Lie Symmetry Solution of Fourth Order Nonlinear Ordinary Differential Equation: $(yy'(y(y')^{-1})'')'=0$

T. J. O. Aminer[1], N. Omolo Ongati[2] and M. E. Oduor Okoya[3]

[1,2,3]School of Mathematics and Actuarial Science, Jaramogi Oginga Odinga University of Science and Technology, P.O. Box 210-40601, Bondo, Kenya

[1]titusaminer @yahoo.com

***Abstract*— The equation $F\left(x, y, y', y'', y''', y^{(4)}\right)=0$ is a one-space dimension version of wave equation. Its solutions can be classified either as analytic or numerical using finite difference approach, where the convergence of the numerical schemes depends entirely on the initial and boundary values given. In this paper, we have used Lie symmetry analysis approach to solve the wave equation given since the solution does not depend on either boundary or initial values. Thus in our search for the solution we exploited a systematic procedure of developing infinitesimal transformations, generators, prolongations (extended transformations), variational symmetries, adjoint-symmetries, integrating factors and the invariant transformations of the problem. The procedure is aimed at lowering the order of the equation from fourth to first order, which is then solved to provide its Lie symmetry solution.**

***Keywords*— Lie Symmetries Analysis, Symmetry Reduction, Ordinary Differential Equations, Infinitesimal Transformations, Transformation Generators, Prolongations (Extended Transformations), Variational Symmetries, Adjoint-Symmetries, Integrating Factors and Invariant Transformations**

FOURTH ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATION

We consider the fourth order nonlinear ordinary differential equation:

$$(yy'(y(y')^{-1})'')'=0 \tag{1}$$

which is a special case of

$$F(x,y,y',y'',y''',y^{(4)})=0 \tag{2}$$

This case arises in studying the group properties of the linear wave equation in an inhomogeneous medium.

To solve equation (1) analytically using Lie symmetry analysis we decompose it in the form

$$y^{(4)}=f(x,y,y',y'',y''')$$

We thus have:

$$y^{(4)}=4y^{-1}(y'')^2-4(y')^{-2}(y'')^3+5(y')^{-1}y''y'''-y^{-2}(y')^2y''-3y^{-1}y'y''' \tag{3}$$

Since the equation is a fourth order differential equation, we use the fourth extension of G, which from the n^{th} extension of the form [5]:

$$G^{[n]}=G+\sum_{i=1}^{n}\left\{\beta^{(i)}-\sum_{j=1}^{i}\binom{i}{j}y^{(i+1-j)}\alpha^{(j)}\right\}\frac{\partial}{\partial y^{(i)}}$$

is given by

$$G^{[4]}=\alpha\frac{\partial}{\partial x}+\beta\frac{\partial}{\partial y}+(\beta'-\alpha'y')\frac{\partial}{\partial y'}+(\beta''-2y''\alpha'-y'\alpha'')\frac{\partial}{\partial y''}$$
$$+(\beta'''-3y'''\alpha'-3y''\alpha''-y'\alpha''')\frac{\partial}{\partial y'''}$$
$$+\left(\beta^{(4)}-4y^{(4)}\alpha'-6y'''\alpha''-4y''\alpha'''-y'\alpha^{(4)}\right)\frac{\partial}{\partial y^{(4)}} \tag{4}$$

Where the generator

$$G=\alpha\frac{\partial}{\partial x}+\beta\frac{\partial}{\partial y} \tag{5}$$

is a symmetry of the differential equation

$$f(x,y,y',y'',\ldots,y^{(n)})=0 \tag{6}$$

if and only if

$$G^{[n]}f|_{f=0}=0, \tag{7}$$

which means that the action of the n^{th} extension of G on f is zero when the original equation is satisfied ([1],[4],[7]).

When $G^{[4]}$ acts on the differential equation (3) we obtain

$$G^{[4]}\left[y^{(4)}-4y^{-1}(y'')^2+4(y')^{-2}(y'')^3-5(y')^{-1}y''y'''\right.$$
$$\left.+y^{-2}(y')^2y''+3y^{-1}y'y'''\right]=0$$

Leading to

$$4\beta y^{-2}(y'')^2-2\beta y^{-3}(y')^2(y'')-3\beta y^{-2}(y')(y''')$$

$$-8\beta'(y')^{-3}(y'')^4+5\beta'(y')^{-2}(y'')^2(y''')+2\beta'y^{-2}(y')(y'')^2+3\beta'y^{-1}(y''')$$

$$+8\alpha'(y')^{-2}(y'')^4-5\alpha'(y')^{-1}(y'')^2(y''')-2\alpha'y^{-2}(y')^2(y'')^2-3\alpha'y^{-1}(y')(y''')$$

$$-8\beta''y^{-1}(y'')(y''')+12\beta''(y')^{-2}(y'')^2(y''')-5\beta''(y')^{-1}(y''')+\beta''y^{-2}(y')^2$$

$$+16\alpha'y^{-1}(y'')^{2}(y''')-24\alpha'(y')^{-2}(y'')^{3}(y''')+10\alpha'(y')^{-1}(y'')(y''')-2\alpha'y^{-2}(y')^{2}(y'')$$

