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PROJECTIVE NORMS AND CONVERGENCE OF NORM-ATTAINABLE OPERATORS

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Abstract

Let H be an infinite dimensional complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. We establish norm-attainability of operators via projective tensor norm. Moreover, we give results on the convergence of norm-attainable operators.

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1 Introduction

Let H be an infinite dimensional complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. Let both A and B be in $B(H)$ and $\mathcal{T} : B(H) \to B(H)$. $\mathcal{T}$ is called an elementary operator if it is representable in the form:$\mathcal{T}(X) = \sum_{i=1}^{n} A_i X B_i$, $\forall\ X \in B(H)$, where A_i, B_i are fixed in $B(H)$ or $\mathcal{M}(B(H))$ where $\mathcal{M}(B(H))$ is the multiplier algebra of $B(H)$. For A, $B \in B(H)$ we have the following examples of elementary operators: (i) the left multiplication operator $L_A(X) = AX$, (ii) the right multiplication operator $R_B(X) = XB$, (iii) the inner derivation $\delta_A = AX - XA$, (iv) the generalized derivation $\delta_{A,B} = AX - XB$, (v) the basic elementary operator $M_{A,\,B}(X) = AXB$, (vi) the Jordan elementary operator $\mathcal{U}_{A,\,B}(X) = AXB + BXA$, $\forall\ X \in B(H)$. Stampfli [**??**] characterized the norm of the generalized derivation $\|\delta_{A,B}\| = \inf_{\beta\in\mathbb{C}}\{\|A-\beta\|+\|B-\beta\|\}$, where $\mathbb{C}$ is the complex plane. In our main result, we prove some necessary and sufficient conditions for norm-attainability of operators in $B(H)$. We shall give some basic definitions first.

Definition 1.1. An operator $A \in B(H)$ is said to be norm-attainable if: There exists a unit vector $x \in H$ such that $\|Ax\| = \|A\|$; there exists a unit functional $\phi \in H^*$ such that $\|A(\phi)\| = \|A\|$. Moreover, the derivation δ_A is norm-attainable if $\exists\ \mu \in B(H) \otimes_p B(H)$ such that $\|\delta_A \mu\|_p = \|\delta_A\|_p$, where $\|\mu\|_p$ is the projective norm such that $\|\mu\|_p = 1$. An operator $\mathcal{T}_{\tilde{A},\tilde{B}}(X) = \sum_{i=1}^{n} A_i X B_i$, is said to be norm-attainable if there exists a contraction X in the unit ball, $(B(H))_1$, such that $\|\mathcal{T}_{\tilde{A},\tilde{B}}(X)\| = \|\mathcal{T}_{\tilde{A},\tilde{B}}\|$, where $\tilde{A} = (A_1, ..., A_n)$ and $\tilde{B} = (B_1, ..., B_n)$ are n-tuples in $B(H)$. If $\tilde{B} = \tilde{A}$ then we have $\mathcal{T}_{\tilde{A},\tilde{A}}$ simply denoted by $\mathcal{T}_{\tilde{A}}$.

We denote the algebra of all norm-attainable operators by $\mathcal{NA}(\mathcal{H})$. See [**??**-**??**] for details on norm-attainable elementary operators and derivations and the references therein.

2 Preliminaries

Proposition 2.1. *Let $u \in H$, $u' \in H^*$. Define an operator in $B(H)$ by $u \otimes u'$ then*
(i)$tr(u \otimes u') = \langle u', u \rangle$, $u' \in H^*$ and $u \in H$.
(ii) trA is independent of the basis chosen in H, and it is the sum of the eigenvalues of A with their order of algebraic multiplicity (in the characteristic polynomial of A), the mapping $\{B(H) \to \mathbb{C}, A \mapsto trA\}$ is linear.

Proof. (i) Let $u = \Sigma_i u^i e_i$, where $\langle e^i, u \rangle u^i$. Since $(u \otimes u')e_i = \langle u', e_i \rangle u$, $tr(u \otimes u') = \langle e^i, (u \otimes u')e_i \rangle = \langle e^i, u \rangle \langle u', e_i \rangle = \langle u', \langle e^i, u \rangle e_i \rangle = \langle u', u \rangle$.
(ii) Since (i) holds, $tr(u \otimes u')$ is independent of the basis chosen. It is evident that $\{A \mapsto trA\}$ is linear. The independence of "tr" relative to the basis is therefore valid for every element in $B(H)$. The trace of A is by definition the sum of the diagonal elements of a matrix [A] of A in the basis $\{e_i,\ i = 1, ..., n\}$. Therefore, if it is in this basis then it is the sum of eigenvalues by taking into account their algebraic multiplicity. □

