ON BIRKHOFF-JAMES ORTHOGONALITY AND NORM-ATTAINABILITY OF OPERATORS IN BANACH SPACES

BY

OTAE LAMECH WASONGA

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SCHOOL OF BIOLOGICAL, PHYSICAL, MATHEMATICS AND ACTUARIAL SCIENCES

JARAMOGI OGINGA ODINGA UNIVERSITY OF SCIENCE AND TECHNOLOGY

Declaration

This thesis is my own work and has not been presented for a degree award in any other institution.

OTAE LAMECH WASONGA

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Signature .... Date ....

This thesis has been submitted for examination with our approval as the university supervisors.

1. Prof. Benard Okelo

Department of Pure and Applied Mathematics

Jaramogi Oginga Odinga University of Science and Technology, Kenya

Signature .... Date ....

2. Prof. Benard Nzimbi

Department of Mathematics

University of Nairobi, Kenya

Signature .... Date ....

3. Prof. Omolo Ongati

Department of Pure and Applied Mathematics

Jaramogi Oginga Odinga University of Science and Technology, Kenya

Signature .... Date ....

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Dedication

To my beloved wife, Maclyne Makungu and sons, Lexy Francis Otae and Cutrone Wallace Otae Odindo.

Abstract

Characterizing geometric properties in Banach spaces in terms of their mappings has been done for a long period of time however it remains a very difficult task to complete due to the complex underlying structures in the Banach spaces. Recently, of interest has been the norm-attainability and orthogonality aspects in Banach space setting in general. To characterize these properties, one requires a geometrical view of the problem and this brings into the picture the concept of Birkhoff-James orthogonality in order to solve the problem. The main objective of this study is to establish norm-attainability conditions of operators via Birkhoff-James orthogonality in Banach spaces. The specific objectives include to: Establish Birkhoff-James orthogonality conditions for operators in Banach spaces; Determine norm-attainability of operators in Banach spaces via Birkhoff-James orthogonality and; Investigate the relationship between the set of norm-attainable vectors and the set of norm-attainable operators via Birkhoff-James orthogonality in Banach spaces. The research methodology involved the use of known orthogonality criterion in normed spaces, technical approaches such as polar decomposition and tensor products and some known inequalities such as triangle inequality and Cauchy-Schwarz inequality. The results show that operators are norm-attainable in Banach spaces via Birkhoff-James orthogonality. Moreover, there is a strong relationship between the set of norm-attainable operators and the set of norm-attainable vectors. The results of this study are useful in understanding the concept of orthogonal projections and has applications in optimization theory and convex analysis.

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Chapter 1

INTRODUCTION

1.1 Mathematical Background

Studies on structural and geometrical properties in mathematical spaces have been studied over the past decades as shown in [53] and the references therein. Some of these properties include norms, numerical ranges, spectra, orthogonality, positivity, invertibility, normality, norm-attainability among others [126]. Recently, a lot of interest has emerged in the study of Banach space particularly the geometrical properties [25]. Characterizing geometrical properties in Banach spaces in terms of their mappings is very difficult due to the complex underlying structures in the Banach spaces (see [106] and [107]). Recently, of interest has been the norm-attainability aspect in Banach space setting in general.

Let $\mathfrak{B}$ be a Banach space and $\mathcal{S}$ be a unit sphere in $\mathfrak{B}$. One of the open questions that has not been answered which was posed by Sain in [117] states as follows: Suppose that $T \in \mathfrak{B}$ is bounded and linear. Find a necessary and sufficient condition for $x \in \mathcal{S}$ to be such that x belongs to the set of norm-attainable vectors. To determine such conditions one requires a geometrical view of the problem and this brings into the picture the concept of orthogonality [27]. In this regard therefore, Birkoff-James orthogonality (BJO) comes into play in order to solve this problem.

In this study we consider two aspects namely BJO and norm-attainability in Banach spaces. These two properties remain interesting since there are a lot of open questions which have not been answered. We scrutinize each of the properties independently then later, we establish the points of convergence. We consider a property in Banach spaces called the norm-attainability. This property has been considered by many authors for instance Okelo [91], characterized norm-attainable operators on Hilbert spaces. Moreover, the study considered elementary operators on Banach algebras. In this study, conditions under which operators become norm-attainable are unveiled. More on norm-attainable operators can be found in [20], [39], [41], [42], [44], [87], [88], [89], [90], [128] and the references therein. In [13], the authors constructed examples of operators in Hilbert spaces where the operators fail to be BJO in the Hilbert space setting. Sain and Paul [117], in the same spirit, characterized operator norm-attainability via BJO. In [8], the authors set out to determine to what extent the BJO can be applied to answer questions about the norm. They studied in more details the properties of the norm that BJO can discern. In their study, they found out that BJO determines when a space is Euclidean. Also, they found out that the BJO, in isolation, knows when the space is finite dimensional and it can be applied to compute the dimensions of the underlying finite dimensional spaces. They equally asserted that BJO can be used to determine when the norm is smooth and when it is strictly convex. Additionally, they discerned that through