$$+8\alpha''y^{-1}(y')(y'')(y''')-12\alpha''(y')^{-1}(y'')^{2}(y''')+5\alpha''(y''')-\alpha''y^{-2}(y')^{1}$$

$$-5\beta'''(y')^{-1}(y'')+15\alpha'(y')^{-1}(y'')(y''')+15\alpha''(y')^{-1}(y'')^{2}+5\alpha'''(y'')$$

$$+3\beta'''y^{-1}(y')-27\alpha'y^{-1}(y')(y''')-9\alpha''y^{-1}(y')(y'')-3\alpha'''y^{-1}(y')^{2}+\beta^{(4)}$$

$$-16\alpha'y^{-1}(y'')^{2}+16\alpha'(y')^{-2}(y'')^{3}-20\alpha'(y')^{-1}y''y'''+4\alpha'y^{-2}(y')^{2}y''+12\alpha'y^{-1}y'y'''$$

$$-6\alpha''y'''-4\alpha'''y''-\alpha^{(4)}y'=0 \qquad (8)$$

We recall that primes in equation (8) refer to total derivatives and so the first, the second, the third and the fourth total derivatives of α can be expressed in terms of partial derivatives as follows:

$$\alpha'=\frac{\partial\alpha}{\partial x}+y'\frac{\partial\alpha}{\partial y}\left\{From\ d(\alpha)=\left(\frac{\partial\alpha}{\partial x}\right)dx+\left(\frac{\partial\alpha}{\partial y}\right)dy\right\} \qquad (9)$$

$$\alpha''=\frac{\partial^2\alpha}{\partial x^2}+2y'\frac{\partial^2\alpha}{\partial x\partial y}+y'^2\frac{\partial^2\alpha}{\partial y^2}+y''\frac{\partial\alpha}{\partial y} \qquad (10)$$

$$\alpha'''=\frac{\partial^3\alpha}{\partial x^3}+3y'\frac{\partial^3\alpha}{\partial x^2\partial y}+3y''\frac{\partial^2\alpha}{\partial x\partial y}+y'''\frac{\partial\alpha}{\partial y}+3y'^2\frac{\partial^3\alpha}{\partial x\partial y^2}$$
$$+3y'y''\frac{\partial^2\alpha}{\partial y^2}+y'^3\frac{\partial^3\alpha}{\partial y^3} \qquad (11)$$

$$\alpha^{4}=\frac{\partial^4\alpha}{\partial x^4}+4y'\frac{\partial^4\alpha}{\partial x^3\partial y}+6y''\frac{\partial^3\alpha}{\partial x^2\partial y}+4y'''\frac{\partial^2\alpha}{\partial x\partial y}+y^{(4)}\frac{\partial\alpha}{\partial y}$$
$$+3y'^2\frac{\partial^4\alpha}{\partial x^2\partial y^2}+9y'y''\frac{\partial^3\alpha}{\partial x\partial y^2}+4y'y'''\frac{\partial^2\alpha}{\partial y^2}+3y''^2\frac{\partial^2\alpha}{\partial y^2}$$
$$+4y'^3\frac{\partial^4\alpha}{\partial x\partial y^3}+6y'^2y''\frac{\partial^3\alpha}{\partial y^3}+3y'^2\frac{\partial^4\alpha}{\partial x^2\partial y^2}+3y'y''\frac{\partial^3\alpha}{\partial x\partial y^2}+y'^4\frac{\partial^4\alpha}{\partial y^4} \qquad (12)$$

and equivalently for β.
Subject to equation (3),
we have

$$4\beta y^{-2}(y'')^{2}-2\beta y^{-3}(y')^{2}(y'')-3\beta y^{-2}(y')(y''')-8(y')^{-3}(y'')^{4}\frac{\partial\beta}{\partial x}$$

$$-8(y')^{-2}(y'')^{4}\frac{\partial\beta}{\partial y}+5(y')^{-2}(y'')^{2}(y''')\frac{\partial\beta}{\partial x}+(y')^{-1}(y'')^{2}(y''')\frac{\partial\beta}{\partial y}$$

$$+2y^{-2}(y')(y'')^{2}\frac{\partial\beta}{\partial x}+2y^{-2}(y')^{2}(y'')^{2}\frac{\partial\beta}{\partial y}+3y^{-1}(y''')\frac{\partial\beta}{\partial x}+3y^{-1}(y')(y''')\frac{\partial\beta}{\partial y}$$

$$+8(y')^{-2}(y'')^{4}\frac{\partial\alpha}{\partial x}+8(y')^{-1}(y'')^{4}\frac{\partial\alpha}{\partial y}-5(y')^{-1}(y'')^{2}(y''')\frac{\partial\alpha}{\partial x}-5(y'')^{2}(y''')\frac{\partial\alpha}{\partial y}$$

$$-2y^{-2}(y')^{2}(y'')^{2}\frac{\partial\alpha}{\partial x}-2y^{-2}(y')^{3}(y'')^{2}\frac{\partial\alpha}{\partial y}-3y^{-1}(y')(y''')\frac{\partial\alpha}{\partial x}-3y^{-1}(y')^{2}(y''')\frac{\partial\alpha}{\partial y}$$

