baire space. Any set containing a dense set is also dense as shown in the following theorem.

Theorem 3.2.
Let $(S_{NA}(T),\tau)$ be an invariant topological space, $S^o_{NA}(T), S^*_{NA}(T)$ and $S'_{NA}(T)$ be subsets of $S_{NA}(T)$, then the following properties hold:

i) A set $S^*_{NA}(T)$ containing a dense set $S^o_{NA}(T)$ is dense.

ii) If $S^o_{NA}(T)$ is a dense set and $S'_{NA}(T)$ is dense in $S^o_{NA}(T)$, then $S'_{NA}(T)$ is also a dense set.

Proof. *i*) Let $S^o_{NA}(T) \subseteq S^*_{NA}(T)$, then from [[8], Theorem 3.24] it follows that $\overline{S^o_{NA}(T)} \subseteq \overline{S^*_{NA}(T)}$. Since $S^o_{NA}(T)$ is dense, we have $\overline{S^o_{NA}(T)} = S_{NA}(T)$, then we can have

$$S_{NA}(T) \subseteq \overline{S^*_{NA}(T)}. \tag{11}$$

Similarly since $S^*_{NA}(T) \subseteq S_{NA}(T)$ and by Lemma 3.1 we have

$$\overline{S^*_{NA}(T)} \subseteq S_{NA}(T). \tag{12}$$

Hence, from Expression 11 and Expression 12 we get $\overline{S^*_{NA}(T)} = S_{NA}(T)$. This therefore shows that $S^*_{NA}(T)$ is dense in $S_{NA}(T)$.
ii) Suppose $S^o_{NA}(T)$ is dense, then $\overline{S^o_{NA}(T)} = S_{NA}(T)$. Since $S'_{NA}(T)$ is also dense in $S^o_{NA}(T)$, invoking Definition 1, this implies that $S^o_{NA}(T) \subseteq \overline{S'_{NA}(T)}$. By closure property from [[8], Theorem 3.24], we have

$$\begin{aligned}\overline{S^o_{NA}(T)} &\subseteq \overline{\overline{S'_{NA}(T)}}\\ &= \overline{S'_{NA}(T)}.\end{aligned}$$

which shows that $\overline{S^o_{NA}(T)} \subseteq \overline{S'_{NA}(T)}$. Thus, $S_{NA}(T) = \overline{S^o_{NA}(T)} \subseteq \overline{S'_{NA}(T)}$. Which implies that

$$S_{NA}(T) \subseteq \overline{S'_{NA}(T)}. \tag{13}$$

But since $S'_{NA}(T) \subseteq S_{NA}(T)$, then we have

$$\overline{S'_{NA}(T)} \subseteq S_{NA}(T). \tag{14}$$

Hence from Expression 13 and 14 we get $\overline{S'_{NA}(T)} = S_{NA}(T)$ which implies that $S'_{NA}(T)$ is dense in $S_{NA}(T)$. □

Corollary 3.3.
Let $(S_{NA}(T), \tau_{S_{NA}(T)})$ be an invariant topological space, $S^o_{NA}(T) \subseteq S'_{NA}(T) \subseteq S_{NA}(T)$ and $S^o_{NA}(T)$ be dense in $S'_{NA}(T)$, then $S^o_{NA}(T)$ is dense in $\overline{S'_{NA}(T)}$.

Proof. Suppose that $S^o_{NA}(T)$ be dense in $\overline{S'_{NA}(T)}$, then we need to show that $\overline{S'_{NA}(T)} \subseteq \overline{S^o_{NA}(T)}$. Let $T \in \overline{S'_{NA}(T)}$ and $\mathcal{O}$ be an open set containing T, $\mathcal{O} \cap S'_{NA}(T) \neq \emptyset$. This implies that $S'_{NA}(T) \subseteq \overline{S'_{NA}(T)}$. Let $T_1 \in \mathcal{O} \cap S'_{NA}(T)$, by intersection properties we have that $T_1 \in S'_{NA}(T)$ and since $S^o_{NA}(T)$ is dense in $S'_{NA}(T)$, by Theorem 3.2 we have $S'_{NA}(T) \subseteq \overline{S^o_{NA}(T)}$ and $S^o_{NA}(T) \subseteq \overline{S^o_{NA}(T)}$, it then follows that also $T_1 \in S^o_{NA}(T)$ and

$$\mathcal{O} \cap S^o_{NA}(T) \neq \emptyset. \tag{15}$$

Expression 15 implies that $T_1 \in \overline{S^o_{NA}(T)}$ and hence T_1 belong to both $\overline{S'_{NA}(T)}$ and $\overline{S^o_{NA}(T)}$. This therefore shows that $S^o_{NA}(T)$ is dense in $\overline{S'_{NA}(T)}$. □