BJO, smooth norms of reflexive Banach spaces up to their linear isometries can be established. They used the concept of directed graphs and ortho-graphs induced by BJO relations on normed spaces to offer proofs to these assertions about the BJO and its knowledge on the norm. In [118], the authors characterized the operators that achieve norms. They studied this property via the weakly convergence sequences that exist in the intersection of closed unit ball and they asserted that any compact operator achieves an absolute norm in the closed unit sphere. They investigated the finite rank operators to check if they also attain norms. In their investigation, they found out that the finite sum of all finite rank operators also achieve norms. In this investigation, one of the conditions that such operators achieve norms is that they first have to be positive. More on finite rank operators can be found in [54], [55], [60], [61] and [63]. Stampfli [125], gave an outstanding result regarding orthogonality of norms of inner derivations, $\delta_A(X)$ and corresponding orthogonality of norms of generalized derivations. Anderson [3] characterized inner derivations with orthogonality and gave a fundamental inequality, $\|AX - XA + T\| \geq \|T\|, \forall X \in B(\mathcal{H})$ which implies that the range of the inner derivation is BJO to its kernel. In this case, orthogonality is defined in the sense of Birkhoff [18] where we take $A, B \in \mathcal{H}$ to be orthogonal if $\|A+\lambda B\| \geq \|A\|$. We note that this sense of orthogonality of the range and kernel of derivations has been achieved via normal operators by quite a number of researchers. More on the orthogonality of the range and kernel can be found in [64], [66], [67], [73], [74], [75] and [76] and their respective references. Mecheri [77] used the Birkhoff-James sense of orthogonality to generalize Anderson's inequality to normal operators A, B and T such that $AT = TB$ and obtained the orthogonality result of the derivation, a study that was later improved on by the authors in studies carried out in [70], [71], [78], [79] and [80].

We now concentrate on the BJO in the general case and explore some studies that have been carried out over the years by different scholars. We note however, that BJO is still interesting since its complete characterization has not been done. To consider particular researches, Pal and Roy [101] studied dilating mappings and BJO and showed that operators on Banach spaces can be characterized in terms of dilations via BJO. The authors further constructed unitary dilations from pairs of commuting contractions and demonstrated that these unitary dilations preserve BJO.

Birkhoff [17], in the study of metric spaces, considered orthogonality and obtained that orthogonality is preserved in real linear metric spaces. The study showed that the maps on the real linear metric spaces are preservers of orthogonality. However, at the time of the study, the work did not consider general Banach spaces. It is worth noting that after the work of [17], James [57], also came up with characterization of orthogonality by considering inner products so that the work was of a higher level of Hilbert spaces, and gave examples in normed spaces(NS). Moreover, in [56], the author considered the dual space of Banach spaces and characterized functionals in terms of orthogonality particularly in normed spaces. Andruchow et al [4], studied BJO of self-adjoint operators in members of the subspace diagonal operators via the operator norm. Also, in [59], the authors characterized BJO of complex Hermitian matrices in the trace norm and some given positive semidefinite operators where they also went ahead to find applications to BJO in other fields such as information systems. In [30], Chmieliński et al extended the study of BJO and studied the notion of ϵ- approximate BJO where they showed the homogeneity of approximate BJO at some point of ϵ where the approximate orthogonality coincides with the BJO. Moreover, in [7], the authors characterized BJO for elements of Hilbert C^*-modules.

In [50], the authors, in a more recent case, further explored the symmetry of BJO of matrices restricted to real Hilbert spaces and explored spaces different from inner product spaces where BJO is symmetric. They asserted that given a Hilbert space, then an operator in the Hilbert space is a left symmetric point provided it is a zero operator. Additionally, Sain [115], extended this study of the symmetry of BJO and explored the left symmetry of BJO of operators defined on Banach spaces. Bhatia and Semrl in [16], did a generalization of the definition of BJO in terms of matrices in any complex space. They studied BJO in finite dimensional Hilbert spaces and asserted that, given two operators in the Hilbert space, then the two operators are BJO to each other provided there exists an element x in the set of all norm-attainable operators such that $T_x \perp_{BJ} A_x$, where $T, A \in \mathcal{H}$. They also advanced the study of BJO via the operator sequences where they asserted that two operators say, A and B are BJO if there exists a sequence such that the limit of the norm of the sequence converges to the operator norm of A and that A_x and B_x are orthogonal in the inner product space. This result opened up many studies on orthogonality of bounded operators in Hilbert spaces. They also introduced the concept of the study of BJO via the Schatten p-norms of operators where they proved that whenever $p = 2$, then the given BJO is equivalent to the usual orthogonality in Hilbert spaces. Additionally, Bhatia and Semrl [16], asserted that given two operators say, A and B in Banach spaces, then there exists necessary and sufficient conditions for the two operators to be BJO to each other, in the set of all complex numbers. They investigated these conditions under some special cases one of which was the condition that the operator B is the identity operator. They applied this result in computations of some distance problems in Banach spaces. In [129], the authors studied BJO of compact linear operators in comparison between normed spaces and Hilbert spaces and developed a characterization of BJO in finite dimensional Hilbert spaces.