$$-8y^{-1}(y'')(y''')\frac{\partial^2\beta}{\partial x^2}-16y^{-1}(y')(y'')(y''')\frac{\partial^2\beta}{\partial x\partial y}-8y^{-1}(y')^{2}(y'')(y''')\frac{\partial^2\beta}{\partial y^2}$$

$$+12(y'')^{2}(y''')\frac{\partial^2\beta}{\partial y^2}+12(y')^{-2}(y'')^{3}(y''')\frac{\partial\beta}{\partial y}-5(y')^{-1}(y''')\frac{\partial^2\beta}{\partial x^2}$$

$$-10(y''')\frac{\partial^2\beta}{\partial x\partial y}-5(y')^{1}(y''')\frac{\partial^2\beta}{\partial y^2}-5(y')^{-1}(y'')(y''')\frac{\partial\beta}{\partial y}+y^{-2}(y')^{2}\frac{\partial^2\beta}{\partial x^2}$$

$$+2y^{-2}(y')^{3}\frac{\partial^2\beta}{\partial x\partial y}+y^{-2}(y')^{4}\frac{\partial^2\beta}{\partial y^2}+y^{-2}(y')^{2}(y'')\frac{\partial\beta}{\partial y}+16y^{-1}(y'')^{2}(y''')\frac{\partial\alpha}{\partial x}$$

$$+16y^{-1}(y')(y'')^{2}(y''')\frac{\partial\alpha}{\partial y}-24(y')^{-2}(y'')^{3}(y''')\frac{\partial\alpha}{\partial x}-24(y')^{-1}(y'')^{3}(y''')\frac{\partial\alpha}{\partial y}$$

$$+10(y')^{-1}(y'')(y''')\frac{\partial\alpha}{\partial x}+10(y'')(y''')\frac{\partial\alpha}{\partial y}-2y^{-2}(y')^{2}(y'')\frac{\partial\alpha}{\partial x}-2y^{-1}(y')^{3}(y'')\frac{\partial\alpha}{\partial y}$$

$$+8y^{-1}(y')(y'')(y''')\frac{\partial^2\alpha}{\partial x^2}+16y^{-1}(y')^{2}(y'')(y''')\frac{\partial^2\alpha}{\partial x\partial y}+8y^{-1}(y')^{3}(y'')(y''')\frac{\partial^2\alpha}{\partial y^2}$$

$$+8y^{-1}(y')(y'')^{2}(y''')\frac{\partial\alpha}{\partial y}-12(y')^{-1}(y'')^{2}(y''')\frac{\partial^2\alpha}{\partial x^2}-24(y'')^{2}(y''')\frac{\partial^2\alpha}{\partial x\partial y}$$

$$-12(y')^{1}(y'')^{2}(y''')\frac{\partial^2\alpha}{\partial y^2}-12(y')^{-1}(y'')^{3}(y''')\frac{\partial\alpha}{\partial y}+5(y''')\frac{\partial^2\alpha}{\partial x^2}+10(y')(y''')\frac{\partial^2\alpha}{\partial x\partial y}$$

$$+5(y')^{2}(y''')\frac{\partial^2\alpha}{\partial y^2}+5(y'')(y''')\frac{\partial\alpha}{\partial y}-y^{-2}(y')^{1}\frac{\partial^2\alpha}{\partial x^2}-2y^{-2}(y')^{2}\frac{\partial^2\alpha}{\partial x\partial y}$$

$$-y^{-2}(y')^{3}\frac{\partial^2\alpha}{\partial y^2}-y^{-2}(y')^{1}(y'')\frac{\partial\alpha}{\partial y}-5(y')^{-1}(y'')\frac{\partial^3\beta}{\partial x^3}-15(y'')\frac{\partial^3\beta}{\partial x^2\partial y}$$

$$-15(y')^{-1}(y'')^{2}\frac{\partial^2\beta}{\partial x\partial y}+5(y')^{-1}(y'')(y''')\frac{\partial\beta}{\partial y}-15(y')(y'')\frac{\partial^3\beta}{\partial x\partial y^2}-15(y'')^{2}\frac{\partial^2\beta}{\partial y^2}$$

$$-5(y')^{2}(y'')\frac{\partial^3\beta}{\partial y^3}+15(y')^{-1}(y'')(y''')\frac{\partial\alpha}{\partial x}$$

$$+15(y'')(y''')\frac{\partial\alpha}{\partial y}+15(y')^{-1}(y'')^{2}\frac{\partial^2\alpha}{\partial x^2}+30(y'')^{2}\frac{\partial^2\alpha}{\partial x\partial y}+15(y')^{1}(y'')^{2}\frac{\partial^2\alpha}{\partial y^2}$$

$$+15(y')^{-1}(y'')^{3}\frac{\partial\alpha}{\partial y}+5(y'')\frac{\partial^3\alpha}{\partial x^3}+15(y')(y'')\frac{\partial^3\alpha}{\partial x^2\partial y}+15(y'')^{2}\frac{\partial^2\alpha}{\partial x\partial y}$$

$$+5(y'')(y''')\frac{\partial\alpha}{\partial y}+15(y')^{2}(y'')\frac{\partial^3\alpha}{\partial x\partial y^2}+15(y')(y'')^{2}\frac{\partial^2\alpha}{\partial y^2}+5(y')^{3}(y'')\frac{\partial^3\alpha}{\partial y^3}$$