Denseness is transitive as demonstrated by the following theorem

Theorem 3.3.
Let $S^o_{NA}(T)$, $S'_{NA}(T)$ and $S''_{NA}(T)$ be subsets of an invariant topological space $(S_{NA}(T), \tau_{S_{NA}(T)})$ such that $S^o_{NA}(T) \subseteq S'_{NA}(T) \subseteq S''_{NA}(T)$. If $S^o_{NA}(T)$ is dense in $S'_{NA}(T)$ and $S'_{NA}(T)$ is dense in $S''_{NA}(T)$, then this implies that $S^o_{NA}(T)$ is also dense in $S''_{NA}(T)$.

Proof. Let $S^o_{NA}(T) \subseteq S'_{NA}(T) \subseteq S''_{NA}(T)$ such that $S^o_{NA}(T)$ is dense in $S'_{NA}(T)$ and $S'_{NA}(T)$ is dense in $S''_{NA}(T)$. Then we need to show that $S^o_{NA}(T)$ is also dense $S''_{NA}(T)$. It suffices to prove that:

i) $S'_{NA}(T) \subseteq \overline{S^o_{NA}(T)}$.

ii) $S''_{NA}(T) \subseteq \overline{S'_{NA}(T)}$.

iii) $S''_{NA}(T) \subseteq \overline{S^o_{NA}(T)}$.

i). Suppose $S^o_{NA}(T)$ is dense in $S'_{NA}(T)$, then $S'_{NA}(T)$ is contained in the closure of $S^o_{NA}(T)$ i.e. $S'_{NA}(T) \subseteq \overline{S^o_{NA}(T)}$. If $T \in S'_{NA}(T)$, the proof is trivial. Let $T \notin S'_{NA}(T)$, then $T \in \overline{S'_{NA}(T)}$, so for every open set $\mathcal{O} \in \tau_{S_{NA}(T)}$ containing T we have $\mathcal{O} \cap S'_{NA}(T) \neq \emptyset$. This implies that $\mathcal{O}$ contain a point T_1 of $S'_{NA}(T)$ and thus of $\overline{S^o_{NA}(T)}$ since $S'_{NA}(T) \subseteq \overline{S^o_{NA}(T)}$. If $T_1 \in S^o_{NA}(T)$ then its trivial. Suppose $T_1 \notin S^o_{NA}(T)$ and $T_1 \in (S^o_{NA}(T))'$, then every limit point of $S^o_{NA}(T)$ is also a point of $S'_{NA}(T)$ since $S^o_{NA}(T) \subseteq S'_{NA}(T)$. This implies that for every open set $\mathcal{O}$ containing T_1 we have $\mathcal{O} \cap S^o_{NA}(T) \neq \emptyset$. But $\overline{S^o_{NA}(T)} \supseteq S'_{NA}(T)$ hence, $\mathcal{O} \cap S^o_{NA}(T) \supseteq \mathcal{O} \cap S'_{NA}(T) \neq \emptyset$. So $T_1 \in (S^o_{NA}(T))'$ implies that $T_1 \in (S'_{NA}(T))'$ thus

$$(S'_{NA}(T))' \subseteq (S^o_{NA}(T))'. \tag{16}$$

This therefore implies that the arbitrary intersection of $\mathcal{O} \cap S'_{NA}(T)$ must contain a point of $S^o_{NA}(T)$. Moreover from Expression 16 we can deduce that $\overline{S'_{NA}(T)} \subseteq \overline{S^o_{NA}(T)}$ and thus $T \in \overline{S^o_{NA}(T)}$. Hence $S'_{NA}(T) \subseteq \overline{S^o_{NA}(T)}$.
The proof of ii) follows similarly as of i).
iii). $S^o_{NA}(T) \subseteq S''_{NA}(T)$ due to the fact that $S^o_{NA}(T) \subseteq S'_{NA}(T)$ and $S^o_{NA}(T) \subseteq S'_{NA}(T)$. From i) we have that since $S'_{NA}(T)$ is contained in the closure of $S^o_{NA}(T)$ and $S'_{NA}(T) \subseteq S''_{NA}(T)$, then this implies that $S''_{NA}(T)$ is contained in the closure of $S^o_{NA}(T)$. Therefore, $S^o_{NA}(T)$ to be dense in $S''_{NA}(T)$. □

Denseness is preserved by continuous functions as shown in the following theorems.