Lummer [69], introduced the concept of Semi-inner-product(s.i.p) in the study of BJO so as to employ the characterization of BJO in the setting of Hilbert spaces extended to normed spaces and as a result, presented every normed linear space as a s.i.p space. In [119], the author characterized BJO on reflexive Banach spaces on connected subsets of the unit sphere and proved BJO on account of existence of a norm-attainable vector in Banach spaces. In [58], the authors elaborated the geometrical properties like smoothness and rotundity in characterizing BJO in Hilbert spaces and extended the study of these notions into some work on geometric properties of the operator spaces. In a more recent study, Paul et al in [97], established sufficient conditions in bounded linear operators for smoothness in the same space via BJO and developed important results that later formed reference points for the study of BJO. Li and Schneider in [67], characterized BJO with reference to the extreme points of the dual unit ball of a convex set and the convex hull of linear operators in Hilbert spaces whenever the orthogonal operators are finite dimensional, and applied this study in finding best approximation of elements in different spaces. They showed that two elements are BJO if and only if the zero operator is the best approximation to one of the elements in the span of the other. The authors, in the same work, proved that the Bhatia-Semrl theorem cannot be extended into the study of BJO in arbitrary finite-dimensional Banach spaces. Closely related to the study by Li and Schneider, Singer [124], used the relationship between best approximation and BJO to advance the study of BJO in linear functionals and the convex hull of extreme points in unit balls of linear functionals.

In [110], the authors characterized orthogonality for any two spaces and the extreme points of the dual ball of some Hilbert space. They did this through the tensor product of the extreme points of reflexive operators in Hilbert spaces. Miguel et al [81], provided a general result characterizing BJO in Banach spaces with regards to the actions of elements in norming sets. They applied the approach of the numerical range which allowed them to characterize smooth points in multilinear maps and polynomials. They also went ahead to obtain results of BJO through the numerical radius of operators in Hilbert spaces. More on numerical range and radius can be found in [106], [107], [108] and [122] and the references therein. The authors also provided results in line with the Bhatia-Semrl Property by omitting the convex hull and the use of limits in establishing BJO in Banach spaces. Their main result was pegged on vector-valued continuous functions on compact Hausdorff spaces. The authors further characterized BJO in Banach spaces of all Lipschitz maps, spaces where tensor products are endowed with norms, which they called injective tensor products, operator normed spaces and finally, the spectral properties of operators such as the numerical radii. They finally studied possible extensions of the BSP on BJO showing results in compact linear operators on reflexive spaces and finite Blaschke products. Their results found very good applications in the study of spear vectors and spear operators where they showed that in a Banach space, there is no smooth point that can be found to be orthogonal to a spear vector. In [98], the authors introduced the notion of the directional orthogonality where they asserted that, given two vectors say p and q, then $p \perp_{BJ} q$ if and only p is directionally orthogonal to q, that is $p \perp_\gamma q$, where $\perp_\gamma$ denotes directional orthogonality.

In [58], the author studied elaborately on the notions of smoothness of operators through BJO and used this study in giving insights in the study of geometrical properties of spaces of operators. The author characterized inner product spaces and used BJO relation to assert that the set of all real numbers of dimension at least three is an inner product space provided that for any three elements, the property of left additivity in BJO is achieved. The author also went ahead to prove that, under the same conditions, even the property of right additivity is obtained in general.

Paul, Sain and Arpita in [98], offered a complete generalization of operators that are bounded and linear in Hilbert spaces. They located an element say y in the norm-attaining set such that given two operators say, Q and R, then it results that Q_y is BJO to R_y.

Benitez et al in [14], proved that a space X qualifies to be called an IPS with a provision that for any two matrices, say P and Q, in a normed space X, if P is BJ-orthogonal to Q and that there exists an element say v picked from a unit sphere such that $\|Pv\| = \|v\|$.

In [132], the authors gave a slightly different notion of BJO using angular projections between two vectors say p and q in normed spaces, and they called this angle, the g-angle. They achieved this characterization through the use of the limits in Gateaux derivative. They developed sequences in their proof and exploited the continuity and the convergence of the sequences so as to achieve their characterization.

In [7], the authors used the theorem due to Gelfand-Mazur to prove the isomorphism between the space of C^*-algebra and the space of all complex numbers, and as a result, led them into the characterization of BJO in the dual of Hilbert spaces.

In the year 2013, Sain and Paul in [117], developed a linkage between Bhatia-Semrl criterion and norm-attaining operators on connected closed subsets of the unit sphere. They asserted that linear operators say K and M, satisfy the condition of BJO if there exists an element in a closed subset of the unit sphere. A converse of the result obtained in [117], was established in [118], where they showed that if a bounded linear operator satisfies BSP, then the set of unit vectors where the linear operator attains its norm is a connected projective space. The slight difference in the results depended on the nature of the set of all norm-attainable operators.

In [100], the authors proved the notion of convexity of normed linear spaces where linear operators attain norms, and used the concept of strong BJO where they proved that this concept of orthogonality implies BJO and the converse is not true in general. They gave an illustration with elements picked from $\mathbb{R}^2$ space to show that BJO does not imply strong BJO.

