WWJMRD 2023; 9(01): 87-90
www.wwjmrd.com
International Journal
Peer Reviewed Journal
Refereed Journal
Indexed Journal
Impact Factor SJIF 2017: 5.182 2018: 5.51, (ISI) 2020-2021: 1.361
E-ISSN: 2454-6615

Odhiambo Francis
Department of Pure and Applied Mathematics Jaramogi Oginga Odinga University of Science and Technology, Bondo-Kenya.

Titus Aminer
Department of Pure and Applied Mathematics Jaramogi Oginga Odinga University of Science and Technology, Bondo-Kenya.

Benard Okelo
Department of Pure and Applied Mathematics Jaramogi Oginga Odinga University of Science and Technology, Bondo-Kenya.

Julius Manyala
Department of Natural Resource Management Jaramogi Oginga Odinga University of Science and Technology, Bondo-Kenya.

Correspondence:
Odhiambo Francis
Department of Pure and Applied Mathematics Jaramogi Oginga Odinga University of Science and Technology, Bondo-Kenya.

Formulation of a Four-Species Food Web System with Prey Refuge and Holling Type III Functional Response

Odhiambo Francis, Titus Aminer, Benard Okelo, Julius Manyala

Abstract
The coexistence of interacting biological species is a vital issue for the management of existing resources and the prediction of the long-term survival of each species. Many species become extinct due to several factors including over-exploitation among others. Suitable measures such as restriction on harvesting, creation of reserved zones among others are key in saving these species. Multi-species models incorporating prey refuge with Holling type I functional response have been studied with recommendations on their extension to include either a type II or type III functional responses. However, type II functional responses are de-stabilizing and can lead to extinction since they cause maximum mortality at low prey density. Hence, this study focused on the adoption of the idea of the reserved zone in formulating an ecological model with a Holling type III functional response.

Keywords: Food web, Prey Refuges, Functional Response, Stability, Holling Types.

1. Introduction
The cohabitation of all species in ecological systems is critical for the management of available resources and the prediction of each species' long-term survival. As a result, one fundamental way to understanding the ecological interplay between prey and predator species is to construct a mathematical model. Many species have become extinct and even others are on the verge of extinction as a result of factors such as over-predation, environmental pollution, mismanagement of natural resources, and over-exploitation, among others [11]. Protecting these species from extinction, according to Diz-Pita and Otero [4], requires well-established measures such as creation of reserve zones and harvesting restrictions among others, to allow them grow without external disruptions. Therefore, more realistic and plausible mathematical models require critical considerations of aspects such as carrying capacity, competition, prey or predator harvesting, and predator functional responses, which determine the dynamical stability of the systems under consideration to a greater extent [7]. Dunn and Hovel [6] state that characterizing a predator-prey interaction necessitates the identification of the predator's functional response (the rate of prey consumption by an average predator). To completely comprehend how predators and prey interact and hence provide a thorough picture of predator population dynamics, knowledge of functional response is essential. In [2], Holling identified three main types of functional responses namely: Holling types I, II and III and their impact on prey killed per unit time. He designated type I as the scenario in which there is a linear relationship between the number of preys consumed and prey density, and type II as the condition in which each consumer's consumption rate decreases as prey density increases, eventually saturating at a constant value of prey consumption. Type III behavior occurs when the gradient of the curve climbs and subsequently drops with increasing prey density, resulting in a sigmoidal behavior linked to the presence of learning behavior in the predator population, according to Mada et al [9]. Because the prey isocline in a type III response always has a stabilizing negative slope at low prey densities, systems with a type III response are more likely to be stable if predators are extremely efficient [2]. Researchers investigating predator-prey dynamics are increasingly

interested in the existence of these protected areas, often known as refuges. The majority of studies have found that refuges have both a stabilizing and a destabilizing influence [3]. According to Peter and Lev [10], refuges which protect a constant number of preys have a stronger stabilizing effect on population dynamics than the refuges which protect a constant proportion of prey. Shireen [12], studied a multi-species model incorporating prey refuge with a prey-predator type of functional response between the prey in the unreserved zone and the specialist predator. In his recommendations, he suggested an extension of his model describing similar multispecies interactions with a standard predator-prey system but with an inclusion of a Holling type II functional response. Later, Dawit [4] studied the effect of prey refuge on a three species food web system with a Holling type II functional response and investigated the impact of prey refuge on the dynamical stability of the system. However, type II functional responses are de-stabilizing and can lead to extinction since they cause maximum mortality at low prey density [4]. In this study, we intended to improve on the existing works on multi-species systems by focusing on the adoption of the idea of the reserved zone in formulating and analyzing an ecological model by allowing a Holling type III functional response between the prey in the unreserved zone and the third component in the web and a linear type functional response between the specialist and top predators and also between the prey in the unreserved zone and generalist predator. The model allows the generalist predator to feed on both the specialist predator and prey in the unreserved zone, resulting into a food web type of species interactions.

