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On Compact Operators Whose Norms Are Eigenvalues and Completeness

J. N. Owino, N. B. Okelo and Omolo Ongati

School of Mathematics and Actuarial Science
Department of Pure and Applied Mathematics
Jaramogi Oginga Odinga University of Science and Technology
P.O. Box 210-40601, Bondo, Kenya

Abstract

Let X be a Banach space and $T : X \to Y$ be a linear operator, then T is compact if it maps bounded sequences in X to sequences in Y with convergent subsequences, that is, if $x_n \in X$ is a bounded sequence, then $Tx_n \in Y$ has a convergent subsequence say, Tx_{n_k} in Y. The eigenvalue of an operator T, is a scalar λ if there is a nontrivial solution x such that $Tx = \lambda x$. Such an x is called an eigenvector corresponding to the eigenvalue λ. A vector space is complete if every every Cauchy sequence in V converges in V. It is known that every finite dimensional normed space is complete and that a Hilbert space is a normed space that is complete with respect to the norm induced by the inner product. In this paper we have established the conditions for completeness of a compact operator T whose norm is an eigenvalue.

Keywords : Norm, Completeness, Uniformly continuous

1 Introduction

The class of compact operators has generated a lot of interest from mathematicians. This could be due to fact that it is fundamental in characterization of operators. A bounded linear operator T acting on a Banach space X to a Banach space Y is compact if it maps every bounded sequence say, $\{x_n\}_{n\in\mathbb{N}} \in X$

into a sequence $\{Tx_n\} \in Y$ with a convergent subsequence, or equivalently if every image $\{Tx_n\} \in Y$ contains a cauchy subsequence. Every bounded function is continuous, the sum of the two continuous functions is continuous, and so is any scalar multiple of a continuous function. Every convergent sequence is cauchy but the converse is not necessarily true. As a result there are sequences which are cauchy but not complete. Compact operators maps weakly convergent sequences into strong convergent sequences. That is, suppose $T \in B(X, Y)$ is a compact operator and (u_n) is a sequence in X such that $u_n \rightharpoonup u$, then $Tu_n \to Tu$ in Y. The set of compact operators $K(X, Y)$ is a closed linear subspace of the Banach space $B(X, Y)$, so that $K(X, Y)$ is also a Banach space [3]. If a linear transformation T maps bounded sets onto precompact sets, then it maps the closed unit ball onto a compact set. A necessary condition for compactness of T is boundedness, which implies that every compact linear operator is bounded [6, 7]. It is also known that the image of a closed unit ball is strongly closed and in addition, if the image is precompact, then the image is actually compact. This however is not universally true for Banach spaces. Compactness conditions are consequences of continuity conditions. Hence compact operators are also completely continuous [3]. Let T be finite dimensional and self-adjoint operator on a Hilbert space $\mathcal{H}$, then T has an eigenvalue λ and the set $x_\lambda = \{x \in X : Tx = \lambda x\}$ is the $\lambda-$eigenspace of T on $\mathcal{H}$. If the eigenspace x_λ is finite-dimensional, then all the eigenvalues are real and $|\lambda| = \|T\|$ or $|\lambda| = -\|T\|$ [6, 4]. Matthew [5] asserted that for any compact self-adjoint operator A on an infinite dimensional Hilbert space $\mathcal{H}$ we may find an orthonormal system of eigenvectors $\{u_n\}_{n\in\mathbb{N}}$ corresponding to the non-zero real eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ such that every $x \in \mathcal{H}$ may be written uniquely as $x = \sum_{n=1}^{\infty} \alpha_n u_n + v$ where $\alpha_n \in \mathbb{C}$ and $Av = 0$. If A has an infinite number of distinct eigenvalues then we have $\lambda_n \to 0$ as $n \to \infty$.

2 Preliminaries

Definition 2.1. [2]. A sequence $\{T_n\} \in B(X, Y)$ converges strongly if $\lim_{n\to\infty} T_n x = Tx, \ \ \forall \ \ n \in \mathbb{N}$ for every $x \in X$ or equivalently, $\lim_{n\to\infty} \|T_n x - Tx\| = 0$ for some $T \in B(X, Y)$ and for every $x \in X$.