In [72], the authors studied BJO through the concept called norm-parallelism where they asserted that two elements are linearly dependent if and only if they are norm-parallel. Through norm-parallelism, the authors introduced a new geometrical property called semi-rotund points of a Hilbert space. They used this new property to study the linkages between strong BJO and BJO itself.

In [130], the authors developed the study of parallelism and its approximations and used this concept to establish BJO of linear and bounded operators in normed spaces. They did this through the establishment of a sequence x_n from the set of all unit vectors and developed limits of convergence to achieve orthogonality in normed spaces.

In [23], the authors introduced a new generalization of BJO of bounded linear operators that gives a clear linkage between BJO and other notions of orthogonality in Banach spaces in terms of boundedness of linear operators and their adjoints, and even the null space of such operators.

In [121], Sain and Tanaka characterized BJO through a re-definition of smooth points to help advance the establishment of the left and right symmetry via BJO and asserted that if an element is both left and right symmetric, then it is called a symmetric point.

In [62], the author gave a different methodology of the proof of BSP by first making a computation of the φ-Gateaux derivative of the individual norms. The author employed this computation of Gateaux derivative to characterize BJO in the attempt to offer a different line of proof to Bhatia-Semrl theorem and he generalized this theorem to Hilbert C^*-modules of the underlying C^*-algebras.

Light and Cheney in [68], explored the theory of approximation as far as approximality in various tensor products is concerned. They established a very interesting link between the theory of approximality and BJO. They achieved the linkage through the application of the concept of cross norms where they asserted that given to sets of elements a_1, a_2 and b_1, b_2 in Banach spaces, then their respective tensor products are BJO.

Mohit and Ranjana in [82], when they explored BJO in certain tensor products of Banach spaces, managed to successfully study the relationship that exists between BJO of elementary tensors in various tensor products and the roles they play in BJO of elements in Banach spaces.

In [65], the authors therein, studied elements in C^*-algebras where BJO is locally symmetric for all elements in a positive cone.

Bose, in [22], developed a study on l_p-direct sums and located the duals of such spaces. The author also characterized the support functionals of elements in normed spaces and studied smoothness and approximate smoothness. These results enabled the author to obtain non-smooth spaces where non-zero elements are approximately smooth. These results further enabled them to characterize BJO and the point-wise symmetry in the spaces so developed.

James, in [57] characterized BJO for support functionals. The author introduced the concept of right-additivity in BJO and claimed that a non-zero point of a Banach space is smooth given the provision that the point satisfies the right-additivity in BJO. The author also went ahead to prove that given the existence of any elements say, a and b in a Banach space, then the elements satisfy BJO if a= 0 or when $f(y) = 0$, provided there exists some support functional say, h of a. James, in [58], again proved that in any normed linear space whose dimension is 3 or more, then BJO is symmetric for all IPS. This establishment enabled the au-

thor to characterize BJO in usual sense in the inner product space.
In [105], the authors solved the operator version of BJO where they developed a compact optimization of Y-valued compact operators for the minimax formula in a distance function and proved BJO for continuous vector valued functions in any separable reflexive Banach spaces.
In [99], the authors used the existence property of BJO and the minimal norm operator to characterize orthogonality in Banach spaces. They showed that, for any bounded linear operators say M and N, there exists a complex scalar α_0 such that orthogonality is achieved between the operators M and N. They proved the existence of the scalar α_0 via the existence of a norm one sequence and they showed that the scalar α_0 is unique if the approximate point spectrum of the operator M does not contain zero. They finally proved that if M is orthogonal to N, then the operator norm of N must exhibit the following norm property, that is , $\|N\| = \sup\{|\langle N_u, v\rangle| : \|u\| = 1, \|v\| = 1, \langle M_u, v\rangle = 1\}$.
In [57], the authors studied different properties of orthogonality that included both right and left uniqueness of orthogonality in Banach spaces. They used some of these properties to characterize properties such as Gateaux differentiability among other techniques. They asserted that there exists a right-uniqueness in BJO given the existence of an element x different from zero and another element y if there exists more than one constant say λ such that $x \perp \lambda x + y$, and for the left-uniqueness, the assertion is left defined as is the right-uniqueness. They went a notch higher and proved that a necessary and sufficient condition for normed linear spaces to be strictly convex is that the orthogonality in the said space must be left-unique.

In [85], the author considered orthogonality preserving properties under sesquilinear maps and studied some characterization of self-adjoint sesquilinear forms. They asserted that sesquilinear forms qualify to be inner products if an operator satisfying the sesquilinear form is a positive invertible operator.

In [51], the authors investigated BJO of operators on the class $(\mathbb{R}^n, \|.\|)$. They proved that orthogonality of two operators is achieved if one of the operators is derived from the class $(\mathbb{R}^n, \|.\|)$ if and only if the operator from the class $(\mathbb{R}^n, \|.\|)$ is norm-attainable. They also proved that for the operators P and Q, then $P \perp_{BJ} Q$ implies that $Q \perp_{BJ} P$ for the operator P derived from the class $(\mathbb{R}^n, \|.\|)$ provided that the operator Q is also norm-attainable.