Proposition 2.2. *Every element $A \in B(H)$ admits a unique decomposition defined by $A = TP$, where T is a partial isometry, P is a positive operator and $KerT = KerP$.*

Proof. If $A = TP$, then $\|Au\| = \|TPu\|,\ \forall\ u \in H$. Indeed, since T is an isometry on the range of P, hence if $Pu = 0$, then $Au = 0$. Therefore, the above equalities are valid for all $u \in H$ and $\langle A^*A, u\rangle = \langle Pu^2, u\rangle$. It is clear that A^*A is positive, self-adjoint and diagonalizable. Let $\{e_i,\ i = 1, ..., n\}$ be an orthonormal basis of H such that $A^*Ae_i = \mu^2 e_i,\ \mu \geq 0$. If P is given by $Pe_i = \mu e_i$, then $P^2 = A^*A$. Hence P exists. Then T is defined by $R(P) \to H,\ Pu \mapsto Au;\ KerP \to H,\ v \mapsto 0$. This is a partial isometry. Uniqueness is clear from the fact that if $G \in B(H)$ is positive then there exists a positive operator P such that $G = P^2$, P is called the square root of G. □

3 Main Results

Theorem 3.1. *Let $S \in B(H),\ \beta \in W_0(A)$ and $\alpha > 0$. There exists an operator $Z \in B(H)$ such that $\|S\| = \|Z\|$, with $\|S - Z\| < \alpha$. Furthermore, there exists a vector $\eta \in H,\ \|\eta\| = 1$ such that $\|Z\eta\| = \|Z\|$ with $\langle Z\eta, \eta\rangle = \beta$.*

Proof. For proof see **??**. □

Corollary 3.2. *Let $S, T \in B(H)$ If both S and T are norm-attainable then the basic elementary operator $M_{S,T}$ and the Jordan elementary operator $\mathcal{U}_{S,\,T}$ are also norm-attainable.*

Proof. The proofs are analogous to that of the main theorem **??** with considerations to Propositions **??** and **??**. □

In the next section we give results on convergence of norm-attainable operators.

4 Convergence of norm-attainable operators

Throughout this section, all the operators are norm-attainable unless otherwise stated. We consider uniform, weak and strong convergence in $NA(H)$.

Definition 4.1. A sequence $\{T_n\}$ of operators in $NA(H)$ converges uniformly, or strongly, or weakly to an operator $T \in NA(H)$ if $\|T_n - T\| \to 0$, or $\|(T_n - T)x\| \to 0$ for every $x \in H$, or $\langle T_n x, y\rangle \to 0$ for every $x, y \in H$ (equivalently, $\langle T_n x, x\rangle \to 0$ for every x in the complex Hilbert space H), and these will be denoted by $T_n \to^u T$, or $T_n \to^s T$, or $T_n \to^w T$, respectively. It is bounded if $\sup_n \|T_n\| < \infty$. Clearly,$T_n \to^u T \Rightarrow T_n \to^s T \Rightarrow T_n \to^w T$. This implies that $\sup_n \|T_n\| < \infty$.

Theorem 4.2. *Let $\{T_n\}$ and $\{S_n\}$ be sequences of operators in $NA(H)$ and $NA(K)$ respectively. Let also $T \in NA(H)$ and $S \in NA(K)$.*

(a) If $T_n \to^u T$ and $S_n \to^u S$, then $T_n \widehat{\otimes} S_n \to^u T \widehat{\otimes} S$.

(b) If $T_n \to^s T$ and $S_n \to^s S$, then $T_n \widehat{\otimes} S_n \to^s T \widehat{\otimes} S$.

(c) If $T_n \to^w T$ and $S_n \to^w S$, then $T_n \widehat{\otimes} S_n \to^w T \widehat{\otimes} S$.

Proof. Recall that $T_n \otimes S_n - T \otimes S = T_n \otimes (S_n - S) + (T_n - T) \otimes S$ for each n , which still holds if $\otimes$ is replaced with $\widehat{\otimes}$.

(a) If $\|T_n - T\| \to 0$ (so that $\{T_n\}$ is bounded) and $\|S_n - S\| \to 0$, then $\|T_n \widehat{\otimes} S_n - T \widehat{\otimes} S\| \leq \sup_n \|T_n\|\|S_n - S\| + \|S\|\|T_n - T\|$, and hence then $\|T_n \widehat{\otimes} S_n - T \widehat{\otimes} S\| \to 0$. That is $T_n \widehat{\otimes} S_n \to^u T \widehat{\otimes} S$.