Definition 2.2. [2]. A sequence (x_n) of a normed space X is to said be Cauchy if for every $\varepsilon > 0$ there is an N such that $\|x_m - x_n\| < \varepsilon$ for every $m, n > N$.

Definition 2.3. [2, 8]. The space X is said to be complete if every Cauchy sequence in X converges (has a limit that is in X).

Definition 2.4. [2]. Let $T : D(T) \to Y$ be any operator ; $D(T) \subset X$ and X and Y are normed spaces . The operator T is said to be uniformly continuous if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\| Tx - Ty \| < \varepsilon$ for all $x, y \in D(T)$ satisfying $\| x - y \| < \delta$.

Definition 2.5. The adjoint of a linear operator $T \in B(X)$ is a linear operator $T^* \in B(X)$ defined by the relation

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \ \ \forall x, y \in X.$$

Definition 2.6. An operator $T : V \to W$ between two vector spaces is said to be of finite-rank if its range is finite-dimensional.

Definition 2.7. If $\{T_n\}_{n\in\mathbb{N}}$ is a sequence of operators in $B(X, Y)$ and $\lim_{n\to\infty} \|T_n - T\| = 0$ for some $T \in B(X, Y)$ then we say that T_n converges uniformly to T or that T_n converges to T in the uniform or operator norm, topology on $B(X, Y)$.

Definition 2.8. [1]. A bounded operator $T : X \to Y$ between Banach spaces is compact (respectively, weakly compact) if and only if for every norm bounded sequence $\{x_n\} \in X$, the sequence $\{Tx_n\}$ has a norm convergent (respectively, weakly convergent) subsequence in Y.

Definition 2.9. [1]. An operator $T \in B(X)$ between Banach spaces is said to be power compact if there exists some $k \in \mathbb{N}$ such that T^k is compact.

3 Main Results

Theorem 3.1. Let X be a Banach space, then there exists an operator $T : X \to X$ which is compact. Moreover, there exists a subspace M of X which is dense in X and complete.

Proof. Let $x, y \in X$, $\alpha, \beta \in \mathbb{C}$ and $\|T\| = |\lambda| \geq 0$, then $\|Tx\| \leq \|T\|\|x\| = |\lambda|\|x\|$. Thus, T is bounded.

Now, $T(x + y) = T(x) + T(y)$ and $T(\alpha x + \beta y) = T(\alpha x) + T(\beta y) = \alpha Tx) + \beta Ty$. Thus, T is linear.

Let $x_n \in X$ be bounded such that $x_n \in X$ is cauchy, then since X is a Banach space there exists an $\epsilon > 0$ and an $N \in \mathbb{N}$ such that

$$\|x_m - x_n\| < \frac{1}{2}\epsilon, \ \forall\ m, n \geq N. \tag{3.1}$$

Then,

$$\|x_n - x\| < \frac{1}{2}\epsilon, \ \forall\ n \geq N. \tag{3.2}$$

Let $\|T\| \leq 1$, then

$$\frac{1}{2}\epsilon > |\lambda|\|x_m - x_n\| = \|T\|\|x_m - x_n\| \geq \|T(x_m - x_n)\| = \|Tx_m - Tx_n\|.$$

That is,

$$\|Tx_m - Tx_n\| < \frac{1}{2}\epsilon, \ \forall\ m, n \geq N. \tag{3.3}$$

Also,

$$\frac{1}{2}\epsilon > |\lambda|\|x_m - x\| = \|T\|\|x_m - x\| \geq \|T(x_m - x)\| = \|Tx_m - Tx\|.$$

That is,

$$\|Tx_m - Tx\| < \frac{1}{2}\epsilon,\ \forall\, n \geq N. \tag{3.4}$$

Without loss of generality, we may assume that $x_{m_k} = x_m$ and $Tx = y \in X$ we have,

$$\begin{aligned}\|Tx_{m_k} - Tx\| &= \|Tx_{m_k} - Tx_n + Tx_n - Tx\| \\ &\leq \|Tx_{m_k} - Tx_n\| + \|Tx_n - Tx\| < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon.\end{aligned}$$

$\|Tx_{m_k} - Tx\| < \epsilon$. This implies that $Tx_{m_k} \to y$. Thus, T is compact.