2. Preliminaries

Functional responses

The functional response measures the rate at which an average predator consumes prey per unit time, meaning that the number of preys consumed per unit time varies with prey density. In [10], Holling discovered the following three basic forms of functional responses and their impact on the number of preys killed per unit time.

(i) Holling type I
(ii) Holling type II
(iii) Holling type III

Holling type I

The Lotka-Volterra type of functional response also known as Holling type I, predicts a linear increase in the rate of prey consumption for each individual predator as prey density increases. Letting N be the number of preys consumed, x be the prey density, T_s the time available for searching the prey and a be the discovery rate, we have

$$N = aT_s x,$$

where $x \geq 0$, and aT_s is the rate of consumption of prey by an individual predator. Other than for manageable populations and passive feeders like web-building spiders and filter-feeding species, Holling type I functional responses are uncommon in most feeding patterns [4].

Holling type II

Michaelis and Manfen presented a reaction in which each consumer's consumption rate rises at a decreasing pace with prey density until it reaches a steady state at satiation [5].

The type II response accounts for handling and ingestion time, b in addition to the time available for searching for each individual prey that is consumed, hence the searching time is reduced to $T_s = T_t - bN$. Combining this equation with $N = aT_s x$ gives

$$N = aT_t x - abxN$$

$$\Rightarrow N = \frac{aT_t x}{1 + abx}$$

Which is the type II formulation.

According to [1], prey consumed saturates, and at low densities, prey has a high per capita mortality rate, which decreases as density grows. Because the type II reaction causes greatest mortality at low prey density, it is destabilizing for predator-prey interactions.

Holling type III

In [8], Holling also proposed the type III response function which can be viewed as a generalization of type II response in the form

$$N = \frac{aT_t x^k}{1 + abx^k}$$

Where k is an integer.

A type-III functional response depicts scenarios in which an individual predator's consumption rate increases with high prey numbers and subsequently drops toward saturation, resulting in a typical S-shaped response curve. Generally, the function

$$N = \frac{x^k}{a + x^k}$$

represents a functional response of predator to prey, which is called Holling type II if k = 1 and Holling type III if k = 2.

Generalist predators

Are consumers who eat more than one type of prey available to them. When a generalist predator chooses to consume one of the available preys, it may have to eat less of the other prey as a result; hence, the consumption rate of one prey by a generalist predator can fluctuate even if the density of this prey remains constant.

Food Webs

A food web is an ecological conceptual tool for demonstrating feeding relationships among species within a community, explaining species interactions and community structure, and comprehending the dynamics of energy transmission in an ecosystem from its source in plants to herbivores to predators [7].

3. Model Formulation

Consider a four-species food web system comprising of two prey species and two predator species, living in an environment divided into two disjoint regions namely, the reserved and unreserved zones, where the predators are not allowed to enter the former.

Let us denote by $x_1(t)$, $x_2(t)$, $y(t)$ and $z(t)$ the biomass population density of prey in the unprotected region, prey in the protected zone, specialist predator and generalist predator respectively at any time, t. The standard predator-prey model with a logistic growth in the prey species $x(t)$ in the presence of predator $y(t)$, would be given by,

$$\frac{dx}{dt} = kx\left(1-\frac{x}{l}\right) - \alpha_1 q(x)y$$
$$\frac{dy}{dt} = -e_1 y + \alpha_2 p(x)y \qquad (1)$$

where the term $q(x)$ represents the interaction between the prey and predator and e_1 represents the natural death rate of the predator in the absence of prey.

