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Binomial mixture based on generalized four parameter beta distribution as prior

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Abstract
A probability distribution can be constructed by mixing two distributions. Binomial distribution when compounded with beta distribution as prior forms a binomial mixture that is a continuous distribution. Skellam 1948, mixed a binomial distribution with its parameter being the probability of success considered as a random variable taking beta distribution. Probability distributions with binomial outcome tend to fail to fit empirical data due to over-dispersion. To address this challenge binomial mixtures are modeled to cater for the influence caused by over-dispersion. This paper focuses on binomial mixture with a four parameter generalized beta mixing distributions. In particular it focuses on application of McDonald generalized and Gerstenkon generalized mixing distributions. The binomial mixture obtained is proved to be a probability density function. Its moments are obtained using probability generating function techniques. The binomial mixture obtained can be applicable to probability distributions whose outcome are binomial in nature.

Keywords: Probability distribution, binomial mixture, Four parameter generalized beta distribution, Over-dispersion, moments

1. Introduction
Mixing two distributions to produce another distribution is one way of constructing probability distributions. Suppose $Y \sim B(a,b)$ and X is another random variable such that its distribution conditional on Y is a binomial distribution with parameters n and p then we are dealing with one continuous random variable Y and one discrete random variable X. To find their joint probability distribution we proceed as if it were a probability density function. Suppose X has a binomial distribution conditional on Y then its conditional probability mass function is

$f(x \mid y) = \binom{n}{x} p^x (1-p)^{n-x}$. And also given that Y has a beta distribution, then its density function is

$$f(y) = \frac{1}{B(a,b)} p^{a-1}(1-p)^{b-1}.$$

Therefore the joint probability density function of x and y is

$$f(x,y) = f(x \mid Y = y) f(y)$$

$$= \binom{n}{x} p^x (1-p)^{n-x} \frac{1}{B(a,b)} p^{a-1}(1-p)^{b-1}$$

$$= \frac{1}{B(a,b)} \binom{n}{x} p^{(a+x)-1} (1-p)^{(n-x+b)-1}$$

$$= \frac{B(a+x, b+n-x)}{B(a,b)} \binom{n}{x} \frac{p^{a+x-1}(1-p)^{(n-x+b)-1}}{B(a+x, n-x+b)}$$

From the above equation the joint probability density function can be factorized as $f(X,Y) = g(x,y)h(y)$ where $g(x,y) = \frac{1}{B(a+x, n-x+b)} p^{a+x-1}(1-p)^{n-x+b-1}$ which is a beta distribution with parameters $(a+x)$ and $(n-x+b)$ while the function $h(y) = \binom{n}{x} \frac{B(a+x, n-x+b)}{B(a,b)}$ which is independent of parameter p. Skellam [7] mixed a binomial distribution with its parameters being the probability of success taking a beta distribution. The mixture is called beta-binomial distribution. In probability distributions many outcome data in real life situations tend to be in the form of binomial distribution. However the binomial distribution sometimes fail to fit these empirical binomial data outcome due to over-dispersion. This effect is caused by the tendency of a group responding in a more similar way to some treatment than members of other group. This creates a challenge of parameter estimation hence the need to cater for the unmodeled influences that affect all the components of the binomial outcome in a given binomial trials. A combination of binomial and beta distribution with the latter considered as prior, makes it easy to analyze data whose distribution is binomial in nature. This paper focusses on binomial mixture with four parameter generalized beta I distribution considered as the mixing distribution. Specifically it focusses on McDonald [5] and Gerstenkon [4] distributions as mixing distributions. Both explicit and expectation methods of construction will be applied. The resulting binomial mixtures will be proved to be a probability density functions and their moments obtained. The mixture obtained is important as it can be used in the analysis of real data which cannot be satisfactorily analyzed using binomial or negative binomial models which are the common models considered in health area.

2. Method

2.1 Formulation of the mathematical problem.

2.1.1 McDonald's Generalized four parameter beta (G4B) mixing distribution

Generalized four parameter beta distribution of first kind was introduced by McDonald [5]. It can be obtained from the classical beta distribution by using the transformation $x = \left(\frac{p}{d}\right)^c$ where χ is a random variable of classical beta distribution with parameters a and b.