For instance, the operator $T^n : X \to X$, where n is a finite natural number is compact. For since if $\|T^n\| \leq 1$, then

$$\frac{1}{2}\epsilon > \|T^n\|\|x_m - x_n\| \geq \|T^n(x_m - x_n)\| = \|T^n x_m - T^n x_n\|.$$

That is,

$$\|T^n x_m - T^n x_n\| < \frac{1}{2}\epsilon,\ \forall\, m, n \geq N. \tag{3.5}$$

Also,

$$\frac{1}{2}\epsilon > \|T^n\|\|x_m - x\| \geq \|T^n(x_m - x)\| = \|T^n x_m - T^n x\|.$$

That is,

$$\|T^n x_m - T^n x\| < \frac{1}{2}\epsilon,\ \forall\, n \geq N. \tag{3.6}$$

Now, let $M = R(T^n)$ such that for all $y \in M$, $y = x - T^n z$ and $x, z \in X$. Then, $T^n x \in M$ and there exists a $q \in \mathbb{N}$ such that $T^{n+q}y = T^q x$. Implying that

$$T^{n+q}y - T^q x = T^q(T^n y - x) = 0 \ \Rightarrow\ T^n y - x = 0 \ \Rightarrow\ x = T^n y.$$

Substituting in y we obtain

$$y = x - T^n z = T^n y - T^n z = T^n(y + (-z)) \in R(T) = M \ \Rightarrow\ y \in M.$$

Let $y_1 = x_1 - T^n z_1$ and $y_2 = x_2 - T^n z_2$ be in M, then

$$y_1 + y_2 = x_1 - T^n z_1 + x_2 - T^n z_2 = x_1 + x_2 - (T^n z_1 + T^n z_2) = x_1 + x_2 - T^n(z_1 + z_2) \in M.$$

That is, $y_1 + y_2 \in M$. Also let $\alpha \in \mathbb{C}$ then,

$$\alpha y_1 = \alpha(x_1 - T^n z_1) = \alpha x_1 - T^n \alpha z_1 \in M.$$

Therefore, M is a subspace of X.

Let $\{y_m\} = \{x_m - T^n z_m\},\ \{y_l\} = \{x_l - T^n z_l\} \in M$ and by 3.1 and 3.5, then

$$\begin{aligned} \|y_m - y_l\| &= \|x_m - T^n z_m - (x_l - T^n z_l)\| \\ &= \|x_m - T^n z_m - x_l + T^n z_l\| \\ &= \|x_m - x_l + T^n z_l - T^n z_m\| \\ &= \|x_m - x_l\| + \|T^n z_l - T^n z_m\| = \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon. \end{aligned}$$

That is, there exists an $\epsilon > 0$ and an $N \in \mathbb{N}$ such that

$$\|y_m - y_l\| < \epsilon,\ \forall\ m, l \geq N. \tag{3.7}$$

So that $\{y_m\} \in M$ is cauchy.

Also, by 3.4 and 3.6, we have

$$\begin{aligned} \|y_m - y\| &= \|x_m - T^n z_m - (x - T^n z)\| \\ &= \|x_m - T^n z_m - x + T^n z\| \\ &= \|x_m - x + T^n z - T^n z_m\| \\ &= \|x_m - x\| + \|T^n z - T^n z_m\| = \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon. \end{aligned}$$

That is, there exists an $\epsilon > 0$ and an $N \in \mathbb{N}$ such that

$$\|y_m - y\| < \epsilon,\ \forall\ m \geq N. \tag{3.8}$$

So that $\{y_m\}$ converges to $y \in M$ which is also an element in X. Thus $\overline{M} = X$. From 3.7 and 3.8, M is complete. □

Theorem 3.2. Let X be a Banach space, Suppose F_β is a finite rank filter of idempotents on X such that $\forall\ x^* \in X^*$ then $F_\beta x^* \mapsto x^*$ in norm and F_β is complete.