The works of [17] and [57] considered together brought about the famous BJO concept and since its introduction a lot of advances have been carried out in this direction.

Next, we consider the aspect of the relationship between the set of norm-attainable vectors and the set of norm-attainable operators. It is a well known fact that BJO can be used to ascertain norm-attainability [113]. However, a clear relationship between the set of all norm-attainable vectors and the set of all norm-attainable operators is not known [57]. There is very little in the literature regarding this relationship. Therefore, in this study we embark on the task of determining this relationship. There has been a strong relationship between BJO and norm-attainability. In fact the latter can be characterized via the former [109]. It is in this perspective that it is interesting to study these two aspects simultaneously in order to answer the question that was posed in the beginning of this section. To carry out this work successfully, we need some basic concepts that are useful in understanding this work.

1.2 Basic Concepts

In this section, we discuss concepts which are of importance to this study.

Definition 1.1. ([88], Definition 2.1). An operator T in on Banach space $\mathfrak{B}$ is said to be norm-attainable if there exists a unit vector $x \in \mathfrak{B}$ such that $\|Tx\| = \|T\|$.

Remark 1.2. We note that $x \in \mathfrak{B}$ in Definition 1.1 is called a norm-attainable vector [94] and the set of all norm-attainable vectors is denoted by $\mathfrak{NA}_v$. Moreover, the set of all norm-attainable operators is denoted by $\mathfrak{NA}_{op}$.

Definition 1.3. ([113], Definition 2.5). Let $\mathcal{H}$ be a Hilbert space. Two elements $x, y \in \mathcal{H}$ are said to be orthogonal denoted by $x \perp y$, if $\langle x, y\rangle = 0$. We say that subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{H}$ are orthogonal written as $\mathcal{A} \perp \mathcal{B}$, if $x \perp y$ for every $x \in \mathcal{A}$ and $y \in \mathcal{B}$.

Definition 1.4. ([86], Definition 3.4) Let $\mathcal{B}$ be a Banach space . Let x be a unit vector in $\mathcal{B}$ and $y \in \mathcal{B}$. Then $x \perp_{BJ} y$ if and only if $\|x + \lambda y\| \geq \|x\|, \forall \lambda \in \mathbb{K}$.

Definition 1.5. ([34], Definition 2.9) Let $\mathcal{B}$ be a Banach space. For any $\epsilon \in [0, 1)$, x is said to be Birkhoff-James ϵ- orthogonal to y denoted by $x \perp_{BJ}^{\epsilon} y$ if $\|x + \lambda y\| \geq \sqrt{1-\epsilon^2}\|x\|$, for all $\lambda \in \mathbb{C}$.

Remark 1.6. (i). In an Inner Product Space (IPS) $(\mathcal{B}, \langle ., . \rangle), x \perp^{\epsilon} y$ if $|\langle x, y \rangle| \leq \epsilon \|x\| \|y\|$.

(ii). For all $n \in \mathbb{N}$, we can extend this definition as follows; $x \perp_{BJ}^{\epsilon} y$ if $\|x + \lambda y\| \geq \sqrt{1 - \epsilon^n} \|x\|, \forall \lambda \in \mathbb{C}$.

Definition 1.7. ([100], Definition 1.8) For any linear space $\mathcal{X}, x \in \mathcal{X}$ is said to be strongly orthogonal to y in the sense of Birkhoff-James if and only if $\|x\| < \|x + \lambda y\|$, for all $\lambda \neq 0$.

Remark 1.8. [100] The notation $x \perp_B^S y$ is used to indicate the strongly BJO.

Definition 1.9. ([113], Definition 3.6) In real Banach space X, the sets x^+ and x^- are defined as follows;

(i) $x^+ = \{y \in X : \|x + \lambda y\| \geq \|x\|, \forall \lambda \geq 0\}$, and

(ii) $x^- = \{y \in X : \|x + \lambda y\| \geq \|x\|, \forall \lambda \leq 0\}$

Definition 1.10. ([130], Definition 2.1) A vector x is approximately parallel to another vector y if the $\inf\{\|x + \lambda y\| : \lambda \in K\} \leq \epsilon \|x\|$, for all $\epsilon \in [0, 1]$.

Definition 1.11. ([114], Definition 4.4) For any complex number z and two matrices A_1 and A_2, $A_1 \perp_{BJ} A_2$ if and only if $\|A_1 + zA_2\| \geq \|A_1\|$ where $\|A_1\|$ is the normal operator norm. Consequently, $\|A_1 + zA_2\|_p \geq \|A_1\|_p$ where $\|A_1\|_p$ is the Schatten-p norm of A given by

$$\|A_1\|_p = |\sum_{i=1}^{n} S_i(A_i)^p|^{\frac{1}{p}}$$

for $1 \leq p < \infty$ and $S_1(A_1) \geq S_2(A_1) \geq \cdots S_n(A_1)$ are the singular values of A_1.