$$+3y^{-1}(y')\frac{\partial^3\beta}{\partial x^3}+9y^{-1}(y')^{2}\frac{\partial^3\beta}{\partial x^2\partial y}+9y^{-1}(y')(y'')\frac{\partial^2\beta}{\partial x\partial y}+3y^{-1}(y')(y''')\frac{\partial\beta}{\partial y}$$

$$+9y^{-1}(y)'^{3}\frac{\partial^3\beta}{\partial x\partial y^2}+9y^{-1}(y')^{2}(y'')\frac{\partial^2\beta}{\partial y^2}+3y^{-1}(y')^{4}\frac{\partial^3\beta}{\partial y^3}-27y^{-1}(y')(y''')\frac{\partial\alpha}{\partial x}$$

$$-27y^{-1}(y')^{2}(y''')\frac{\partial\alpha}{\partial y}-9y^{-1}(y')(y'')\frac{\partial^2\alpha}{\partial x^2}-18y^{-1}(y')^{2}(y'')\frac{\partial^2\alpha}{\partial x\partial y}-9y^{-1}(y')^{3}(y'')\frac{\partial^2\alpha}{\partial y^2}$$

$$-9y^{-1}(y')(y'')^{2}\frac{\partial\alpha}{\partial y}-3y^{-1}(y')^{2}\frac{\partial^3\alpha}{\partial x^3}-9y^{-1}(y')^{3}\frac{\partial^3\alpha}{\partial x^2\partial y}-9y^{-1}(y')^{2}(y'')\frac{\partial^2\alpha}{\partial x\partial y}$$

$$-8y^{-1}(y'')^{2}(y''')\frac{\partial\beta}{\partial y}+12(y')^{-2}(y'')^{2}(y''')\frac{\partial^2\beta}{\partial x^2}+24(y')^{-}\ldots$$

$$-3y^{-1}(y')^{2}(y''')\frac{\partial\alpha}{\partial y}-9y^{-1}(y')^{4}\frac{\partial^3\alpha}{\partial x\partial y^2}+9y^{-1}(y')^{3}(y'')\frac{\partial^2\alpha}{\partial y^2}-3y^{-1}(y')^{5}\frac{\partial^3\alpha}{\partial y^3}$$

$$+\frac{\partial^4\beta}{\partial x^4}+4y'\frac{\partial^4\beta}{\partial x^3\partial y}+6y''\frac{\partial^3\beta}{\partial x^2\partial y}+4y'''\frac{\partial^2\beta}{\partial x\partial y}$$

$$+4y^{-1}(y'')^2\frac{\partial\beta}{\partial y}-4(y')^{-2}(y'')^3\frac{\partial\beta}{\partial y}+5(y')^{-1}(y'')(y''')\frac{\partial\beta}{\partial y}-y^{-2}(y')^2(y'')\frac{\partial\beta}{\partial y}$$
$$-3y^{-1}(y')(y''')\frac{\partial\beta}{\partial y}+3(y')^2\frac{\partial^4\beta}{\partial x^2\partial y^2}+9(y')(y'')\frac{\partial^3\beta}{\partial x\partial y^2}+4(y')(y''')\frac{\partial^2\beta}{\partial y^2}$$
$$+3(y'')^2\frac{\partial^2\beta}{\partial y^2}+4(y')^3\frac{\partial^4\beta}{\partial x\partial y^3}+6y'^2y''\frac{\partial^3\beta}{\partial y^3}+3y'^2\frac{\partial^4\beta}{\partial x^2\partial y^2}$$
$$+3(y')(y'')\frac{\partial^3\beta}{\partial x\partial y^2}+(y')^4\frac{\partial^4\beta}{\partial y^4}-16y^{-1}(y'')^2\frac{\partial\alpha}{\partial x}-16y^{-1}(y')(y'')^2\frac{\partial\alpha}{\partial y}$$