If we then divide the prey species $x(t)$ into $x_1(t)$, $x_2(t)$ representing the prey in the unreserved and reserved zones respectively and only allow the prey species to freely move in and out of the reserved zone, we add one more equation to System (1) and end up with

$$\frac{dx_1}{dt} = k_1x_1\left(1-\frac{x_1}{l_1}\right) - \beta_1x_1 + \beta_2x_2 - \alpha_1 q_1(x_1)y$$
$$\frac{dx_2}{dt} = k_2x_2\left(1-\frac{x_2}{l_2}\right) + \beta_1x_1 - \beta_2x_2 \qquad (2)$$
$$\frac{dy}{dt} = \alpha_2 q_1(x_1)y - e_1y$$

where β_1 is the migration rate coefficient of the prey species from the unreserved to reserved zone, while β_2 is the migration rate coefficient of prey species from the reserved to unreserved area and both prey species grow logistically in the absence of predators.

Let's now allow the predator $y(t)$ to only feed on the prey in the unreserved zone $x_1(t)$, hence termed as specialist predator and introduce another top predator $z(t)$ which feeds both on the specialist predator $y(t)$ and prey in the unreserved zone $x_1(t)$, a fourth equation is added to the system (2) to form,

$$\frac{dx_1}{dt} = k_1x_1\left(1-\frac{x_1}{l_1}\right) - \beta_1x_1 + \beta_2x_2 - \alpha_1 q_1(x_1)y - \sigma_1 q_2(x_1)z$$
$$\frac{dx_2}{dt} = k_2x_2\left(1-\frac{x_2}{l_2}\right) + \beta_1x_1 - \beta_2x_2 \qquad (3)$$
$$\frac{dy}{dt} = \alpha_2 q_1(x_1)y - e_1y - \delta_1 q_3(y)z$$
$$\frac{dz}{dt} = \sigma_2x_1z + \delta_2yz - e_2z,$$

In which e_2 is the natural death rate of the generalist predator and $q_3(y)$ is the interaction term between the specialist and generalist predators. $q_2(x_1)$ is the interaction term between generalist predator and prey in the unreserved region.

It can be noted from System (3) that the specialist predator $y(t)$ only feeds on the unprotected prey $x_1(t)$ according to a linear type functional response. Similarly, the generalist predator $z(t)$ feeds on both unprotected prey $x_1(t)$ and specialist predator $y(t)$ according to a linear type functional response, that is, $q_2(x_1)\,z$ and $q_3(y)\,z$ respectively.

If we now let the specialist predator $y(t)$ to feed on unreserved prey according to a Holling Type III functional response, then $q_1(x_1)y$ modifies to

$$q_1(x_1)y = \frac{x_1^2y}{b+x_1^2}$$

where b is the average saturation rate and the generalist predator retains a linear type feeding pattern on both specialist predator and prey in the unreserved zone. Finally, we end up with our system given by:

$$\frac{dx_1}{dt} = k_1x_1\left(1-\frac{x_1}{l_1}\right) - \beta_1x_1 + \beta_2x_2 - \alpha_1\left(\frac{x_1^2y}{b+x_1^2}\right) - \sigma_1x_1z$$
$$\frac{dx_2}{dt} = k_2x_2\left(1-\frac{x_2}{l_2}\right) + \beta_1x_1 - \beta_2x_2 \qquad (4)$$
$$\frac{dy}{dt} = \alpha_2\left(\frac{x_1^2y}{b+x_1^2}\right) - e_1y - \delta_1yz$$
$$\frac{dz}{dt} = \sigma_2x_1z + \delta_2yz - e_2z,$$

where $x_1(0) \geq 0$, $x_2(0) \geq 0$, $y(0) \geq 0$ *and* $z(0) \geq 0$, and all the parameters of the model are assumed to be positive. The functions σ_1x_1 and δ_1y represent the linear type functional responses while the function $\left(\frac{x_1^2}{b+x_1^2}\right)$ represents the Holling type 3 functional response.

Remark

The System (4) is an improvement of system studied by Shireen [12] in the sense that, it:

a) Has one prey species which is classified as $x_1(t)$ and $x_2(t)$ representing the prey in the unreserved and reserved zones respectively.
b) Incorporates a Holling type III functional response $f(x)=\left(\frac{x_1^2}{b+x_1^2}\right)$ between the specialist predator and prey in the unreserved zone and a linear type of functional response between the generalist and prey in the unreserved zone and between the generalist and specialist predators
c) Allows a selection of more than one functional response as suggested by Mada et al [9], which would probably enhance the stability of the system (4) (a Holling type III between a specialist predator and unpreserved prey, and a linear type functional response between the generalist predator and both specialist and unprotected prey.)