Definition 1.12. ([16], Definition 2.3) A point $x \in \mathcal{B}$ is said to be a smooth point if there exists a unique linear functional $\psi \in \mathcal{B}^*$ such that $\|\psi\| = 1$ and $\psi(x) = \|x\|$.

Definition 1.13. ([16], Definition 2.9) Let $T \in \mathcal{B}$ be norm-attainable. Then T satisfies BSP if for any $A \in \mathcal{B}, T \perp_{BJ} A$, an implication that there exists $x \in \nu : T_x \perp_{BJ} A$.

Definition 1.14. ([43], Definition 3.7) Let $\mathcal{B}$ be a Banach space and $x, y \in \mathcal{B}$ with $y \neq 0$ for any $\epsilon \in [0, 1)$, the Birkhoff-James ϵ- orthogonality set of x with respect to y is denoted and defined by $\mathcal{O}_{BJOS}^{\epsilon:x,y} = \{\alpha \in \mathbb{C} : y \perp_{BJ}^{\epsilon} (x - \alpha y)\}$.

Definition 1.15. ([43], Definition 1.9) Let $\overline{\mathcal{B}}$ be a Banach space of operators. let $A_1, A_2 \in \mathcal{B}$ where $A_1 \neq 0$. For any usual operator norm $\|.\|$ and $\epsilon \in [0, 1)$ the BJ ϵ-orthogonality set of A_1 with respect to A_2 is denoted and defined by $O_{BJOS}^{\epsilon:A_1,A_2} = \{\alpha \in C : A_2 \perp_{BJ}^{\epsilon} (A_1 - \alpha A_2)\}$.

Definition 1.16. ([119], Definition 3.5) Given a unit sphere $S_x = \{x \in X : \|x\| = 1\}$, then a sequence $\{x_n\} \in S_x$ is said to be a norming sequence for a bounded linear operator T if $\|Tx_n\| \longrightarrow \|T\|$.

1.3 Statement of the Problem

Let $\mathfrak{B}$ be a Banach space and $\mathcal{S}$ be a unit sphere in $\mathfrak{B}$. Characterizing geometrical properties in Banach spaces in terms of their mappings is very difficult due to the complex underlying structures in the Banach spaces. Some of these properties are norms, numerical ranges, spectra, orthogonality among others. Recently, of interest has been the norm-attainability aspect in Banach space setting in general. One of the open questions that has not been answered which was posed by [114] states as follows: Suppose that $T \in \mathfrak{B}$ is bounded and linear. Find a necessary and sufficient condition for $x \in \mathcal{S}$ to be such that x belongs to the set of norm-attainable vectors. To determine such conditions one requires a geometrical view of the problem and this brings into the picture the concept of orthogonality. In this regard therefore, BJO comes into play in order to solve this problem. Further, it is a well known fact that BJO can be used to ascertain norm-attainability [118]. However, a clear relationship between the set of all norm-attainable vectors and the set of all norm-attainable operators is not known [57]. There is very little in the literature regarding this relationship. Therefore, in this study we embark on the task of determining this relationship.

1.4 Objectives of the Study

To solve the stated problem we aim to achieve the following objectives.

1.5 Main Objective

The main objective of this study is to establish norm-attainability conditions via Birkhoff-James Orthogonality in Banach spaces

1.6 Specific Objectives

The specific objectives of the study are to:

(i). Establish Birkhoff-James orthogonality conditions for operators on Banach spaces.

(ii). Determine norm-attainability of operators on Banach spaces via Birkhoff-James orthogonality.

(iii). Investigate the relationship between the set of norm-attainable vectors and the set of norm-attainable operators via Birkhoff-James orthogonality on Banach spaces.

1.7 Significance of the Study

The results of this study are useful in understanding the concept of orthogonal projections which are important in constructing spectral decompositions of operators which play a crucial role in characterizing properties of operators in general Banach space setting. Moreover, the results are also useful in optimization theory and convex analysis.

Chapter 2

LITERATURE REVIEW

2.1 Introduction

Related literature to this study with fundamental results are reviewed in this chapter. We consider various studies and give a critique of the same. We also indicate the relevance of the reviewed work to our study.

2.2 Norm-Attainability

We review some literature related to norm-attainability in normed spaces. We begin with self-adjoint normal operators.

Proposition 2.1. *[88] A normal operator on a Hilbert space $\mathfrak{H}$ is norm-attainable if and only if it is self-adjoint.*

Proposition 2.1 characterizes Hilbert space operators in terms of self adjointedness. In our work we consider other operators like unitary, isometry and projections to determine the norm-attainable vectors for these operators. Turnsek[126], characterized the existence of normal operators in irreducible algebra. Our study considered norm-attainable operators and determine whether they attain their norms in a general Banach space setting. The author further characterized the existence of a rank one operator on a closed maximal triangular algebra but we characterize normal operators in other Banach spaces such as l_p spaces. Finally, the study characterizes the existence of normal operators on a closed maximal triangular algebra using decomposability of finite rank operators. In our study we considered other techniques such as known inequalities.

