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Research Article

Characterization of Inner Derivations induced by Norm-attainable Operators

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Abstract

In the present paper, results on characterization of inner derivations in Banach algebras are discussed. Some techniques are employed for derivations due to Mecheri, Hacene, Bounkhel and Anderson**.** Let H be an infinite dimensional complex Hilbert space and B(H) the algebra of all bounded linear operators on H. A generalized derivation δ: $B(H) \to B(H)$ is defined by $\delta_{A,B}(X) = AX - XB$, for all $X \in B(H)$ and A,B fixed in B(H). An inner derivation is defined by $\delta_A(X) = AX - XA$, for all $X \in B(H)$ and A fixed in $B(H)$. Norms of inner derivations have been investigated by several mathematicians. However, it is noted that norms of inner derivations implemented by norm-attainable operators have not been considered to a great extent. In this study, we investigate properties of inner derivations which are strictly implemented by norm-attainable and we determine their norms. The derivations in this work are all implemented by norm-attainable operators. The results show that these derivations admit tensor norms of operators.

Keywords: Banach space; Hilbert space; Inner Derivation; Norms; Tensor Products.

Introduction

Derivation has been an area of interest for many mathematicians and researchers particularly their properties. There are several results on the studies of norms of inner derivations, aspects of underlying algebras of these derivations and the structures of the operators inducing these derivations [1]. A good number of these studies are on the conditions necessary for derivations (generalized) to be inner derivations. To begin with, Sakai [2] proved that every derivation on a von Neumann algebra or on a simple C*-algebra is inner. Elliot [3] used a generalized Sakai"s theorem (every derivation on a simple $C*$-algebra is determined by a multiplier) to show that every derivation in the class of separable approximately finite-dimensional $C*$-algebra can be approximated arbitrarily closely in norm by a derivation which is determined by a multiplier on a non-zero closed two-sided ideal and that the multiplier may be chosen to have norm bounded fixed multiple of the norm of the derivation.

The fact that the underlying algebra significantly affects the behaviour of the derivation in question, Elliot [4,5], later proved that every derivation of a $AW*$-algebra of type III (or of type I) is inner using properties of derivations of continuous fields of $C*$-algebras and that if a given quotient of an $AW*$-algebra is known to have only inner derivations, then its tensor product with a separable commutative $C*$-algebra with unit also has this property (i.e every derivation in it is inner). Kadison did a lot of research on the relationship between derivation and inner derivations of a $C*$-algebra and also extension of these to automorphisms. Indeed, [6,7] proved that each derivation of a $C*$-algebra $B(H)$ extends to an inner derivation of the weak operator closure of $B(H)$.

In [8] Kadison noted that since with respect to automorphisms, a derivation δ on a $C*$-algebra $B(H)$ of all bounded linear operators acting on a Hilbert space H is spatial when there is a bounded operator B on H such that an inner derivation $\delta A = BA - AB$, then there is no non-spatial derivations of $C*$-algebras and non-inner derivations of von Neumann algebras. In particular, he showed that each derivation of a hyperfinite von Neumann algebra is inner. In [8] used the idea of locality to formulate that every

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norm-continuous locally inner derivation on a von Neumann algebra into itself is an inner derivation. In [9] they discussed extensively on the range of the elementary operators. Some authors used topological approaches in their analysis of inner derivations while focusing on the underlying algebras. Other properties like continuity, linearity, trace, measurability, normality, spectra of inducing operator have been used in the analysis of derivations. For example, In [10] it was proved that any z-linear derivation on $L(M)$ is linear and hence is automatically continuous in the measure topology for a type I von Neumann algebra M with center z and a faithful normal semi-finite trace τ , so that $L(M)$ is the algebra of all τ-measurable operators affiliated with M. In [11], they used the spectrum of a generalized derivation $\delta A,B(X) = AX - XB$ to prove that the generalized derivation is convexoid if and only if the inducing operators A,B are convexoid and also investigated cases when generalized derivations are inner. He further proved that this is true for standard operator algebras.

Research Methodology

The key terms used in the methodology are defined here.

Definition 2.1 ([12], Definition 1.2) A Banach space is a complete normed space.

Definition 2.2 ([13], Definition 33.1) A Hilbert space H is an inner product space which is complete under the norm induced by its inner product.

Definition 2.3 ([14,] Definition 3.1) Let f be a function on an open subset U of a Banach space X into Banach space Y. f is Gateaux differentiable at $x \in U$ if there is bounded and linear operator $T: X \to Y$ such that $T_x(h) = \lim_{t\to 0}\frac{f(x+th)-f(x)}{t}$ for every $h \in X$. The operator T is called the Gateaux derivative of f at x.