Let

$$x = \left(\frac{p}{d}\right)^c, \tag{1}$$

implying $\left|\frac{dx}{dp}\right| = \left|\frac{cp^{c-1}}{d^c}\right|$.

Therefore

$$g(p) = \left|\frac{x^{a-1}(1-x)^{b-1}|c|p^{c-1}}{B(a,b)d^c}\right|$$

$$= \left(\frac{p}{d}\right)^{c(a-)} \frac{\left[1-\left(\frac{p}{d}\right)^c\right]^{b-1} |c|p^{c-1}}{B(a,b) \quad d^c}$$

$$= \frac{|c| p^{ca-c+c-1}\left[1-\left(\frac{p}{d}\right)^{c}\right]^{b-1}}{d^{c} \quad d^{ca-c} \quad B(a,b)}$$

$$g(p) = \frac{|c| p^{ca-1}\left[1-\left(\frac{p}{d}\right)^{c}\right]^{b-1}}{d^{ca-1} \quad B(a,b)}, \; 0 < p < d, a, b, c, d > 0 \tag{2}$$

The j^th moment becomes

$$E\left[P^{j}\right] = \int_0^d p^{j+ca-1}\left[1-\left(\frac{p}{d}\right)^{c}\right]^{b-1} \frac{|c|}{d^{ca}B(a,b)}\,dp$$

Let, $y = \left(\frac{p}{d}\right)^{c}, \qquad \frac{c}{d}\left(\frac{p}{d}\right)^{c-1} dp = dy, \qquad p = dy^{\frac{1}{c}}$

$$E[P^{j}] = \int_0^1 \left(dy^{\frac{1}{c}}\right)^{j+ca-1} (1-y)^{b-1} \frac{c}{d^{ca} \quad B(a,b)\,\frac{c}{d}\left(\frac{p}{d}\right)^{c-1}}\,dy$$

$$= \int_0^1 \frac{d^{\,j+ca-1} y^{\frac{1}{c}(j+ca-1)} (1-y)^{b-1}}{d^{ca-1} d \left(\frac{p}{d}\right)^{c-1} \quad B(a,b)}\,dy$$

$$= \frac{d^{\,j}}{B(a,b)} \int_0^1 \frac{y^{\frac{j}{c}+a-\frac{1}{c}} (1-y)^{b-1}}{y^{\frac{1}{c}(c-1)}}\,dy$$

Therefore

$$E[P^{j}] = \frac{d^{\,j} B\left(\frac{j}{c}+a, b\right)}{B(a,b)}. \tag{3}$$

2.1.2 Gerstenkon four parameter generalized beta (G4B) mixing distribution.
Gerstenkon [4] re-parameterized the classical beta distribution as

$$h(y) = \frac{y^{\frac{r}{a}-1} (1-y)^{w-1}}{B\left(\frac{r}{a}, w\right)} \qquad 0 < y < 1 \; a, r, w > 0 \tag{4}$$

Let

$$y = \frac{p^{a}}{bw}; \qquad \left|\frac{dy}{dp}\right| = \frac{ap^{a-1}}{bw}$$

And

$0 < y < 1; \Rightarrow 0 < \frac{p^a}{bw} < 1; \Rightarrow 0 < p^a < bw; \Rightarrow 0 < p < (bw)^{\frac{1}{a}}$.