(b) Take an arbitrary $\Sigma_{i=1}^n x_i \otimes y_i$ in $H \otimes K$ and observe that
$\|T_n \otimes S_n - T \otimes S \Sigma_{i=1}^n x_i \otimes y_i\| \leq \sup_n \|T_n\| \Sigma_{i=1}^n \|x_i\| \Sigma_{i=1}^n \|(S_n - S)y_i\|$
$+\|S\| \Sigma_{i=1}^n \|y_i\| \Sigma_{i=1}^n \|(T_n - T)x_i\|$.
If If $T_n \to^s T$ and $S_n \to^s S$, then $\|(T_n \otimes S_n - T \otimes S)\Sigma_{i=1}^n x_i \otimes y_i\| \to 0$ and so $T_n \otimes S_n \to^s T \otimes S$. Moreover, $\{T_n \widehat{\otimes} S_n\}$ is bounded (because $\sup_n \|T_n \widehat{\otimes} S_n \leq \sup_n \|T_n\| \sup_n \|S_n\| < \infty$). As it is well-known, if a sequence of operators converges strongly in a normed space, and if its extension is bounded in the completion, then convergence holds in the completion of the space. Thus $T_n \widehat{\otimes} S_n \to^s T \widehat{\otimes} S$.

(c) Similarly, and applying the Schwarz inequality, $|\langle T_n \otimes S_n - T \otimes S \Sigma_{i=1}^n x_i \otimes y_i, \Sigma_{i=1}^n x_i \otimes y_i\rangle| \leq \sup_n \|T_n\| \Sigma_{i=1}^n \Sigma_{j=1}^n \|x_i\|\|x_j\| \Sigma_{i=1}^n \Sigma_{j=1}^n |\langle (S_n - S)y_i, y_j\rangle|$
$+\|S\| \Sigma_{i=1}^n \Sigma_{j=1}^n \|y_i\|\|y_j\| \Sigma_{i=1}^n \Sigma_{j=1}^n |\langle (T_n - T)x_i, x_j\rangle|$.
Thus $|\langle T_n \otimes S_n - T \otimes S \Sigma_{i=1}^n x_i \otimes y_i, \Sigma_{i=1}^n x_i \otimes y_i\rangle| \to 0$, whenever $T_n \to^w T$ and $S_n \to^w S$, and so $T_n \otimes S_n \to^w T \widehat{\otimes} S$. The same argument applies for weak convergence so that $T_n \widehat{\otimes} S_n \to^w T \widehat{\otimes} S$.

$\square$

Remark 4.3. The result of part (c) in Theorem **??** does not mirror the ordinary product counterpart. Indeed, $T_n \rightarrow^w T$ and $S_n \rightarrow^w S$ do not imply $T_nS_n \rightarrow^w TS$. In fact, even $T_n \rightarrow^s T$ and $S_n \rightarrow^w S$ do not imply $T_nS_n \rightarrow^w TS$. We give the following example. If V is a unilateral shift, then if we put $T_n^* = S_n = V^n$ so that $T_n \rightarrow^s 0$, $S_n \rightarrow^w 0$, we have $T_nS_n = I$ for every n.

Theorem 4.4. *Let $\{T_n\}$ and $\{S_n\}$ be sequences of operators in $NA(H)$ and $NA(K)$ respectively. If one of them converges to zero uniformly (strongly, weakly) and the other is bounded, then $\{T_n\widehat{\otimes}S_n\}$ converges to zero uniformly (strongly, weakly).*

Proof. If $\|T_n\| \rightarrow 0$ and $\sup_n \|S_n\| < \infty$ or vice versa, then $\|T_n\widehat{\otimes}S_n\| \rightarrow 0$ because $\|T_n\widehat{\otimes}S_n\| = \|T_n \otimes S_n\| = \|T_n\|\|S_n\|$ for every $n \geq 1$, which proves the claimed result for uniform convergence. For strong and weak convergences take an arbitrary vector $\Sigma_{i=1}^n x_i \otimes y_i$ in $H \otimes K$. Note that

$$\|(T_n \otimes S_n)\Sigma_{i=1}^n x_i \otimes y_i\| \leq \sup_n \|S_n\|\Sigma_{i=1}^n\|T_nx_i\|\Sigma_{i=1}^n\|y_i\|.$$