Definition 2.4 ([15], Definition 0.1) Let X be a complex Banach space. Then $y \in X$ is orthogonal to $x \in X$ if for all complex λ there holds $\| x + \lambda y \| \geq \| x \|$.

Definition 2.5 ([16], Section 2) Let $T \in B(H)$ be compact. Then $s_1(T) \geq s_2(T) \geq \ldots \geq 0$ are the singular values of T i.e the eigenvalues of $\|T\| = (T^*T)^{\frac{1}{2}}$ counted according to multiplicity and arranged in descending order. For $1 \leq p \leq \infty$, $C_p = C_p(H)$ is the set of those compact $T \in B(H)$ with finite p-norm, $\| T \|_p = (\sum_{i=1}^{\infty} s_i(T)^p)^{\frac{1}{p}} = (tr|T|^p)^{\frac{1}{p}} < \infty$.

Results and Discussions

Lemma 3.1. Let H, K be Hilbert spaces and suppose that $u \in B(H)$ and $v \in B(K)$. Then there is a unique operator $(u \hat{\otimes} v \in B(H \hat{\otimes} K)$ such that $(u \hat{\otimes} v)(x \otimes y) = u(x) \otimes v(y)$ $(x \in H, y \in K)$. Moreover, $\|u \hat{\otimes} v\| = \|u\|\|v\|$.

Proof. The map $(u, v) \to u \otimes v$ is bilinear, so to show that $u \otimes v : H \otimes K \to H \otimes K$ is bounded, we may assume that u and v are unitaries [14], since the unitaries span the C*-algebras $B(H)$ and $B(K)$. If $z \in H \otimes K$, then we may write $z = \Sigma x_i \otimes yi$ where $y_1, \ldots, y_n$ are orthogonal. Hence, $\|(u \otimes v)(z)\|^2 = \Sigma_n u(x_i) \otimes {}_v(y_i)^2 = \|(x_i)\|^2\|(y_i)\|^2 = \|z\|^2$. Consequently, $\|u \otimes v\| = 1$. Thus, for all operators u, v on H, K respectively, the linear map $u \otimes v$ is bounded on $H \otimes K$ and hence has an extension to a bounded linear map $u \hat{\otimes} v$ on $H \hat{\otimes} K$. The maps $B(H) \to B(H \hat{\otimes} K)$ defined by $u \mapsto u \otimes idk$ (where idK is identity in K) and $B(K) \to B(H \hat{\otimes} K)$ defined by $v \to idK \otimes v$ (where idH is identity in H) are * homomorphisms and therefore isometric. Hence, $\|u \hat{\otimes} id\| = \|u\|$ and $\|id \hat{\otimes} v\| = \|v\|$. Therefore, $\|u \hat{\otimes} v\| = \|(u \hat{\otimes} id)(id \hat{\otimes} v)\| \leq \|u \hat{\otimes} id\|\|id \hat{\otimes} v\| = \|u\|\|v\|$.

If ϵ is a sufficiently small positive number, and if u, v are not zero, then there are unit vectors x and y such that $\|u(x)\| > \|u\| - \epsilon > 0$ and $\|v(y)\| > \|v\| - \epsilon > 0$. Hence,

$\|(u \hat{\otimes} v)(x \otimes y)\| = \|u(x)\|\|v(y)\| > (\|u\| - \epsilon)(\|v\| - \epsilon)$

$\Rightarrow \|u \hat{\otimes} v\| > (\|u\| - \epsilon)(\|v\| - \epsilon)$. As $\epsilon \to 0$ we obtain $\|u \hat{\otimes} v\| \geq \|u\|\|v\|$. *This completes the proof.*

Theorem 3.2 Let $T : H_1 \to H_2$ and $S : K_1 \to K_2$ be bounded operators between Hilbert spaces. Then there exists a unique bounded operator $T \hat{\otimes} S : H_1 \hat{\otimes} K_1 \to H_2 \hat{\otimes} K_2$ such that

$(T \hat{\otimes} S)(x \otimes y) = T(x) \otimes S(y)$ $\forall x \in H_1$ and $\forall y \in K_1$. Moreover, $\|T \hat{\otimes} S\| = \|T\|\|S\|$.

Proof. Since the algebraic tensor product $H_1 \otimes K_1$ is dense in $H_2 \otimes K_2$, there may exists at most one bounded operator satisfying the desired condition. Further, by the identity

$\|x \otimes y\| = \|x\|\|y\|$ for the norm in the Hilbert tensor product, for this hypothetical operator

$T \otimes S$ we would have from the definition of norm,

$\|T \hat{\otimes} S\| \geq \sup\{\|(T \hat{\otimes} S)(x \otimes y)\| : x \in B(H)_1, y \in B(K)_1\} = \sup\{\|T(x)\|\|S(y)\| : x \in BH1, y \in BK1\} = \|T\|\|S\|$. We must show that this operator indeed exists and $\|T \hat{\otimes} S\| \leq \|T\|\|S\|$. We state the following lemma which gives a solution.

