BONDO UNIVERSITY COLLEGE

SCHOOL OF MATHEMATICS AND ACTUARIAL SCIENCE

FIRST YEAR FIRST SEMESTER EXAMINATION FOR THE DEGREE OF MASTER OF SCIENCE IN PURE MATHEMATICS

SMA 823: COMMUTATIVE ALGEBRA

INSTRUCTIONS:

1. This paper consists of FIVE questions

2. Attempt any THREE questions.

3. Observe further instructions on the answer booklet.

QUESTION 1 [20 Marks]

(a) Consider the polynomial ring $\mathbf{C}[x_1, ..., x_n]$. What are the ideals of this ring? **[5 mks]**

(b) Show that any maximal ideal is prime. **[5 mks]**

(c) Show that a commutative ring with $1 \neq 0$ is a field iff it has only the ideal (1) and the maximal ideal (0). **[5 mks]**

(d) Suppose a ring R is a module over itself and S is a subset of R. Find the ring structure of $S^{-1}R$. **[5 mks]**

QUESTION 2 [20 Marks]

(a) Let R be any commutative ring and $J \subset R$ a prime ideal. Show that R_J is a local ring. **[5 mks]**

(b) Let I be an ideal of a ring R. Define the radical of I and prove that the radical of I is an ideal if I is an ideal. **[5 mks]**

(c) Let M and N be $R-$ modules. Suppose $f : M \to N$ is an $R-$ module homomorphism, show that f is injective iff for every maximal ideal $J \subset R$, $f_J : M_J \to N_J$ is injective. **[10 mks]**

QUESTION 3 [20 Marks]

(a) Let R be a commutative ring and Spec$R$ be the collection of prime ideals in R. Define $U_f \subseteq$ Spec$R$ as subset of Spec$R$ consisting of primes not containing f. Show that U_f forms a basis for the Zariski topology. **[6 mks]**

(b) Suppose $f : M \to N$, $g : N \to P$ and $M \to N \to P$ is exact. Let $S \subset R$ be multiplicatively closed. Proving all the results involved, show that $S^{-1}M \to S^{-1}N \to S^{-1}P$ is exact. **[14 mks]**

QUESTION 4 **[20 Marks]**

(a) Let R be a commutative ring and M and N be $R-$modules. Define the tensor product of M and N. Hence or otherwise, show that:
i) the tensor product is symmetric;
ii) there is a canonical isomorphism $M \to \otimes_R R$, for each M.
iii) the tensor product is associative. **[10 mks]**

(b) Let R be a commutative ring. Show that an irreducible submodule $N \subset M$ is coprimary. **[10 mks]**

QUESTION 5 **[20 Marks]**

(a) Let R be a ring and $S \subset R$ be a multiplicatively closed subset. Show that if R is noetherian, then $S^{-1}R$ is noetherian. **[5 mks]**

(b) Using an example, demonstrate that the formation of tensor products is in general, not exact. **[5 mks]**

(c) i) Let (R, J) be a local artinian ring. Show that J is nilpotent **[7 mks]**.
ii) Show that if R is artinian, then there are only finitely many maximal ideals. **[3 mks]**