$$+16(y')^{-2}(y'')^3\frac{\partial\alpha}{\partial x}+16(y')^{-1}(y'')^3\frac{\partial\alpha}{\partial y}-20(y')^{-1}(y'')(y''')\frac{\partial\alpha}{\partial x}-20(y'')(y''')\frac{\partial\alpha}{\partial y}$$
$$+4y^{-2}(y')^2(y'')\frac{\partial\alpha}{\partial x}+4y^{-2}(y')^3(y'')\frac{\partial\alpha}{\partial y}+12y^{-1}(y')(y''')\frac{\partial\alpha}{\partial x}+12y^{-1}(y')^2(y''')\frac{\partial\alpha}{\partial y}$$
$$-6(y''')\frac{\partial^2\alpha}{\partial x^2}-12(y')(y''')\frac{\partial^2\alpha}{\partial x\partial y}-6(y')^2(y''')\frac{\partial^2\alpha}{\partial y^2}-6(y'')(y''')\frac{\partial\alpha}{\partial y}$$
$$-4(y'')\frac{\partial^3\alpha}{\partial x^3}-12(y')(y'')\frac{\partial^3\alpha}{\partial x^2\partial y}-12(y'')^2\frac{\partial^2\alpha}{\partial x\partial y}-4(y'')(y''')\frac{\partial\alpha}{\partial y}$$
$$-12(y')^2(y'')\frac{\partial^3\alpha}{\partial x\partial y^2}-12(y')(y'')^2\frac{\partial^2\alpha}{\partial y^2}-4(y')^3(y'')\frac{\partial^3\alpha}{\partial y^3}-(y')\frac{\partial^4\alpha}{\partial x^4}$$
$$-4(y')^2\frac{\partial^4\alpha}{\partial x^3\partial y}-6(y')(y'')\frac{\partial^3\alpha}{\partial x^2\partial y}-4(y')(y''')\frac{\partial^2\alpha}{\partial x\partial y}-4y^{-1}(y')(y'')^2\frac{\partial\beta}{\partial y}$$
$$+4(y')^{-1}(y'')^3\frac{\partial\beta}{\partial y}-5(y'')(y''')\frac{\partial\beta}{\partial y}+y^{-2}(y')^3y''\frac{\partial\beta}{\partial y}+3y^{-1}(y')^2(y''')\frac{\partial\beta}{\partial y}$$
$$-3(y')^3\frac{\partial^4\alpha}{\partial x^2\partial y^2}-9(y')^2(y'')\frac{\partial^3\alpha}{\partial x\partial y^2}-4(y')^2(y''')\frac{\partial^2\alpha}{\partial y^2}-3(y')(y'')^2\frac{\partial^2\alpha}{\partial y^2}$$
$$-4(y')^4\frac{\partial^4\alpha}{\partial x\partial y^3}-6(y')^3y''\frac{\partial^3\alpha}{\partial y^3}-3(y')^3\frac{\partial^4\alpha}{\partial x^2\partial y^2}-3(y')^2(y'')\frac{\partial^3\alpha}{\partial x\partial y^2}$$
$$-(y')^5\frac{\partial^4\alpha}{\partial y^4}=0 \qquad (13)$$

Equation (13) is an identity in x, y, y', y'' and y''', i.e., it holds for any arbitrary choice of x, y, y', y'' and y''' [2]. Since α and β are functions of x and y only, we must equate the coefficients of the powers of y', y'', y''' and their combinations to zero. We obtain the following systems of partial differential equations known as determining equations ([2], [3])

$$(y')^3y''y''':8y^{-1}\frac{\partial^2\alpha}{\partial y^2}=0 \qquad (14)$$

$$(y')^2y''y''':-8y^{-1}\frac{\partial^2\beta}{\partial y^2}+16y^{-1}\frac{\partial^2\alpha}{\partial x\partial y}=0 \qquad (15)$$

$$(y')^1y''y''':-16y^{-1}\frac{\partial^2\beta}{\partial x\partial y}+8y^{-1}\frac{\partial^2\alpha}{\partial x^2}=0 \qquad (16)$$

$$(y')^0y''y''':-8y^{-1}\frac{\partial^2\beta}{\partial x^2}+5\frac{\partial\alpha}{\partial y}-5\frac{\partial\beta}{\partial y}=0 \qquad (17)$$

Integrating equation (14), we find

$$8y^{-1}\frac{\partial^2\alpha}{\partial y^2}=0\Rightarrow\frac{\partial^2\alpha}{\partial y^2}=0$$

$$\Rightarrow\frac{\partial\alpha}{\partial y}=c_1\Rightarrow\alpha=c_1y+c_2 \qquad (18)$$

where c_1 and c_2 are arbitrary functions of x. We substitute equation (18) in (15) and solve to find

$$-8y^{-1}\frac{\partial^2\beta}{\partial y^2}+16y^{-1}\frac{\partial^2\alpha}{\partial x\partial y}=0\Rightarrow\frac{\partial^2\beta}{\partial y^2}-2\frac{\partial^2\alpha}{\partial x\partial y}=0$$

$$\frac{\partial^2\beta}{\partial y^2}=2\frac{\partial}{\partial x}(c_1)\Rightarrow\frac{\partial^2\beta}{\partial y^2}=2c_1^1\Rightarrow\frac{\partial\beta}{\partial y}=2c_1^1y+c_3$$

$$\Rightarrow\beta=c'_1y^2+c_3y+c_4 \qquad (19)$$

where c_3 and c_4 are arbitrary functions of x. Substituting (18) and (19) in (16) we have

$$-2\frac{\partial^2\beta}{\partial x\partial y}+8y^{-1}\frac{\partial^2\alpha}{\partial x^2}=0\Rightarrow-16y^{-1}\frac{\partial^2\beta}{\partial x\partial y}+\frac{\partial^2\alpha}{\partial x^2}=0$$

$$\Rightarrow-2\frac{\partial}{\partial x}(2c'_1y+c_3)+c''_1y+c''_2=0$$
$$\Rightarrow-2(2c''_1y+c'_3)+c''_1y+c''_2=0$$
$$\Rightarrow-4c''_1y-2c'_3+c''_1y+c''_2=0$$
$$\Rightarrow-3c''_1y-2c'_3+c''_2=0 \qquad (20)$$