Proof. Let X be a Banach space, Suppose F_β is a finite rank filter of idempotents on X such that $\forall\ x^* \in X^*$ and $z^* \in X^*$ then $\exists$ a $y^* = x^* - T^{n^*} z^* \in M^*$.

Let $F_\beta : M^* \to X^*$ then, $F_\beta = y^* \otimes x$ and $F_\beta y^* = F_\beta = (y^* \otimes x)y^*$

$$\begin{aligned}
\langle F_\beta y^*, x^*\rangle &= \langle (y^* \otimes x)y^*, x^*\rangle \\
&= \langle (y^* \otimes x)y^*, x^*\rangle \\
&= y^*(y^*)x^*(x) \\
&= \langle (y^*, (y^* \otimes y)x^*)\rangle \\
&= \langle (y^*, (y^*x^* \otimes y))\rangle \\
&= \langle (y^*, y^*(x^* \otimes y))\rangle \\
&= \langle (y^*y, (x^* \otimes y))\rangle \\
&= \langle (I, (x^* \otimes I)(I \otimes y))\rangle \\
&= \langle ((x \otimes I), (I \otimes y))\rangle \\
&= \langle x, y\rangle = \overline{\langle x, y\rangle} = \langle y^*, x^*\rangle.
\end{aligned}$$

Thus,

$$\langle F_\beta y^*, x^*\rangle = \langle y^*, x^*\rangle \Rightarrow \langle F_\beta y^*, x^*\rangle - \langle y^*, x^*\rangle = 0 \Rightarrow \langle F_\beta y^* - y^*, x^*\rangle = 0,$$

$\Rightarrow F_\beta y^* \to y^*$ weakly $\Rightarrow F_\beta(x^* - T^{n^*}z^*) = F_\beta x^* - F_\beta T^{n^*}z^* = F_\beta x^* - F_\beta T^{n^*}z^* \to x^* - T^{n^*}z^* \Rightarrow F_\beta x^* - x^* + (T^{n^*}z^* - F_\beta T^{n^*}z^*) \to 0$. That is, $F_\beta x^* \to^w x^*$ and $T^{n^*}z^* - F_\beta T^{n^*}z^* = T^{n^*}z^* - T^{n^*}F_\beta z^* = T^{n^*}(z^* - F_\beta z^*) = T^*(z^* - F_\beta z^*)$. Since T is compact, it maps weakly convergent sequences to strong ones. Hence $\|T^*(z^* - F_\beta z^*)\| \leq \|T^*\|\|(z^* - F_\beta z^*)\|$. We choose a $z^* \in X^*$ and a λ associated with $z \in X$ with $|\lambda| \leq 1$ such that
$\|T^*(z^* - F_\beta z^*)\| = \|T^*\|\|z^* - F_\beta z^*\| = \|T^*\|\|F_\beta z^* - z^*\| = \|T\|\|F_\beta z^* - z^*\| = |\lambda|\|F_\beta z^* - z^*\| \to 0 \Rightarrow \|F_\beta z^* - z^*\| \to 0$ and since $z^* \in X^*$ is arbitrary, $\|F_\beta x^* - x^*\| \to 0$ for all $x^* \in X^*$. □

Theorem 3.3. Let X be a Banach space then there exists a compact operator T whose norm is an eigenvalue which forms a complete space.

Proof. Let X be a Banach space then there exists a bounded sequence $\{x_n\} \in X$ an $\epsilon > 0$ and an $N \in \mathbb{N}$ such that $\|x_m - x_n\| < \epsilon$, $\forall\, m, n \geq N$ and $\|x_n - x\| < \epsilon$, $\forall\, n \geq N$.