Therefore

$$g(p)=h(y)\left|\frac{dy}{dp}\right|=\frac{y^{\frac{r}{a}-1}(1-y)^{w-1}}{B(\frac{r}{a},w)}\frac{ap^{a-1}}{bw}$$

$$g(p)=\frac{\left(\frac{p^a}{bw}\right)^{\frac{r}{a}-1}\left[1-\frac{p^a}{bw}\right]^{w-1}}{B(\frac{r}{a},w)}\frac{ap^{a-1}}{bw}$$

$$=\frac{ap^{r-a+a-1}\left[1-\frac{p^a}{bw}\right]^{w-1}}{(bw)^{\frac{r}{a}}B(\frac{r}{a},w)}$$

$$g(p)=\frac{ap^{r-1}\left[1-\frac{p^a}{bw}\right]^{w-1}}{(bw)\frac{r}{a}B(\frac{r}{a},w)},\quad 0<p<(bw)^{\frac{1}{a}}\ a,b,r,w>0 \tag{5}$$

The j^{th} moment becomes

$$E[P^j]=\int_0^{(bw)^{\frac{1}{a}}}p^{j+r-1}\left(1-\frac{p^a}{bw}\right)^{w-1}\frac{a}{(bw)^{\frac{r}{a}}B(\frac{r}{a},w)}dp$$

Let

$$\frac{p^a}{bw}=t\ ,\Rightarrow\quad \frac{ap^{a-1}}{bw}dp=dt \text{ and } p=(bw)\frac{1}{a}$$

Therefore

$$E[P^j]=\int_0^1\frac{a(bwt)^{\frac{1}{a}(j+r-1)}[1-t]^{w-1}bw}{(bw)^{\frac{r}{a}}B(\frac{r}{a},w)\quad a\quad p^{a-1}}dt$$

$$=\frac{a(bwt)^{\frac{1}{a}(j+r-1)+1}}{(bw)^{\frac{r}{a}}B(\frac{r}{a},w)}\int_0^1\frac{t^{\frac{1}{a}(j+r-1)}(1-t)^{w-1}}{a(bw)^{\frac{1}{a}(a-1)}}dt$$

$$=\frac{a(bwt)^{\frac{1}{a}(j+r-1)+1}}{(bw)^{\frac{r}{a}}B(\frac{r}{a},w)}\int_0^1\frac{t^{\frac{1}{a}(j+r-1)}(1-t)^{w-1}}{a(bw)^{1-\frac{1}{a}}}dt$$

$$=\frac{a(bwt)^{\frac{1}{a}(j+r-1)+1}}{(bw)^{\frac{r}{a}}B(\frac{r}{a},w)a(bw)^{1-\frac{1}{a}}}\int_0^1\frac{t^{\frac{1}{a}(j+r-1)}(1-t)^{w-1}}{t^{1-\frac{1}{a}}}dt$$

$$E[P^j]=\frac{(bw)^{\frac{j}{a}}B(\frac{j+r}{a},w)}{B(\frac{r}{a},w)} \tag{6}$$

2.2 Construction of the Binomial Mixture
2.3 Binomial-McDonald G4B distribution
The Binomial-McDonald G4B distribution can be constructed using direct integration as

$$f(x) = \frac{\binom{n}{x}}{B(a,b)} \int_0^d p^x (1-p)^{n-x} \frac{|c|}{d^{ca}} \left(1 - \left(\frac{p}{d}\right)^c\right)^{b-1} dp$$

$$= \frac{\binom{n}{x}|c|}{d^{ca} B(a,b)} \int_0^d p^{x+ca-1} (1-p)^{n-x} \left(1 - \left(\frac{p}{d}\right)^c\right)^{b-1} dp$$

Let

$$\left(\frac{p}{d}\right)^c = y; \;\; \frac{c}{d}\left(\frac{p}{d}\right)^{c-1} dp = dy$$

Therefore

$$\frac{cp^{c-1}}{d^c} dp = dy \text{ and } \frac{p}{d} = y^{\frac{1}{c}} \Rightarrow p = y^{\frac{1}{c}} d$$