Since c_1, c_2 and c_3 depend on x only, we can now equate the coefficients of powers of y to zero. This yields

$$y^1:-3c''_1=0 \qquad (21)$$

$$y^0:-2c'_3+c''_2=0 \qquad (22)$$

Now we substitute (18) and (19) in (17) we obtain

$$-8y^{-1}\frac{\partial^2\beta}{\partial x^2}+5\frac{\partial\alpha}{\partial y}-5\frac{\partial\beta}{\partial y}=0$$

$$\Rightarrow-8y^{-1}\frac{\partial^2}{\partial x^2}(c_1^1y^2+c_3y+c_4)+5c_1-5(2c'_1y+c_3)=0$$

$$\Rightarrow-8c'''_1y-8c''_3-8c''_4y^{-1}+5c_1-10c'_1y-5c_3=0 \qquad (23)$$

Again we can equate the coefficients of powers of y to zero and we obtain

$$y^1:-8c'''_1-10c'_1=0 \qquad (24)$$

$$y^0:-8c''_3+5c_1-5c_3=0 \qquad (25)$$

$$y^{-1}:-8c''_4=0 \qquad (26)$$

We can now solve the differential equations (21), (22), (24), (25) and (26) as follows.
From (26) we have

$$-8c''_4=0\Rightarrow c''_4=0\Rightarrow c'_4=H_1\Rightarrow c_4=H_1x+H_2 \qquad (27)$$

Now from (21)

$$3c''_1=0\Rightarrow c''_1=0\Rightarrow c'_1=H_3\Rightarrow c_1=H_3x+H_4 \qquad (28)$$

Then considering (25), we have

$$-8c''_3+5(H_3x+H_4)-5c_3=0\Rightarrow 8c''_3+5c_3=5H_3x+5H_4 \quad (29)$$

since $c_1=H_3x+H_4$. The equation (28) is a nonhomogeneous differential equation. So on solving by using undetermined coefficients method, we find the complimentary solution from

$$8c''_3+5c_3=0 \quad (30)$$

The characteristic equation for this differential equation and its roots are

$$8r^2+5=0\Rightarrow r=\sqrt{\frac{5}{8}}i \quad (31)$$

The complimentary solution is then given by

$$c_{3_H}=H_5\cos\sqrt{\frac{5}{8}}x+H_6\sin\sqrt{\frac{5}{8}}x \quad (32)$$

Now proceeding with a particular solution

$$c_{3_P}=Ex+F \quad (33)$$

where E and F are arbitrary constants, we find

$$c'_{3_P}=E\Rightarrow c''_{3_P}=0 \quad (34)$$

Applying (32) and (33) into (28), we have

$$8(0)+5Ex+5F=5H_3x+5H_4$$
$$\Rightarrow 5Ex=5H_3x\Rightarrow E=H_3$$
$$\Rightarrow 5F=5H_4\Rightarrow F=H_4$$
$$\Rightarrow c_{3_P}=H_3x+H_4 \quad (35)$$

Hence

$$c_3=H_5\cos\sqrt{\frac{5}{8}}x+H_6\sin\sqrt{\frac{5}{8}}x+H_3x+H_4 \quad (36)$$

Finally from equation (22), we find

$$c''_2=2c'_3$$
$$c''_2=2\left(H_5\cos\sqrt{\frac{5}{8}}x+H_6\sin\sqrt{\frac{5}{8}}x+H_3x+H_4\right)'$$
$$c''_2=2\left(-\sqrt{\frac{5}{8}}H_5\sin\sqrt{\frac{5}{8}}x+\sqrt{\frac{5}{8}}H_6\cos\sqrt{\frac{5}{8}}x+H_3\right)$$
$$c''_2=-\sqrt{\frac{5}{2}}H_5\sin\sqrt{\frac{5}{8}}x+\sqrt{\frac{5}{2}}H_6\cos\sqrt{\frac{5}{8}}x+2H_3$$
$$c'_2=\frac{5}{4}H_5\cos\sqrt{\frac{5}{8}}x+\frac{5}{4}H_6\sin\sqrt{\frac{5}{8}}x+2H_3x+H_7$$

$$c_2=\frac{5}{4}\sqrt{\frac{5}{8}}H_5\sin\sqrt{\frac{5}{8}}x-\frac{5}{4}\sqrt{\frac{5}{8}}H_6\cos\sqrt{\frac{5}{8}}x+H_3x^2+H_7x+H_8$$

$$c_2=\frac{5}{16}\sqrt{10}H_5\sin\sqrt{\frac{5}{8}}x-\frac{5}{16}\sqrt{10}H_6\cos\sqrt{\frac{5}{8}}x+H_3x^2+H_7x+H_8 \quad (37)$$

where H_1, H_2, H_3, H_4, H_5, H_6, H_7 and H_8 are arbitrary constants. Substituting (28) and (37) in (18) to find

$$\alpha(x,y)=(H_3x+H_4)y+\frac{5}{16}\sqrt{10}H_5\sin\sqrt{\frac{5}{8}}x-\frac{5}{16}\sqrt{10}H_6\cos\sqrt{\frac{5}{8}}x+H_3x^2+H_7x+H_8 \quad (38)$$

$$\alpha(x,y)=H_3xy+H_4y+\frac{5}{16}\sqrt{10}H_5\sin\sqrt{\frac{5}{8}}x-\frac{5}{16}\sqrt{10}H_6\cos\sqrt{\frac{5}{8}}x+H_3x^2+H_7x+H_8 \quad (39)$$