Theorem 3.4.
Let $(S_{NA}(T), \tau_{S_{NA}(T)})$, $(S'_{NA}(T), \tau_{S'_{NA}(T)})$ be an invariant topological spaces, $S^o_{NA}(T)$ be a dense subset of $S_{NA}(T)$ and $T : S_{NA}(T) \to S'_{NA}(T)$ be continuous, then $T(S^o_{NA}(T))$ is dense in $S'_{NA}(T)$.

Proof. Let $(S_{NA}(T), \tau_{S_{NA}(T)})$ be an invariant topological space. From Definition 2.3, $S^o_{NA}(T) \subseteq S_{NA}(T)$ is said to be dense in $S_{NA}(T)$ if for any non-empty set $O_1 \in \tau_{S_{NA}(T)}$, we have $O_1 \cap S^o_{NA}(T) \neq \emptyset$. Suppose that O_2 is a non-empty set in $\tau_{S'_{NA}(T)}$, then from Lemma 3.1 it suffices to show that $O_2 \cap T(S^o_{NA}(T)) \neq \emptyset$. By continuity of T, this implies that $T^{-1}(O_2)$ is a non-empty open subset of $S_{NA}(T)$. Moreover, since $S^o_{NA}(T)$ is dense in $S_{NA}(T)$, then we have $S^o_{NA}(T) \cap T^{-1}(\mathcal{O}_2) \neq \emptyset$. Then it follows that $\mathcal{O}_2 \cap T(S^o_{NA}(T)) \neq \emptyset$ which completes the proof. □

Theorem 3.5.
Let $T_1, T_2 : S'_{NA}(T) \to S''_{NA}(T)$ be continuous functions on an invariant topological spaces $(S_{NA}(T), \tau_{S_{NA}(T)})$, $(S'_{NA}(T), \tau_{S_{NA}(T)})$ and $S''_{NA}(T)$ be T_2-spaces. In particular, if $S^o_{NA}(T)$ is a dense subset of $S'_{NA}(T)$ such that $T_1 |_{S^o_{NA}(T)} = T_2 |_{S^o_{NA}(T)}$, then $T_1 = T_2$.

Proof. Let $A \in S'_{NA}(T) \backslash S^o_{NA}(T)$ and $T_1(A(x)) \neq T_2(A(x))$, for all x. Since $S''_{NA}(T)$ is a T_2-spaces, there exist disjoint open subsets $\mathcal{O}_1$ and $\mathcal{O}_2$ of $S''_{NA}(T)$ such that $T_1(A(x)) \in \mathcal{O}_1$ and $T_2(A(x)) \in \mathcal{O}_2$. Hence $T_1^{-1}(\mathcal{O}_1)$ and $T_2^{-1}(\mathcal{O}_2)$ are open in $S'_{NA}(T)$ since T_1 and T_2 are continuous. Since the intersection of open sets is an open set, then $T_1^{-1}(\mathcal{O}_1) \cap T_2^{-1}(\mathcal{O}_2) = \mathcal{O}_3$ is an open set containing A and thus $\mathcal{O}_3$ is a non-empty subset of $S'_{NA}(T)$. But $A \in S'_{NA}(T) \backslash S^o_{NA}(T)$ and $S^o_{NA}(T)$ is dense, then from Proposition 3.1, this implies that A is the limit point of $S^o_{NA}(T)$, invoking Definition 2.3, it follows that $\mathcal{O}_3$ contains at least one point B of $S^o_{NA}(T)$. Then we have $B \in T_1^{-1}(\mathcal{O})_1$, $B \in T_2^{-1}(\mathcal{O})_2$, then $T_1(B(x)) \in T_1 T_1^{-1}(\mathcal{O})_1 \subseteq \mathcal{O}_1$, $T_2 T_2^{-1}(\mathcal{O})_2 \subseteq \mathcal{O}_2$, for all x and $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$ since they are disjoint which gives that $T_1(B(x)) \neq T_2(B(x))$ a contradiction since $B \in S^o_{NA}(T)$. Therefore $T_1(B(x)) = T_2(B(x))$ and hence $T_1 = T_2$. □

It's known that similarity orbits forms a sequence of operators which can be of the form $T^n = \{T \circ ... \circ T\}$. Then, the theory of dynamical systems can be introduced on similarity orbits. In this case we let $T \in S_{NA}(T)$ be a continuous linear operator and $((H,T),\tau_H)$ a topological space. The ordered pair (H,T) constitute a dynamical system.