Okelo [87], showed that there exists normal operators in a Banach algebra. The author used the method of tensor products in the characterization but in our study we employed the same technique and other methods such as polar decomposition. The work was extended to characterize norm-attainable operators in a Banach algebra but the result was limited to other Banach spaces such as l_p-spaces. Finally, the work characterized operators that are norm-attainable in Lie ideas but our study considered l_p-spaces and determines whether there exist normal operators in those spaces.

Okelo et al [93], showed the relation between compactness and boundedness of an operator when they are norm-attainable in Hilbert spaces. In our study, we considered the notion of compactness and boundedness for norm-attainable operators in Banach spaces in general. Okelo et al [92], illustrated that certain operators attain their norms on Banach spaces and went ahead to give necessary conditions for the existence of such norms on a Banach space $\mathcal{Q}$ especially those operators with rank two from a Banach space $\mathcal{Q}$ to a Banach space of dimension two. For instance, one of the conditions was the existence of a cone that is not trivially comprising of continuous functionals that attain their norms on $\mathcal{Q}$. The authors further discussed denseness of rank-two operators that attain their norms, which is true, for example, whenever there exists a dense linear subspace comprising of functionals that attain their norms on $\mathcal{Q}$. Specifically, the authors considered operators on Hilbert spaces where these properties were completely characterized and obtained. Our work however, sought to utilize the compactness property of operators to establish whether such operators attain their norms in Banach spaces. From the work of [91], it is shown that compact operators are norm-attainable but in our study we determined if operators can be approximated by those that attain their norms.

Lemma 2.2. *[90] Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces and consider J as a compact operator. If $[kerJ]^{\perp} \subset NA(X)$, then J has norm attainment property.*

Lemma 2.2 characterizes norm attainability of compact operators but in our study, we characterized Banach space operators in a general setting. Finally, we showed that every operator which is compact can be approximated by operators that attain their norms. Sain et al [120], proved norm-attainability via BJO attached to a unit sphere. The authors further showed that if an operator satisfies Bhatia-Semrl criterion, then the operator is norm-attainable. Contrastingly, in our work, we used the Bhatia-Semrl property as a tool in determining the operators that attain norms at given points in a Banach space. We also delved into other classes and subclasses of operators to establish the specific conditions under which they attain their norms. In [112], the authors studied norm-attainability via approximate ϵ-BJO. Through the ϵ-BJO, they proved for reflexivity and norm-attainability in spaces whose dimension is at least three. Our work, however, characterized norm-attainability of operators via BJO in a wider generalization as opposed to restricting the determination to approximate ϵ-BJO . We also determined norm-attainability via BJO of other different classes of operators in Banach space as opposed to bounded linear operators only.

In [118], the authors showed that the set of all unit vectors in a unit sphere at which an operator attains norm is a countable set where the space in which the set exists is smooth. In this case, they claimed that, the operator does not satisfy the Bhatia-Semrl property on orthogonality. They partitioned the the set of the unit vectors where operators attain norms into two non-void sets contained in the inverse of the Banach space which disqualified the satisfaction of the Bhatia-Semrl property in any normed space, different from any smooth spaces. In the same work, the authors conjectured that a linear operator in a normed space satisfies the BS property provided that the set of unit vectors where an operator attains norm is connected in a well defined projective space. In our work, we established norm attainability conditions for operators in Banach spaces provided that the Banach space under operation is a smooth reflexive Banach space. In most cases, we also dealt with the closure of the SRB-classes in determining norm-attainability of operators in such spaces.

In [104], the authors characterized the norm-attainability of operators in inner product spaces. They investigated the norm-attainability of operators under such conditions that included fixing positive constant integers in the dimension at least three. They utilized the concept of fixed theorem of compact convex subsets of a subspace in a Hilbert space to establish the norm-attainability in inner product spaces. The authors also used the idea of strong convergence of sequences of positive integers to a fixed point of an operator to establish norm-attainability conditions of operators in inner product spaces. Our work established the norm-attainability of operators on account of SB-spaces . We also used the concept of orthonormal sequences of unit vectors to characterize norm-attainability of operators via the BJO.

In [116], the authors characterized the norm attainment set of all bounded linear operators on Hilbert spaces. They did this through the study of extreme contractions on Euclidean and convex spaces. In this study, they gave an alternative proof to the concept of contractions of operators in both real and complex cases. The study of extreme contractions in two dimensional subspaces enabled the authors to offer a proof of norm-attanability conditions via BJO in Banach spaces. In their work, they proved that, for a finite dimensional Hilbert space, given a bounded linear operator in the space, then there exists an orthonormal basis of the Hilbert space such that the operator which attains its norm in a unit sphere preserves orthogonality on the orthonormal basis. The authors further proved the necessary condition for an operator to be a contraction in a Banach space, and for this case, they asserted that the operator is an extreme contraction provided that the operator is norm-attainable at linearly independent unit vectors. In our work, we characterized the norm-attainability conditions of operators on SRB-spaces. We characterized these conditions via approximate BJO where the operators involved were approximately semi-orthogonal in the SRB-spaces.