On substitution we obtain

$$f(x) = \frac{|c|\binom{n}{x}}{d^{ca} B(a,b)} \int_0^1 \frac{\left(dy^{\frac{1}{c}}\right)^{x+ca-1} \left(1 - dy^{\frac{1}{c}}\right)^{n-x} (1-y)^{b-1} d^c}{c \;\; d^{c-1} \;\; y^{1-\frac{1}{c}}} dy$$

$$= \frac{|c|\binom{n}{x}}{d^{ca} B(a,b)} \int_0^1 \frac{d^{x+ca-1} y^{\frac{x}{c}+a-\frac{1}{c}} \left(1 - dy^{\frac{1}{c}}\right)^{n-x} (1-y)^{b-1} d^c}{c \;\; d^{c-1} \;\; y^{1-\frac{1}{c}}} dy$$

$$= \frac{|c| d^x \binom{n}{x}}{B(a,b)} \int_0^1 y^{\frac{x}{c}+a-1} (1-y)^{b-1} \left(1 - dy^{\frac{1}{c}}\right)^{n-x} dy$$

$$= \frac{|c| d^x \binom{n}{x}}{B(a,b)} \sum_{k=0}^{n-x} (-1)^k \binom{n-x}{k} \left(dy^{\frac{1}{c}}\right)^k \int_0^1 y^{\frac{x}{c}+a-1} (1-y)^{b-1} dy$$

Therefore

$$f(x) = \frac{|c|\binom{n}{x} d^{x+k}}{B(a,b)} \sum_{k=0}^{n-x} (-1)^k \binom{n-x}{k} B\left(\frac{k+x}{c} + a, b\right) \qquad (7)$$

By the method of moments, we have

$$f(x) = \binom{n}{x} \sum_{j=x}^{n} (-1)^{j-x} \binom{n-x}{j-x} E[P^j]$$

$$= \binom{n}{x} \sum_{j=x}^{n} (-1)^{j-x} \binom{n-x}{j-x} d^j \frac{B\left(\frac{j}{c} + a, b\right)}{B(a,b)}$$

$$f(x)=\binom{n}{x}\sum_{k=0}^{n-x}(-1)^k\binom{n-x}{k}d^{k+x}\frac{B\left(\frac{k+x}{c}+a,b\right)}{B(a,b)} \tag{8}$$

taking $k=j-x$.

Not that equation [7] and [8] are the same, hence no proof of identity.

2.2.2 Binomial-Gerstenkon G4B distribution.

The Binomial-Gerstenkon G4B distribution is obtained by direct method as

$$f(x)=\int_0^{(bw)^{\frac{1}{a}}}\binom{n}{x}p^x(1-p)^{n-x}\frac{ap^{r-1}\left[1-\frac{p^a}{bw}\right]^{w-1}}{(bw)^{\frac{r}{a}}B\left(\frac{r}{a},w\right)}dp$$

$$=\frac{a\binom{n}{x}}{(bw)^{\frac{r}{a}}B\left(\frac{r}{a},w\right)}\int_0^{(bw)^{\frac{1}{a}}}p^{r+x-1}(1-p)^{n-x}\left(1-\frac{p^a}{bw}\right)^{w-1}dp$$

Let

$$\frac{p^a}{bw}=t\ ;\Rightarrow\ p^a=bwt\ ;\Rightarrow\ p=(bwt)^{\frac{1}{a}}\ \text{and}\ \frac{dp}{dt}=\frac{(bwt)^{\frac{1}{a}-1}}{abw}\ .$$

Therefore

$$f(x)=\frac{\binom{n}{x}a}{aB\left(\frac{r}{a},w\right)(bw)^{\frac{r}{a}}}\int_0^1(bwt)^{\frac{1}{a}(x+t-1)}\left[1-(bwt)^{\frac{1}{a}}\right]^{n-x}[1-t]^{w-1}(bwt)^{\frac{1}{a}-1}bwdt$$

$$=\frac{\binom{n}{x}(bw)^{\frac{1}{a}(x+r-1)+\frac{1}{a}-1+1}}{(bw)^{\frac{r}{a}}B\left(\frac{r}{a},w\right)}\int_0^1t^{\frac{1}{a}(x+r-1)+\frac{1}{a}-1}(1-t)^{w-1}\left[1-(bwt)^{\frac{1}{a}}\right]^{n-x}dt$$

