JARAMOGI OGINGA ODINGA UNIVERSITY OF SCIENCE AND TECHNOLOGY

SCHOOL OF BIOLOGICAL, PHYSICAL, MATHEMATICS AND ACTUARIAL SCIENCES

UNIVERSITY EXAMINATION FOR DEGREE OF BACHELOR OF EDUCATION AND ACTUARIAL SCIENCE

4TH YEAR 2ND SEMESTER 2023/2024 ACADEMIC YEAR

MAIN CAMPUS

COURSE CODE: WMB9402

COURSE TITLE: MEASURE THEORY

EXAM VENUE: **STREAM: BED AND ACT SCIENCE**

DATE: **EXAM SESSION:**

TIME: 2.00 HOURS

Instructions:

1. **Answer question one (compulsory) and any other two questions.**
2. **Candidates are advised not to write on the question paper.**
3. **Candidates must hand in their answer booklets to the invigilator while in the examination room.**

QUESTION ONE (30 MARKS) [*COMPULSORY*]

a) Compute the outer Lebesgue measure of the set $W = (4, 17] \cup \{7\}$. (5mks)

b) Show that the Dirichlet function $f: [0,1] \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$$

fails to have a Riemann integral over any interval [0, 1]. (5mks)

c) Suppose g and h are measurable functions and k is a scalar, prove measurability of the following: (i). *kh* (ii). *g-h*. (5mks)

d) Let $f(x)$ be a step function defined as: $g(x) = \begin{cases} 1 & -2 < x < 4 \\ 2 & 4 \leq x < 8 \\ 3 & 8 \leq x \leq 16 \\ 0 & otherwise \end{cases}$

Calculate the Lebesgue integral of $g(x)$. (5mks)

e) Let $(f_n(x))_{n=1}^{\infty}$ be a sequence of nonnegative Lebesgue measurable functions defined on a Lebesgue measurable set A. Suppose that:

i) $0 \leq f_1(x) \leq f_2(x) \leq \cdots \leq f_n(x) \leq f(x)$ for all $x \in A$,

ii) $(f_n(x))_{n=1}^{\infty}$ converges pointwise to $f(x)$ on A.

Show that: $\lim_{n\to\infty} \int_A f_n \, d\mu = \int_A f d\mu$. (5mks)

f) Diagrammatically describe translation invariance as a property of outer measure. (5mks)

QUESTION TWO (20 MARKS)

Consider the measure space $([0, 1), \mathcal{B}([0, 1)), \lambda)$, where $\mathcal{B}([0, 1))$ is the Borel σ- algebra restricted to [0, 1) and λ is the restriction of Lebesgue measure on [0, 1). Define the transformation T: $[0, 1) \rightarrow [0, 1)$ given by:

$$T(x) = \begin{cases} 3x, & 0 < x < 1/3 \\ \frac{3x}{2} - \frac{1}{2}, & \frac{1}{3} < x < 1 \end{cases}$$

(a) Show that T is $\mathcal{B}([0, 1))/\mathcal{B}([0, 1))$- measurable, and determine the image measure $T(\lambda) = \lambda \circ T^{-1}$. (7mks)

(b) Let $C = \{A \in \mathcal{B}([0, 1)) : \lambda(T^{-1}A \Delta A = 0\}$. Show that C is a σ-algebra.(8mks)

(c) Suppose $A \in \mathcal{B}([0, 1))$ satisfies the property that $T^{-1}(A) = A$ and $0 < \lambda(A) < 1$. Define μ_1, μ_2 on $\mathcal{B}([0, 1))$ that satisfy the properties of a measure. (5mks)

QUESTION THREE (20 MARKS)

Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra over $\mathbb{R}$ and λ is Lebesgue measure. Define f on $\mathbb{R}$ by $f(x) = 2x\mathbf{1}_{[0,1)}(x)$.

(a) Show that f is $B(\mathbb{R})/B(\mathbb{R})$-measurable. (5mks)

(b) Find a sequence (f_n) in $\mathbb{R}^+$ such that $f_n \longrightarrow f$. (5mks)

(c) Determine the value of $\int f \, d\mu$ using Lebesque integral technique. (5mks)

(d) Let $\mathcal{C} = \sigma(\ \{\{x\} : x \in [0, 1)\ )$ and $\mathcal{A} = \{A \in [0, 2) : A$ is countable or A^c is countable .$\}$ Show that f is $\mathcal{C}/\mathcal{A}$-measurable. (5mks)

QUESTION FOUR (20 MARKS)

Consider the measurable space $([0, 1], \mathcal{B}([0, 1]))$, and λ is the restriction of Lebesgue measure to $[0, 1]$, and let $A \quad \mathcal{B}([0, 1])$ be such that $\lambda(A) = 1/2$. Consider the real function defined on $[0, 1]$, by $f(x)= \lambda(A \cap [0,x])$

a) Show that for any $x, y \in [0, 1]$, we have $|f(x)-f(y)|\leq|x-y|$. (10mks)

b) Show that f is $B(\mathbb{R})/B(\mathbb{R})$-measurable. (10mks)

QUESTION FIVE (20 MARKS)

One of the applications of Measure Theory is in the study of fixed points. In topology, the application is seen in Brouwer's fixed point theorem which states as follows: Let X be a non-empty, compact, convex subset of a Euclidean space, and let $f:X\to X$ be a continuous function. Then, there exists a point $x\in X$ such that $f(x)=x$. In mathematical analysis, we have Banach's fixed point theorem which states that if you have a mapping T on a complete metric space X, and this mapping is a contraction (meaning it brings points closer together), then there exists a unique fixed-point x^* in X that remains unchanged under the action of T. These fundamental results are useful in real life situations. Analyze the two pictures below and describe mathematically how the fixed-point theorems are useful in the two scenarios under integration. If the porcupine is disturbed, is there an equivalence with the whirlwind? Can one comb the porcupine until all the spines lie flat?

Picture 1: Disturbed porcupine

Picture 2: Whirlwind