The System (4) is formulated under the following assumptions:

Assumptions

1. The environment contains one prey species and two predator species. The prey species is separated into two classes $x_1(t)$, which is the population density of prey in the non-protected zone and $x_2(t)$, which is the density of the prey in the reserved zone where predators are not allowed to enter.
2. The prey species grow logistically in the absence of the predators.
3. The rate of predator populations' increase depends on the amount of prey biomass converted as food.
4. External factors such as drought, fires, epidemics are either stable or have similar effects on the interacting species.
5. There is constant prey growth hence constant prey biomass conversion by predators.
6. Not all consumed prey is converted into newborn predators.
7. The rate of movement of prey species in and out of the protected zones is constant.
8. Generalist predators feed on both specialist predator and unprotected prey.
9. The population is in dynamic equilibrium

Parameter Descriptions of System (4)

$x_1(t)$, $x_2(t)$	population density of prey in the unreserved and reserved regions respectively at any time, t
$y(t)$	population density of specialist predators who only feed on unprotected prey.
$z(t)$	population density of generalist predators who feed on both specialist predators and unprotected prey.
α_1	attack rate of the specialist predator on the prey in the unreserved zone.
σ_1	attack rate of the generalist predator on unprotected prey.
δ_1	attack rate of top predator on specialist predator.
α_2	conversion rate of the unprotected prey to a specialist predator.
σ_2	conversion rate of unprotected prey to a generalist predator.
$\delta 2$	conversion rate of specialist predator to generalist predator.
l_1, l_2	carrying capacities for preys $x_1(t)$ and $x_2(t)$ respectively.
k_1, k_2	per capita intrinsic growth rates for preys $x_1(t)$ and $x_2(t)$ respectively.
β_1	migration rate coefficient of prey species from unreserved to reserved regions.
β_2	migration rate coefficient of prey species from the reserved area to unreserved area
e_1, e_2	specialist and generalist predators' natural death rates respectively.

Conclusion

In this paper, we have formulated a four-species food web system with prey refuge and Holling type III functional response.

References

1. Ahmed B. A. and Dawit M. G., (2017). Mathematical Modeling of a Predator-Prey Model with Modified Leslie-Gower and Holling II Schemes, *Global Journal of Science Frontier Research: F Mathematics and Decision Sciences*, 17(3).
2. Dawes J. H. P. and Souza M. O., (2013). A derivation of Holling's type I, II and III functional responses in predator-prey systems, *Journal of Theoretical Biology*.
3. Dawit M. and Abraha H., (2021). The effect of prey refuge on the dynamics of three species food web system, *Ethiop. J. Sci. Technol*, 14(2), 105-121.
4. Diz-Pita E., Otero-Espinar M.V., (2021). Predator-prey Models: *A Review of Some Recent Advances; Mathematics (MDPI)*, 9, 17-83.
5. Cabello T. and Gamez M., (2007). An Improvement of the Holling type III Functional Response in Entomophagous Species Model, *Journal of Biological Systems*: 15(4), 515-524.
6. Dunn R. P. and Hovel K. A., (2020). Predator type influences the frequency of functional responses to prey in marine habitats, *Biol. Lett.* 16.
7. Efrem T. W., (2015). *Mathematical Modelling of food web system with prey harvesting*, A Thesis submitted to the department of mathematics, college of natural science, school of graduate studies Arba Minch University, a partial ful_llment of the requirements for the Master of Science.
8. Jeschke J. M, Kopp M. and Tollrian R., (2004). Consumer food systems: Why type I functional responses are exclusive to filter feeders, *Biol. Rev*, 79, 337-349.
9. Mada S. et al., (2013). Mathematical Model of Three Species Food Chain with Holling Type-III Functional response, *International Journal of Pure and Applied Mathematics*, 89(5), 647-657.
10. Peter A. A. and Lev R. G., (2000). The nature of predation: prey dependent, ratio dependent or neither? *TREE*, 15(8).
11. Raid K. N. and Shireen R. J., (2016). The dynamics of prey-predator model with a reserved zone, *World Journal of Modelling and Simulation*, 12(3), 175-188.
12. Shireen R. J., (2018). *Modelling, Dynamics and Analysis of Multi-Species Systems with Prey Refuge*, A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy, Department of Mathematics Brunel University London.