Therefore

$$f(x)=\frac{\binom{n}{x}(bw)^{\frac{x}{a}+\frac{r}{a}-\frac{1}{a}+\frac{1}{a}}}{(bw)^{\frac{r}{a}}B\left(\frac{r}{a},w\right)}\int_0^1t^{\frac{x}{a}+\frac{r}{a}-\frac{1}{a}+\frac{1}{a}-1}[1-t]^{w-1}\left[1-(bwt)^{\frac{1}{a}}\right]^{n-x}dt$$

$$=\frac{\binom{n}{x}(bw)^{\frac{x}{a}}}{B\left(\frac{r}{a},w\right)}\int_0^1t^{\frac{x+r}{a}-1}[1-t]^{w-1}\left[1-(bwt)^{\frac{1}{a}}\right]^{n-x}dt$$

$$=\frac{\binom{n}{x}(bw)^{\frac{x}{a}}}{B\left(\frac{r}{a},w\right)}\int_0^1t^{\frac{x+r}{a}-1}[1-t]^{w-1}\sum_{k=0}^{n-x}\binom{n-x}{k}\left(-(bwt)^{\frac{1}{a}}\right)^k dt$$

$$=\frac{\binom{n}{x}(bw)^{\frac{x}{a}}}{B\left(\frac{r}{a},w\right)}\sum_{k=0}^{n-x}(-1)^k\binom{n-x}{x}\int_0^1(bw)^{\frac{k}{a}}t^{\frac{k+x+r}{a}-1}[1-t]^{w-1}dt$$

$$= \binom{n}{x} \sum_{k=0}^{n-x} (-1)^k \binom{n-x}{k} (bw)^{\frac{k+x}{a}} \frac{B\left(\frac{k+x+r}{a}, w\right)}{B\left(\frac{r}{a}, w\right)}$$

Therefore

$$f(x) = \binom{n}{x} \sum_{j=x}^{n} (-1)^{j-x} \binom{n-x}{j-x} (bw)^{\frac{j}{a}} \frac{B\left(\frac{j+r}{a}, w\right)}{B\left(\frac{r}{a}, w\right)} \text{ where } j = k + x. \qquad 7(9)$$

By the method of moments, we have

$$f(x) = \binom{n}{x} \sum_{j=x}^{n} (-1)^{j-x} \binom{n-x}{j-x} E\left[P^j\right]$$

$$= \binom{n}{x} \sum_{j=x}^{n} (-1)^{j-x} \binom{n-x}{j-x} (bw)^{\frac{j}{a}} \frac{B\left(\frac{j+r}{a}, w\right)}{B\left(\frac{r}{a}, w\right)} \qquad (10)$$

Both methods give the same results, hence identity not proved.

3. Results

3.1 Binomial-McDonald G4B distribution

One of the properties of a distribution of a random variable is that $\int_{-\infty}^{\infty} f(x)dx = 1$.

$$f(x) = \frac{\binom{n}{x}|c|}{d^{ca}B(a,b)} \int_0^d p^{x+ca-1}(1-p)^{n-x}\left[1-\left(\frac{p}{d}\right)^c\right]^{b-1} dp$$

$$\frac{|c|}{d^{ca}B(a,b)} \binom{n}{x} p^x (1-p)^{n-x} \int_0^d p^{ca-1}\left[1-\left(\frac{p}{d}\right)^c\right]^{b-1} dp$$

But

$$\sum_{x=0}^{n-x} \binom{n}{x} p^x (1-p)^{n-x} = 1.$$

And putting $c = d = 1$ we have

$$f(x) = \frac{1}{B(a,b)} \int_0^1 p^{a-1}(1-p)^{b-1} dp$$

$$= \frac{B(a,b)}{B(a,b)}$$

= 1

Indicating that the mixture is a pdf.