Lemma 3.3. There exists a bounded operator $T \hat{\otimes} \mathrm{I} : H_1 \hat{\otimes} K_1 \to H_2 \otimes K_1$ such that

$(T \hat{\otimes} \mathrm{I})(x \otimes y) = T(x) \otimes y$ for all $x \in H_1$ and $y \in K_1$. Moreover, $\|T \hat{\otimes} \mathrm{I}\| \leq \|T\|$.

Proof. Consider the bilinear operator R: $H_1 \times K_1 \to H_2 \hat{\otimes} K_1 : (x, y)\ 7\to T(x) \otimes y$. Suppose $R' : H_1 \otimes K_1 \to H_2 \hat{\otimes} K_1$. Take $u \in H_1 \times K_1$, and a representation $u = \Sigma_n\ x_i \otimes y_i$. Without loss of generality, we can assume that the system $y_1, ..., y_n \in K_1$ is orthonormal.

The system $x_1 \otimes y_1, ..., x_n \otimes y_n \in H_1 \otimes K_1$ and $T(x_1) \otimes y_1, ..., T(x_n) \otimes y_n \in H_1 \otimes K_1$ is orthogonal in $H_2 \hat{\otimes} K_1$. Therefore, using the Pythagorean equality we have

$\|R'(u)\|^2 = \Sigma_n T(x_i) \otimes y_i^2 \|T(x_i)\|^2 \leq \|T\|^2 = \|T\|^2 \|u\|^2$.

Thus, R' is a bounded operator from the pre-Hilbert space $H_1 \otimes K_1$ to the Hilbert space $H_2 \hat{\otimes} K_1$, and $\|\mathbf{R}\| \leq \|T\|$. Extending this by continuity to the whole $H_1 \hat{\otimes} K_1$, we obtain the operator $T \hat{\otimes} \mathrm{I}$ with required properties. Now we complete the proof of the theorem. Similarly to the lemma, we obtain a bounded linear operator $\mathrm{I} \hat{\otimes} S : H_2 \hat{\otimes} K_1 \to H_2 \hat{\otimes} K_2$ such that $(\mathrm{I} \hat{\otimes} S)(x \otimes y) = x \otimes S(y)$ for all $x \in H_2$ and $y \in K_1$ and $\|\mathrm{I} \hat{\otimes} S\| \leq \|T\|$.

Put $T \hat{\otimes} S := (\mathrm{I} \hat{\otimes} S)(T \hat{\otimes} \mathrm{I}) : H_1 \hat{\otimes} K_1 \to H_2 \hat{\otimes} K_2$. By the multiplicative inequality for the operator norm, this operator is bounded and $\|T \hat{\otimes} S\| \leq \|T\|\|S\|$ but from the definition, $\|T \hat{\otimes} S\| \geq \|T\|\|S\|$ so $\|T \hat{\otimes} S\| = \|T\|\|S\|$.*This completes the proof.*

Theorem 3.4. If $a, b \in B(H)$, and let $a \otimes b$ denote the tensor product

of a and b then $\|a \otimes b + b \otimes a\| \leq \sqrt{(2\|a\|^2\|b\|^2 + 2\|b^*a\|^2.)}$

Proof.

$\|a \otimes b + b \otimes a\|^2 = \langle a \otimes b + b \otimes a, a \otimes b + b \otimes a\rangle$

$= \langle a \otimes b, a \otimes b\rangle + \langle a \otimes b, b \otimes a\rangle + \langle b \otimes a, a \otimes b\rangle + \langle b \otimes a, b \otimes a\rangle$

$= \langle a, a\rangle\langle b, b\rangle + \langle a, b\rangle\langle b, a\rangle + \langle b, a\rangle\langle a, b\rangle + \langle b, b\rangle\langle a, a\rangle$

$= \|a\|^2\|b\|^2 + \|b\|^2\|a\|^2 + \langle a, b\rangle\langle b, a\rangle + (\langle a, b\rangle)(\langle b, a\rangle)\ (\langle a, b\rangle = \langle b, a\rangle)$

$= \|a\|^2\|b\|^2 + \|b\|^2\|a\|^2 + 2\mathrm{Re}\langle a, b\rangle\langle b, a\rangle$

So by Cauchy-Schwarz inequality,

$\|(a \otimes b) + (b \otimes a)\|^2 \leq \|a\|^2\|b\|^2 + \|b\|^2\|a\|^2 + 2\|a\|\|b\|\|b\|\|a\| \leq 2\|a\|^2\|b\|^2 + 2\|a\|\|b\|\|b\|\|a\|$.

