JARAMOGI OGINGA ODINGA UNIVERSITY OF SCIENCE AND TECHNOLOGY

SCHOOL OF BIOLOGICAL, PHYSICAL, MATHEMATICS AND ACTUARIAL SCIENCES

UNIVERSITY EXAMINATION FOR DEGREE OF BACHELOR OF EDUCATION

3RD YEAR 1ST SEMESTER 2023/2024 ACADEMIC YEAR

(MAIN)

COURSE CODE: WMB 9312

COURSE TITLE: GROUP THEORY

EXAM VENUE: **STREAM: (EDUCATION)**

DATE: **EXAM SESSION:**

TIME: 2.00 HOURS

Instructions:

1. **Answer question 1 (Compulsory) and ANY other 2 questions**
2. **Candidates are advised not to write on the question paper.**
3. **Candidates must hand in their answer booklets to the invigilator while in the examination room.**

QUESTION ONE (30 marks)

a) Define the following terms as used in Group Theory
   i) A homomorphism from a group $(G,*)$ to another group $(H,\dagger)$. (2 marks)
   ii) The centre of a group G. (2 marks)
   iii) The index of a subgroup. (1 mark)

b) Let $(G,*)$ be a group. Then prove that for each element $x \in G$, the inverse $\bar{x}$ in $(G,*)$ is unique. (4 marks)

c) Determine whether the set $(\mathbb{Z},\cdot)$ is a group or not, where $\cdot$ is the usual multiplication on $\mathbb{Z}$. (5 marks)

d) Explain why the following subset of $\mathbb{Z}$ is not subgroup of $(\mathbb{Z},+)$:
   The set $\{n \in \mathbb{Z}: n \geq 0\}$ of all non-negative integers. (2 marks)

e) Let G be a group. If $a, b, c \in G$ with $ab = cb$, then show that $a = c$. (4 marks)

f) If n is a positive integer, then show that the set $n\mathbb{Z} := \{nk: k \in \mathbb{Z}\}$ of all multiples of n *is a subgroup of* $(\mathbb{Z},+)$. (4 marks)

g) Determine the order of the following
   i) $2 \in (\mathbb{Z}_7,\cdot)$. (2 marks)
   ii) $(\mathbb{R},+)$. (1 mark)

h) Show that every subgroup of an abelian group in a normal subgroup. (3 marks)

QUESTION TWO (20 marks)

a) Consider the group $(\mathbb{Z}_5 \setminus \{0\},\times)$.
   i) Draw the Cayley table of the cyclic group, $(\mathbb{Z}_5 \setminus \{0\},\times)$. (4 marks)
   ii) Determine the identity element of the group. (1 mark)
   iii) Determine the inverse of each element of the group. (4 marks)
   iv) Determine the subgroups of the group,$(\mathbb{Z}_5 \setminus \{0\},\times)$. (4 marks)
   v) Determine the generators of $(\mathbb{Z}_5 \setminus \{0\},\times)$. Is the group cyclic? (3 marks)

b) If a is an element of a group of order n, then show that $a^n = e$, where e is the identity element of the group. (4 marks)

QUESTION THREE (20 marks)

a) Define the term permutation in group theory (2 marks)

b) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 2 & 1 & 5 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 5 & 1 \end{pmatrix}$ be permutations.

i) Compute $\alpha \circ \beta$ and $\beta \circ \alpha$ giving your answer both in matrix and cycle notation. (6 marks)

ii) Determine α^{-1} giving your answer both in matrix and cycle notation. (2 marks)

iii) Draw the graphs for α and β. (4 marks)

iv) Determine the sign of β (2 marks)

c) Write the following permutation $(1,2,3)(4,6,8)$ in matrix notation. (2 marks)

d) State the First Isomorphism Theorem. (2 marks)

QUESTION FOUR (20 marks)

a) Fill in the missing parts of the Cayley table of a group given below. (4 marks)

$*$	a	b	c
a	c		
b			c
c			

b) Determine the identity element of the group in a) above. (1 mark)

c) Determine the inverse of each element in the group in a) above. (4 marks)

d) Let $f: G \to H$ be a homomorphism from a group $(G,*)$ and $(H,\dagger)$. Define

i) The image of f. (2 marks)

ii) The kernel of f. (2 marks)

e) Use the subgroup test to prove that the centre of a group, $Z(G)$ is a subgroup of a group.G. (7 marks)

QUESTION FIVE (20 marks)

a) Let $(G,*)$, $(H,\odot)$ and $(M,\dagger)$ be groups, $f: G \to H$ and $g: H \to M$ be homomorphisms. then prove that $f \circ g: G \to M$ and $g^{-1}: M \to H$ are also homomorphisms. (10 marks)

b) Let H be a normal subgroup of G. Then show that $G \setminus H$ is a group under multiplication of left cosets and that the mapping $\phi: G \to G \setminus H$ defined by $\phi(a) = aH, \forall a \in G$ is a homomorphism. (10 marks)