The expected value of binomial-McDonald G4B distribution can be obtained by using rth factorial moment which is

$$E(X^r) = E\left[X(X-1)(X-2)\ldots(X-r+1)\right] = n^{(r)}E(P^r).$$

Where

$$n^{(r)} = n(n-1)(n-2)(n-3)\ldots(n-r+1).$$

Therefore the mean becomes

$$E(X) = nE(P)$$

$$= \frac{ndB\left(\frac{1}{c}+a,b\right)}{B(a,b)} \tag{11}$$

And the variance is

$$Var(X) = n^2E(P^2) + nE(P) - (nE(P))^2$$

$$= \frac{nd^2B\left(\frac{2}{c}+a,b\right)(n-1)}{B(a,b)} + \frac{ndB\left(\frac{1}{c}+a,b\right)}{B(a,b)}\left\{1 - \frac{ndB\left(\frac{1}{c}+a,b\right)}{B(a,b)}\right\}$$

$$= n^2d^2\left\{\frac{B\left(\frac{2}{c}+a,b\right)}{B(a,b)} - \left[\frac{B\left(\frac{1}{c}+a,b\right)}{B(a,b)}\right]^2\right\} + nd\left\{\frac{B\left(\frac{1}{c}+a,b\right)}{B(a,b)} - \frac{dB\left(\frac{2}{c}+a,b\right)}{B(a,b)}\right\} \tag{12}$$

3.2 Binomial-Gerstenkon G4B distribution.
Binomial-Gerstenkon G4B distribution is a pdf. Given

$$f(x) = \frac{a\binom{n}{x}}{(bw)^{\frac{r}{a}}B\left(\frac{r}{a},w\right)}\int_0^{(bw)^{\frac{1}{a}}}\left(\frac{p}{bw}\right)^{r-1}\left(1-\left(\frac{p}{bw}\right)^a\right)^{w-1}dp$$

$$= \frac{a\binom{n}{x}p^x(1-p)^{n-x}}{(bw)^{\frac{r}{a}}B\left(\frac{r}{a},w\right)}\int_0^{(bw)^{\frac{1}{a}}}\left(\frac{p}{bw}\right)^{r-1}\left(1-\left(\frac{p}{bw}\right)^a\right)^{w-1}dp$$

But

$$\sum_{x=0}^{n-x}\binom{n}{x}p^x(1-p)^{n-x} = 1$$

And putting $a = b = w = 1$ we get

$$g(p) = \int_0^1 \frac{p^{r-1}(1-p)^{1-1}}{B(r,1)}dp$$

$$= \frac{B(r,1)}{B(r,1)}$$

= 1. Hence a pdf.

The expected value of Binomial-Gerstenkon becomes

$$E(X) = nE(P)$$

$$= \frac{n(bw)^{\frac{1}{a}}B\left(\frac{1+r}{a},w\right)}{B\left(\frac{r}{a},w\right)} \tag{13}$$

And the variance becomes

$$Var(X) = n^2E(P^2) + nE(P) - [nE(P)]^2$$

$$= n^2\left\{\frac{(bw)^{\frac{2}{a}}B\left(\frac{2+r}{a},w\right)}{B\left(\frac{r}{a},w\right)} - (bw)^{\frac{2}{a}}\left[\frac{B\left(\frac{1+r}{a},w\right)}{B\left(\frac{r}{a},w\right)}\right]^2\right\} + n\left\{(bw)^{\frac{1}{a}}\frac{B\left(\frac{1+r}{a},w\right)}{B\left(\frac{r}{a},w\right)} - (bw)^{\frac{2}{a}}\frac{B\left(\frac{r+2}{a},w\right)}{B\left(\frac{r}{a},w\right)}\right\} \tag{14}$$

4. Discussion

From beta-Binomial distribution obtained above, it is noted that binomial distribution has the parameter n and p in which p is treated as a random variable. From generalized beta distributions four parameters are obtained making the prior distribution be much more flexible and improves the capacity of the binomial mixture to fit the empirical data more effectively. The mixture has been proved to be a pdf. Its properties such as expected value and variance have been obtained for both binomial-McDonald and Binomial-Gerstenkon G4B distributions. The two methods of constructions that is explicit and expectation forms have resulted into the same outcome of binomial mixture.

5. Conclusion

The mixed distribution can be obtained from binomial and generalized four parameter beta distributions of the first kind. Both distributions of the first kind. Both McDonald and Gerstenkon G4B distributions have been applied to form the mixed distributions accordingly. The mixture obtained can be used to model probability distributions whose data outcome are binomial.

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