We also apply (27), (28) and (36) in (19) to produce

$$\beta(x,y)=\left(H_3x+H_4\right)'y^2+\left(H_5\cos\sqrt{\frac{5}{8}}x+H_6\sin\sqrt{\frac{5}{8}}x+H_3x+H_4\right)y+H_1x+H_2 \quad (40)$$

$$\beta(x,y)=H_3y^2+H_5y\cos\sqrt{\frac{5}{8}}x+H_6y\sin\sqrt{\frac{5}{8}}x+H_3xy+H_4y+H_1x+H_2 \quad (41)$$

$$+H_6y\sin\sqrt{\frac{5}{8}}x \quad (42)$$

As a result, the generator G of the infinitesimal transformation is

$$G=\left(H_3x^2+H_3xy+H_4y+\frac{5}{16}\sqrt{10}H_5\sin\sqrt{\frac{5}{8}}x-\frac{5}{16}\sqrt{10}H_6\cos\sqrt{\frac{5}{8}}x+H_7x+H_8\right)\frac{\partial}{\partial x}+\left(H_1x+H_2+H_3xy+H_3y^2+H_4y+H_5y\cos\sqrt{\frac{5}{8}}x+H_6y\sin\sqrt{\frac{5}{8}}x\right)\frac{\partial}{\partial y} \quad (43)$$

$$\begin{aligned}G=&H_1\left(x\frac{\partial}{\partial y}\right)\\&+H_2\left(\frac{\partial}{\partial y}\right)\\&+H_3\left(xy\frac{\partial}{\partial x}+xy\frac{\partial}{\partial y}+y^2\frac{\partial}{\partial y}\right)\\&+H_4\left(y\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\\&+H_5\left(\left[\frac{5}{16}\sqrt{10}\sin\sqrt{\frac{5}{8}}x\right]\frac{\partial}{\partial x}+\left[\cos\sqrt{\frac{5}{8}}x\right]y\frac{\partial}{\partial y}\right)\\&+H_6\left(\left[-\frac{5}{16}\sqrt{10}\cos\sqrt{\frac{5}{8}}x\right]\frac{\partial}{\partial x}+\left[\sin\sqrt{\frac{5}{8}}x\right]y\frac{\partial}{\partial y}\right)\\&+H_7\left(x\frac{\partial}{\partial x}\right)\\&+H_8\left(\frac{\partial}{\partial x}\right)\end{aligned} \quad (44)$$

which is an eight parameter symmetry.

Any n- parameter symmetry may be separated into n- one-parameter symmetry by letting particular parameters take on specific values. Usually we set one parameter equal to one and the rest equal to zero in turn. If we do this in (44) we generate eight one parameter symmetries [5] as follows:

$$G_1=\frac{\partial}{\partial x}$$

$G_2=\frac{\partial}{\partial y}$

$G_3=x\frac{\partial}{\partial x}$

$G_4=x\frac{\partial}{\partial y}$

$G_5=y\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$

$G_6=xy\frac{\partial}{\partial x}+xy\frac{\partial}{\partial y}+y^2\frac{\partial}{\partial y}$

$G_7=\left[\frac{5}{16}\sqrt{10}sin\sqrt{\frac{5}{8}}x\right]\frac{\partial}{\partial x}+\left[cos\sqrt{\frac{5}{8}}x\right]y\frac{\partial}{\partial y}$

$G_8=\left[-\frac{5}{16}\sqrt{10}cos\sqrt{\frac{5}{8}}x\right]\frac{\partial}{\partial x}+\left[sin\sqrt{\frac{5}{8}}x\right]y\frac{\partial}{\partial y}$ (45)

with nonzero Lie brackets given by

$[G_1,G_3]=G_1$

$[G_1,G_4]=G_2$

$[G_3,G_4]=G_4$

(46)

Consider the three-dimensional sub algebra

$W_1=\frac{\partial}{\partial x}$

$W_3=x\frac{\partial}{\partial x}$

$W_4=x\frac{\partial}{\partial y}$ (47)

which are the Lie solvable algebra of the admitted eight one parameter symmetries (45).

Using the extended generators (prolongations) up to the fourth order:

$$G^{[4]}=G^{[3]}+\left(\beta^{(4)}-4y^{(4)}\alpha'-6y'''\alpha''-4y''\alpha'''-y'\alpha^{(4)}\right)\frac{\partial}{\partial y^{(4)}}$$

Yield:

(i) For the operator $W_1=\frac{\partial}{\partial x}$ the required fourth order prolongation which is

$$W_1^{[4]}=1.\frac{\partial}{\partial x}+0.\frac{\partial}{\partial y} \quad (48)$$

In order to integrate the fourth order equation (3), we must solve the following equation for the characteristic:

$$\frac{dx}{1}=\frac{dy}{0} \quad (49)$$

We obtain the following differential invariant:

$$y=u \quad (50)$$

(ii) For the operator $W_3=x\frac{\partial}{\partial x}$ the required fourth order prolongation which is

$$W_3^{[4]}=x\frac{\partial}{\partial x}-y'\frac{\partial}{\partial y'}-2y''\frac{\partial}{\partial y''}-3y'''\frac{\partial}{\partial y'''}-4y^{(4)}\frac{\partial}{\partial y^{(4)}} \quad (51)$$