Therefore, $\|(a \otimes b) + (b \otimes a)\|^2 \leq 2\|a\|^2\|b\|^2 + 2\|a\|\|b\|\|b\|\|a\|$..........................(1)

But $\|b\| = \|b^*\|$ so replacing $\|b\|$ by $\|b^*\|$ in the second summand on the right hand side of Equation (1), we get $\|a \otimes b + b \otimes a\|^2 \leq 2\|a\|^2\|b\|^2 + 2\|b^*a\|^2$. Taking the positive square root on both sides yields the desired result.

Theorem 3.5. Let H be a complex Hilbert space, $B(H)$ the algebra of bounded linear operators on H. Let $\delta_{a,b} : B(H) \to B(H)$ be defined by $\delta_{a,b}(x) = ax - xb$, $\forall x \in B(H)$ where a, b are fixed in $B(H)$. Then $\|\delta_{a,b}\| = \|a\|\|b\|$.

Proof. By definition, $\|\delta_{a,b}|B(H)\| = \sup\{\|\delta_{a,b}(x)\| : x \in B(H), \|x\| = 1\}$. This implies that $\|\delta_{a,b}|B(H)\| \geq \|\delta_{a,b}(x)\|$, $\forall x \in B(H)$, $\|x\| = 1$. So $\forall \epsilon > 0$, $\|\delta_{a,\ b|}B(H)\| - \epsilon < \|\delta_{a,b}(x)\|$, $\forall x \in B(H)$, $\|x\| = 1$. But, $\|\delta_{a,b}|B(H)\| - \epsilon < \|ax - xb\| \leq \|a\|\|b\|$. Since ϵ is arbitrary, this implies that $\|\delta_{a,b}|B(H)\| \leq \|a\|\|b\|$. ....(2)

On the other hand, let $\xi, \eta \in H$, $\|\xi\| = \|\eta\| = 1$, $\phi \in H*$.

Now, $\|\delta_{a,b}|B(H)\| \geq \|\delta_{a,b}(x)\| : \forall x \in B(H)$, $\|x\| = 1$.

But, $\|\delta_{a,b}(x)\| = \sup\{\|(\delta_{a,b}(x))\eta\| : \forall \eta \in H, \|\eta\| = 1\} = \sup\{\|(ax - xb)\eta\| : \eta \in H, \|\eta\| = 1\}$.

Setting $a = (\phi \otimes \xi_1)$, $\forall \xi_1 \in H$, $\|\xi_1\| = 1$ and $b = (\varphi \otimes \xi_2)$, $\forall \xi_2 \in H$, $\|\xi_2\| = 1$, we have,

$\|\delta_{a,b}|B(H)\| \geq \|\delta_{a,b}(x)\| \geq \|(\delta_{a,b}(x))\eta\|$

$= \|(ax - xb)\eta\|$

$= \|((\phi \otimes \xi_1)x(\varphi \otimes \xi_2))\eta\|$

$= \|(\phi \otimes \xi_1)x - x(\varphi(\eta)\xi_2)\|$

$= \|(\phi \otimes \xi_1)\varphi(\eta)x - x(\xi_2)\|$

$= |\varphi(\eta)|\|(\phi \otimes \xi_1)x - x(\xi_2)\|$

$= \|a\|\|b\|$.

Therefore,

$\|\delta_{a,b}|B(H)\| \geq \|a\|\|b\|$. ....(3)

Hence by Inequalities (2) and (3), $\|\delta_{a,b}|B(H)\| = \|a\|\|b\|$.

Corollary 3.6. Let H be a complex Hilbert space, $B(H)$ the algebra of bounded linear operators on H. Let $\delta_a : NA(H) \to NA(H)$ be defined by $\delta_{a,b}(x) = ax - xa$, $\forall x \in NA(H)$ where a is fixed in $NA(H)$. Then $\|\delta_a\| = 2\|a\|$.

Proof. This follows from an analogous proof of Stampfli and the theorem above.

Conclusions

A number of research studies have been done on inner derivations and norm-attainability of operators and they have obtained fundamental results. Norms of derivations is a very interesting area of study in functional analysis and it has not been exhausted. In our case, we considered norms of generalized and inner derivations. Efforts thus can be directed on determining the lower estimate of the norm of a general elementary operator acting on general Banach algebras. The study of inner derivations is applicable in video imagery in near shore oceanographic field study, velocity spectral-digital computer derivation, embryonic stem cell line derived from human blastocysts, study on creep fatigue evaluation procedures for high-chronium steels and the study and interpretation of the chemical characteristics of natural water amongst others. This study forms a recipe for understanding quantum mechanics.

Conflicts of Interest

The authors hereby declare that they have no conflict of interest.

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