Now, $\|Tx_m - Tx_n\| \leq \|T\|\|x_m - x_n\| = |\lambda|\|x_m - x_n\|$. Let $|\lambda| \leq 1$ then, $|\lambda|\|x_m - x_n\| \leq \|x_m - x_n\| < \epsilon$.

Thus, $\|Tx_m - Tx_n\| < \epsilon$, $\forall\, m, n \geq N$.

Similarly, $\|Tx_n - Tx\| \leq \|T\|\|x_n - x\| = |\lambda|\|x_n - x\| \leq \|x_n - x\| < \epsilon \Rightarrow \|Tx_n - Tx\| < \epsilon$, $\forall\, n \geq N$. Thus the space TX is complete.

Without loss of generality, assume that there is $N_1, N_2 \in \mathbb{N}$ such that

$$\|x_{m_k} - x_n\| < \frac{1}{2}\epsilon,\ \forall\, m_k, n \geq N_1$$

and

$$\|x_n - x\| < \frac{1}{2}\epsilon, \ \forall \ n \geq N_2.$$

Let $N = \max\{N_1, N_2\}$, then $\|Tx_{m_k} - Tx\| = \|Tx_{m_k} - Tx_n + Tx_n - Tx\| \leq \|Tx_{m_k} - Tx_n\| + \|Tx_n - Tx\| \leq \|T\|\|x_{m_k} - x_n\| + \|T\|\|x_n - Tx\| = |\lambda|\|x_{m_k} - x_n\| + |\lambda|\|x_n - Tx\| < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon.$

$$\|Tx_{m_k} - Tx\| < \epsilon, \ \forall \ m_k \geq N.$$

Implying that T is compact. □

Theorem 3.4. Let T be a compact operator, on Banach space X, whose norms are eigenvalues. Then the following conditions are equivalent:

(i) T is uniformly continuous.

(ii) T is bounded.

(iii) T is limit point.

Proof. **(i)** We show that (i) $\Rightarrow$ (ii). Let T be uniformly continuous, $|\lambda| = \|T\| \geq 0$ and X be a Banach space, then for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in X$ with $\|x - y\| < \delta$ we have $\|Tx - Ty\| < \epsilon$. Thus,

$$\|Tx - Ty\| = \|T(x - y)\| \leq \|T\|\|x - y\| \leq |\lambda|\|x - y\|.$$

Hence T is bounded.

(ii) That (ii) $\Rightarrow$ (iii). Let T be bounded such that $\|T_n - T_{n+1}\| < \frac{1}{2^n}$. Let also $S \in B(X)$ such that $S = T_{n+1}, \ n \in \mathbb{N}$ then

$$\begin{aligned} \|T_n - T_m\| &= \|T_n - S + S - T_m\| \\ &\leq \|T_n - S\| + \|S - T_m\| \\ &= \|T_n - T_{n+1}\| + \|T_{m+1} - T_m\| \\ &< \frac{1}{2^n} + \frac{1}{2^m}. \end{aligned}$$

$$\|T_n - T_m\| < \frac{1}{2^n} + \frac{1}{2^m}. \tag{3.9}$$

Taking limits as $m, n \to \infty$ on both sides of inequality 3.9, we have

$$\begin{aligned} \lim_{m,n\to\infty} \|T_n - T_m\| &< \lim_{m,n\to\infty} \left\{\frac{1}{2^n} + \frac{1}{2^m}\right\} \\ &= \lim_{n\to\infty} \frac{1}{2^n} + \lim_{m\to\infty} \frac{1}{2^m} = 0. \end{aligned}$$

Also, $\lim_{m,n\to\infty}\|T_n - T_m\| \leq \lim_{m,n\to\infty}\{\|T_n\| + \|T_m\|\} = \lim_{m,n\to\infty}\{|\lambda_n| + |\lambda_m|\} = \lim_{n\to\infty}|\lambda_n| + \lim_{m\to\infty}|\lambda_m| = 0$.