In order to integrate the fourth order equation (3), we must solve the following equations for the characteristics:

$$\frac{dx}{x}=\frac{dy'}{-y'}=\frac{dy''}{-2y''}=\frac{dy''}{-3y'''}=\frac{dy^{(4)}}{-4y^{(4)}} \quad (52)$$

Leading to the following differential invariants:

$k_1=xy'$ (where k_1 is a constant)

$v_1=\frac{y''}{y'^2}$, $v_2=\frac{y'''}{y'^3}$, $v_3=\frac{y^{(4)}}{y'^4}$, $v_4=\frac{y'''^2}{y''^3}$, $v_5=\frac{y^{(4)}}{y''^2}$ and $v_5=\frac{y^{(4)3}}{y''^4}$.

(iii) For the operator $W_4=x\frac{\partial}{\partial y}$ the required fourth order prolongation which is

$$W_4^{[4]}=x\frac{\partial}{\partial y}+\frac{\partial}{\partial y'} \quad (53)$$

Let us now reduce equation (3) to a differential equation of lower order by using Lie algebra. We employ the method of invariant differentiation [6].

Two lower-order differential invariants of Lie algebra applicable are equations:

$$y=u,\ v_2=\frac{y'''}{y'^3} \quad (54)$$

The equation (3) can be reduced to the first order ODE as follows

$$\frac{dv}{du}=\frac{D_x(v)}{D_x(u)}$$

$$=\frac{D_x\left(y'''/y'^3\right)}{D_x(y)}$$

$$=4u^{-1}\frac{y''^2}{y'^4}-4\frac{y''^3}{y'^6}+5\frac{y''y'''}{y'^5}-u^{-2}\frac{y'^2y''}{y'^4}-3u^{-1}\frac{y'''}{y'^3}-3\frac{y''y'''}{y'^5} \quad (55)$$

or

$$\frac{dv}{du}=4u^{-1}u''^2u'^{-4}-4u''^3u'^{-6}+2u''u'^{-2}v-u^{-2}u'^{-2}u''-3u^{-1}v \quad (56)$$

and hence

$$\frac{dv}{du}+\left(3u^{-1}-2u''u'^{-2}\right)v=4u^{-1}u''^2u'^{-4}-4u''^3u'^{-6}-u^{-2}u'^{-2}u'' \quad (57)$$

Let

$$P(u)=3u^{-1}-2u''u'^{-2} \quad (58)$$

and

$$Q(u)=4u^{-1}u''^{2}u'^{-4}-4u''^{3}u'^{-6}-u^{-2}u'^{-2}u'' \qquad (59)$$

then our equation (57) is in the form

$$\frac{dv}{du}+P(u)v=Q(u) \qquad (60)$$

which is a first order linear equation.

Hence equation (3) reduces to first order linear equation (60) which is directly integrable.

Let the integrating factor be $I(u)$.
Hence

$$I(u)=e^{\int(3u^{-1}-2u''u'^{-2})du}$$
$$=e^{3\ln u+2u'^{-1}}$$
$$=u^{3}+e^{2u'^{-1}} \qquad (61)$$

Therefore

$$v=\frac{1}{u^{3}+e^{2u'^{-1}}}\int(u^{3}+e^{2u'^{-1}})(4u^{-1}u''^{2}u'^{-4}-4u''^{3}u'^{-6}-u^{-2}u'^{-2}u'')du \qquad (62)$$

completes the integration procedure and hence the general solution of the fourth order nonlinear wave equation (1).

CONCLUSION

In this work, we have looked at methods of group invariant solutions, based on the theory of continuous group of transformations, better known as 'Lie groups', acting on the space of independent and dependent variables of the system. We have reduced the equation to first order linear ordinary differential equation which we have then solved to find the general solution (62) of our problem given in equation (3).

REFERENCES

[1] Bluman, G.W., Kumei, S. (1989). *Symmetries and Differential Equations*; Springer-Verlag: New York, NY, USA.

[2] Dresner, L. (1999), *Applications of Lie's Theory of Ordinary and Partial Differential Equations*; London, Institute of Physics.

[3] Hydon, P. T. (2000), *Symmetry Methods for Differential Equations*: A Beginner's Guide, Cambridge, Cambridge University Press.

[4] Ibragimov, N. H. (1999). *Elementary Lie Group Analysis and Ordinary Differential Equations*; John Wiley and Sons Ltd, England.

[5] Mahomed, F. M., and Leach P G L. (1990). Symmetry Lie algebras of nth order ordinary differential equations *J. Math. Anal. Appl.* 151. 80-107.

[6] Nucci, M. C., Cerquetelli, T. and Ciccoli, N. (2002). Fourth Dimensional Lie Symmetry Algebra and Fourth Order Ordinary Differential Equations, *Journal of Nonlinear Mathematical Physics*, Volume 9, 24-35.

[7] Olver, P. J. (1986). *Applications of Lie Groups to Differential Equations*; Springer: New York, NY, USA.