Proposition 3.2.
Let $T:(H,\tau_H)\to(H,\tau_H)$ be a dynamical system. Then the following conditions are equivalent.

i) if $M,N\subseteq H$, such that M is T-invariant, M and N have nonempty interior, then $H\neq M\cup N$.

ii) for any pair U,V subsets of H, there exists some $n\geq 0$ such that

$$T^n(U)\cap V\neq\emptyset. \tag{17}$$

iii) If $U\neq\emptyset$ is an open subset of H, then the set $\cup_{n=0}^{\infty}T^n(U)$ is dense in H.

iv) for any nonempty open set subset U of H the set $\cup_{n=0}^{\infty}T^{-n}(V)$ is dense in H.

Proof. $i)\Rightarrow iii)$ Let $M=\cup_{n=0}^{\infty}T^n(U)$ and $N=H\backslash M$. Hence, M and N are disjoint, that is; $M\cap N=\emptyset$. Since $M\subseteq H$, then $T:M\to M$, this implies that M is T-invariant and has a nonempty interior since it contains U. Moreover, since $H\neq M\cup N$, then N must have an empty interior. Invoking [[1], Theorem 3.8], this implies that M is dense in H.
$ii)\Rightarrow i)$ Let $H=M\cup N$, $M\cap N=\emptyset$ and $T(M)\subseteq M$. Then $int(M)$ and $int(N)$ are open sets with

$$T^n(intM)\cap int(N)\subseteq M\cap N=\emptyset, \tag{18}$$

$\forall\, n\geq 0$. By Equation 17, Equation 18 can only hold if either M or N has has an empty interior. Hence, the proof.
$ii)\Leftrightarrow iii)$ The proof is an immediate consequence of definition 2.3.
$ii)\Leftrightarrow iv)$ Since T is continuous, then applying T^{-n} on both sides of Equation 17 we get

$$\begin{aligned} T^{-n}[T^n(U)\cap V]\neq T^{-n}(\emptyset) &\Rightarrow (T^{-n}T^n)(U)\cap T^{-n}(V)\neq\emptyset \\ &= U\cap T^{-n}(V)\neq\emptyset. \end{aligned}$$

By continuity of T and Definition 2.7 it completes the proof. □

Remark 3.2.
Let $T\in S_{NA}(T)$ and $T:H\to H$ be a dynamical system, then $Orb(x,T):=\{x,Tx,T^2,...\}$ is the orbit of x under T for all $x\in H$.

4. Conclusion

In this paper, we introduce the concept of dense sets of similarity orbits of norm-attainable operators on invariant topological spaces and study their properties. The results obtained in our paper may be applied in numerical analysis and approximation theory.

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References

[1] T. M. Al-shami, Somewhere dense orbits and ST_1-space, Indiana Univ. Math. J., 52 (2003) 811-819.
[2] E. Bishop, R.R. Phelps, A proof that every Banach space is subreflexive, Bull. Math Soc, 67 (1961) 97-98.
[3] P.S. Bourdon, N.S. Feldman, Somewhere dense orbits are everywhere dense, Indiana Univ. Math. J., 52 (2003) 811-819.
[4] L.A. Fialkow, The similarity orbit of a normal operator, Transactions of the Amer. Math. Soc, 210 (1975) 129-137.

[5] Gasiński N. and Papageorgiou N.S, Exercises in Analysis part 1, Springer Int. publishing Switzerland, 2014.
[6] D. W. Hadwin, E. A.Nordgren, H. Radjavi, P. Rosentha, Most similarity orbits are strongly dense, American Math Soc., (1979) 250-252.
[7] D.A. Herrero, Closure of similarity orbits of Hilbert space operators. II:Normal operators, J. London Math. Soc., 13 (1976) 299-316.
[8] C. S. Kubrusly, Elements of operator theory, Birkhäuser Boston. Basel. Berlin, 2001.
bibitemLee J. L. Lee J, On the norm-attaining operators, Korean J. Math., 20 (2012) 485-491.
[9] J. Lindenstrauss, Operators which attain their norms.111, Istael J. Math, 1 (1963) 139-148.
[10] R.A. Martínez-Avendaño, P. Rosenthal P: An introduction to operators on the Hardy-Hilbert space, Springer, 2007.
[11] S. A. Morris, Topology without tears, 2001.
[12] V. Müller: Orbits, weak orbits and local capacity of operators, Integral Equations operator theory, 41 (2000) 230-253.
[13] Rao N. S., Kalyani K. and Ramacharyulu N. C. P, Result on fixed point theorem in Hilbert space, Int. J. of Advances in Applied Math and Mechanics, 41 (2000) 230-253.
[14] S. Sompong, Dense sets on Bigeneralized topological spaces, Int. J. of math., 7 (2013) 999-1003.
[15] M. S. Wabomba, P.M. Ohuru, K. Musyoka, On analytical approach to semi-open/semi-closed sets, Int.J. of discrete math, 2 (2017) 54-58.