JARAMOGI OGINGA ODINGA UNIVERSITY OF SCIENCE AND TECHNOLOGY

SCHOOL OF BIOLOGICAL, PHYSICAL, MATHEMATICS AND ACTUARIAL SCIENCES

UNIVERSITY EXAMINATION FOR DEGREE OF BACHELOR OF EDUCATION AND ACTUARIAL SCIENCE

SPECIAL RESITS DECEMBER 2022

MAIN CAMPUS

COURSE CODE: WMB9402

COURSE TITLE: MEASURE THEORY

EXAM VENUE: **STREAM: BED AND ACT SCIENCE**

DATE: **EXAM SESSION:**

TIME: 2.00 HOURS

Instructions:

1. **Answer question one (compulsory) and any other two questions.**
2. **Candidates are advised not to write on the question paper.**
3. **Candidates must hand in their answer booklets to the invigilator while in the examination room.**

QUESTION ONE (30 MARKS)

a) i) State two properties of Lebesgue outer measure (2mks)

ii) Given the set $E = [0, 1]$, calculate the value of $m(E')$ where E' is the set of irrational number in E. (3mks)

iii) Prove that the outer measure of a singleton set is zero (3mks)

b) i) Show that for any sequence of set E_n, $m^*(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} m^*(E_n)$ (6mks)

ii) Calculate the outer measure of the following set $\bigcup_{y=1}^{\infty} \left\{x: \frac{1}{y+1} < x \leq \frac{1}{y}\right\}$ (3mks)

c) Show that if function $g(x)$ is measurable on a measurable set A, then $|g(x)|$ is also measurable (5mks)

d) i) Prove that if E is a countable set, then $m^*(E) = 0$ (5mks)

ii) Give an example of a set with outer measure zero but not countable. (1mks)

iii)Show that interval $[a, b]$ is not countable. (2mks)

QUESTION TWO (20 MARKS)

a) i) Describe two differences and similarities between the Riemann and Lebesgue Integrals. (4mks)

ii)Prove that the Dirichlet function defined by

$$f(x) = \begin{cases} 1, x \text{ rational} \\ 0, x \text{ irrational} \end{cases}$$

fails to have a Riemann integral over any interval $[a, b]$. Prove further that the Lebesgue intergral of $f(x)$ of any measurable set A exist and is equal to zero (8mks)

b) i) State Caratheodory's measurability criteria (3mks)

ii) Prove that if $m^*(E) = 0$, then $m^*(E \cup F) = m^*(F)$ for any set F (5mks)

QUESTION THREE (20 MARKS)

a) Define a σ −algebra (3mks)

b) Show that if E_1 and E_2 are measurable, then $E_1 \cup E_2$ is measurable (7mks)

c) Prove that if $E$is measurable, then$E + x_o$is measurable. (5mks)

d) Prove that if $g(x)$ and □(x) are equivalent functions on a set A and $g(x)$ is measurable, then □(x) is also measurable.
(5mks)

QUESTION FOUR (20 MARKS)

a) i) Describe three forms of measure (3mks)

ii) Define a property of almost everywhere in a set (2mks)

b) Let $f(x)$ be defined in the intervals $0 \leq x \leq 1$ as follows;

$$f(x) = \begin{cases} 2, x\ rational \\ 3, x\ irrational \end{cases}$$

Calculate the Lebesgue integral of this function. (7mks)

c) Show that if$f$ ismeasurable function, then$\{x: f(x) = \alpha\}$is measurable for each extended real number α. (4mks)

d) Show that if a set E has positive outer measure, then there is a bounded subset of E that also has positive outer measure. (4mks)

QUESTION FIVE (20 MARKS)

a) Show that the outer measure of an interval equals its length (10mks)

b) State and prove Monotone convergence theorem (10mks)