If $\{T_n\}$ converges strongly to zero and if $\{S_n\}$ is bounded (or vice versa), then $\|(T_n \otimes S_n)\Sigma_{i=1}^n x_i \otimes y_i\| \rightarrow 0$. Applying the same argument in the proof of (b) of Theorem **??** above we get $T_n\widehat{\otimes}S_n \rightarrow^s 0$. Similarly,
$|\langle(T_n\otimes S_n)\Sigma_{i=1}^n x_i\otimes y_i, \Sigma_{i=1}^n x_i\otimes y_i\rangle| \leq \sup_n \|S_n\|\Sigma_{i=1}^n\Sigma_{j=1}^n|\langle T_nx_i, x_j\rangle|\Sigma_{i=1}^n\Sigma_{j=1}^n\|y_i\|\|y_j\|$
If $\{T_n\}$ converges weakly to zero and if $\{S_n\}$ is bounded (or vice versa), then $\langle(T_n \otimes S_n)\Sigma_{i=1}^n x_i \otimes y_i, \Sigma_{i=1}^n x_i \otimes y_i\rangle \rightarrow 0$. Again, applying the same argument in the proof of (c) of Theorem **??**, it follows that $T_n\widehat{\otimes}S_n$ converges weakly to zero. $\square$

At this point we give some results on convergence of power sequences of Hilbert space operators.We shows that, unlike the above example, convergence to zero of power sequences (or, equivalently, of sequences having the semigroup property) is transferred from the tensor product to one of the factors. First we consider the following auxiliary result.

Proposition 4.5. *If the power sequence $\{T^n\widehat{\otimes}S^n\}$ is bounded, then so is one of the power sequences $\{T^n\}$ or $\{S^n\}$.*

Remark 4.6. Let $T \in NA(H)$ and let $S \in NA(K)$. Consider the power sequences $\{T^n\}$ and $\{S^n\}$. If $\{T^n \widehat{\otimes} S^n\}$ converges to zero uniformly or strongly, then so does one of the sequences $\{T^n\}$ or $\{S^n\}$. If $\{T^n \widehat{\otimes} S^n\}$ converges to zero weakly, and one of T or S is power incremented, then one of the sequences $\{T^n\}$ or $\{S^n\}$ converges to zero strongly or uniformly.

References

[1] N. B. Okelo, J .O. Agure and D. O. Ambogo, Norms of elementary operators and characterization of norm - attainable operators, Int. Journal of Math. Analysis, **24** (2010), 687-693.

[2] N. B. Okelo, M. O. Okongo and S. A. Nyakiti, On Projective Tensor Norm and Norm-Attainable ?-Derivations, Int. J. Contemp. Math. Sciences, Vol. 5, no. 40 (2010), 1969 - 1975.

[3] N. B. Okelo, D. O. Ambogo and S. A. Nyakiti, On the Constants $C(\Omega)$ and $C_s(\Omega)$ of a C^*-algebra and Norms of Derivations, International Mathematical Forum, Vol. 5, no. 53 (2010), 2647 - 2653.

[4] N. B. Okelo and J. O. Agure, A two-sided multiplication operator norm, Gen. Math. Notes, Vol. 2, No. 1, (2011), 18-23.

[5] N. B. Okelo, J. O. Agure and P. O. Oleche, Various Notions Of Orthogonality in Normed Spaces, Acta Mathematica Scientia, Vol.33 No. 5 (2013), 1387-1397 .

[6] O. T. Mewomo and N. B. Okelo, On Approximate Character Amenability of Banach Algebras, Journal of the Nigerian Mathematical Society, Vol. 32, (2013), 303-315.

[7] N. B. Okelo, The norm-attainability of some elementary operators, Applied Mathematics E-Notes, Vol.13 (2013), 1-7.

[8] N. B. Okelo, J. O. Agure and P. O. Oleche, Certain conditions for norm-attainability of elementary operators and derivations, Interna-

tional Journal of Mathematics and Soft Computing, Vol.3, No.1 (2013), 53 - 59.

[9] N. B. Okelo, J. O. Agure and P. O. Oleche, On norm-attainability of elementary operators, International Journal of Science Research and Technology, Vol.1, No.1 (2013), 6-10.

[10] N. B. Okelo, On Dvoretsky's theorem and Norms of elementary operators, Int. Journal of pure and applied Sciences and technology, Vol 2. N0. 2 (2011), 46-53.

[11] A. Seddik, On the numerical range and norm of elementary operators, Linear Multilinear Algebra 52 (2004), 293-302.

[12] J. G Stampfli, The norm of a derivation, Pacific journal of mathematics, Vol. 33, 3 (1970), 737-747.