Therefore, $\{T_n\}_{n\in\mathbb{N}}$ is Cauchy and since $\{T_n\}_{n\in\mathbb{N}}$ is complete by theorem 4.19, every Cauchy sequence converges. Hence T_n converges say, to an operator T, that is, for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $\|T_n - T\| < \epsilon$, $\forall\, n \geq N$. Thus, T_n converges uniformly.

Also, since $\{T_n\}_{n\in\mathbb{N}}$ is Cauchy, then for every $\epsilon > 0$ there is an $N_1 \in \mathbb{N}$ such that

$$\|T_n x - T_m x\| < \frac{1}{2}\epsilon,\ \forall\, m, n \geq N_1.$$

Since $\{T_n\}$ is bounded and compact, there is a subsequence $T_{n_k} \in B(X)$ such that for every $\epsilon > 0$ there is an $N_2 \in \mathbb{N}$ such that

$$\|T_{n_k} x - Tx\| < \frac{1}{2}\epsilon,\ \forall\, n_k \geq N_2.$$

Choose $N = \max\{N_1, N_2\}$, then

$$\begin{aligned}\|T_n x - Tx\| &= \|T_n x - T_{n_k}x + T_{n_k}x - Tx\| \\ &\leq \|T_n x - T_{n_k}x\| + \|T_{n_k}x - Tx\| \\ &= \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon.\end{aligned}$$

Thus, $T_n \to T$ strongly.

(iii) (iii) $\Rightarrow$ (iv). Let T be a limit point. Then $\|T_n - T\| \to 0$ as $n \to \infty$. Let $x, y \in X$ and since T and T_n are compact and X is complete, there is a $n, m \in \mathbb{N}$ such that $Tx_n = Tx$ and $Tx_m = Ty$. Choose $x_n \in X$ such that $T_n x_n = y$. We have,

$$\begin{aligned}\|Tx - Ty\| = \|Tx_n - Tx_m\| &= \|Tx_n - y + y - Tx_m\| \\ &\leq \|Tx_n - y\| + \|y - Tx_m\| \\ &= \|Tx_n - T_n x_n\| + \|T_m x_m - Tx_m\| \\ &= \|(T - T_n)x_n\| + \|(T_m - T)x_m\| \\ &\leq \|T - T_n\|\|x_n\| + \|T_m - T\|\|x_m\| \to 0.\end{aligned}$$

Thus, for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\|Tx - Ty\| < \epsilon$ whenever $\|x - y\| < \delta$ and choose a $\delta = \frac{\epsilon}{|\lambda|}$, where $|\lambda| = \|T\|$. Hence the assertion follows.

□

Notes and Comments

Let X be a Banach space, then there exists an operator $T : X \to X$ whose norm is an eigenvalue which is compact. Moreover, there exists a subspace M of X which is dense in X and complete. T is uniformly bounded, continuous and a limit point of a sequence $\{T_n\}_{n\in\mathbb{N}}$ of compact operators whose norm is an eigenvalue.

References

[1] Y. A. Abramorich and C. D. Aliprantis, *An Invitation to Operator Theory*, Graduate Studies in Mathematics, Vol. 50, 2002. https://doi.org/10.1090/gsm/050

[2] J.B. Conway, *A Course in Functional Analysis*, Second Edition, Springer-Verlag, Berlin, 1990.

[3] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag Berlin Heidelberg New York, 1980.

[4] C. S. Lin, On Operator whose Norm is an Eigenvalue, *Bull. of Institute of Math. Acad. Sinica*, **31** (2003), 1-9.

[5] R. G. Matthew *Hilbert Space Theory and Applications in Basic Quantum Mechanics*, Cal Poly San Luis Obispo, 2013.

[6] G. Paul Lecture Notes on Compact Operators on Hilbert Space, 2012.

[7] V.I. Paulsen, *Completely Bounded Maps and Operator Algebras*, Cambridge Studies in Advance Mathematics, Vol. 78, Cambridge University Press, Cambridge, 2003. https://doi.org/10.1017/cbo9780511546631

[8] M. Sizwe, *Real Analysis Notes*, Dept. of Math., Rhodes University, 2009.

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