2.3 Birkhoff-James Orthogonality

Regarding orthogonality, we considered this aspect in general then we narrowed down to BJO. From the work of [13], various notions of orthogonality are given. Many researchers have henceforth studied these different notions of orthogonality in different spaces.
Choi and Kim [33], studied norm-attainment of multilinear mappings in Hilbert spaces. They showed that denseness of operators does not hold when the domain space is not null for arbitrary range.

Theorem 2.3. *[33] Every multilinear map attains its norm if it is the identity.*

Theorem 2.3 characterizes norm-attainment in terms of the identity. In our work, considered other nontrivial operators like the unitary and isometries. Since multilinear operators are sometimes products of other operators, we found it interesting to study product operator of other operators. At this juncture, let's put our effort to reviewing BJO. We begin with the following proposition.

Proposition 2.4. *[128] Every operator on a Hilbert space which satisfies BJO condition is approximately BJ-orthogonal. However, the converse is not true.*

Proposition 2.4 gives the relationship between BJO condition and approximate BJO. It confirms that every operator which satisfies BJO condition is approximately BJ-orthogonal. It further shows that the converse is not true. Approximate orthogonality is weaker but an important notion of orthogonality that gives a picture of the geometry of Banach spaces. We also considered approximate BJO but in a general Banach space setting. Miguel et al [81], provided characterization of BJO in a number of families of Banach spaces in terms of the elements of significant subsets of the unit ball. They used the approach of the numerical range to advance this study. They also studied BJO of the duals of operators in Banach spaces. They used the techniques of the tensors, norm-attainability and the maximum values of the numerical range to advance their study. However, our work sought to establish BJO conditions on other nontrivial operators on Banach spaces. We also employed the proof due to Bhatia to advance our establishments. Our work also concentrated on the classes of bounded linear operators due to the rich geometrical structures they posses so as to help us achieve the desired results.

In [15], Bhatacharya and Priyanka explained the BJO in C^*-modules in IPS. They proved the infinite dimensional case in operators that achieve BJO in terms of sequences that converge to the operator norms. They also proved the real Bhatia-Semrl theorem where they asserted that if $A, B \in M_{(n)}$, then picking $t \in \mathbb{R}$, we have that $\|A + tB\| \geq \|A\|$. They proved that this is only possible if the operator norm and the real part of the inner product are both zero. They showed that this assertion is always true but the reverse fails the affirmative test. However, in our work we considered both the set of real numbers and the set of complex numbers to prove for BJ-orthogonality in Banach spaces.

Morrel proved in [84], orthogonality via tensor products of vectors in Banach spaces. They made an attempt to answer the question that if $x_1 \perp_{BJ} x_2$ and $y_1 \perp_{BJ} y_2$, then is $(x_1 \otimes y_1) \perp_{BJ} (x_2 \otimes y_2)$? However, in our work, we employed the techniques of the tensor products to establish

BJO conditions on Banach spaces. We used tensor products as a tool but not in perturbation of different classes. We also restricted our work to the class of Banach spaces as opposed to the other classes advanced in the work of Morrel.

Bajracharya and Ojha [11], established BJO via the approach of subdifferential of continuous linear functions. They used this approach to advance on the Bhatia-Semrl property and the composition of the subdifferential of the norms of functions $f(t) = \|A + tB\|$ to establish BJO. They used composition of functions to advance their characterization of BJO via subdifferential of functions. Finally, they discussed the inclusion of semidefinite operators in the field of values of functions. In our work, we established BJO conditions using approaches such as numerical ranges of operators, tensor products of Banach spaces, strong orthogonality and approximate orthogonality among other techniques. We established these orthogonality conditions on operators that attain norms at given points in Banach spaces.

Priyanka and Sushil [102], studied the cases of orthogonality in such special cases when it is symmetric and when it is both right and left additive as well as establishing BJO in extreme points of operators among others. They extended the study of BJO and its connections to hyperplanes and support hyperplanes where they made a claim that every hyperplane is orthogonal to its subspace. To prove this assertion, they employed the technique of the Hahn-Banach theorem to offer a proof of the claim. They connected the claim to symmetry of BJO and confirmed that indeed, BJO is right additive provided the existence of a norm which is Gateaux differentiable at each non-zero point. This work goes ahead to show right-additivity in other spaces. However, in our work, we established BJO from the premise that BJO is homogenous, nonsymmetric and non-additive, both left and right, in normed spaces. We utilized the techniques used in their proofs to help us develop our results in Banach spaces.

Saidi [111], also considered the study of symmetry in Banach spaces. The author gave examples of operators that achieve symmetry in Banach spaces and proved left-symmetry in such spaces. Our work established BJO in general sense in some subclasses of Banach algebras. We also concentrated in homogeneity property of BJO in normed spaces. Chmielnski and W$\acute{o}$j$\acute{c}$ik in [31], studied approximate BJO and the approximate symmetry of BJO. Their result was based on the non-symmetry of BJO in normed spaces. They showed that orthogonality due to Birkhoff-James is symmetric up to some point in the dual space of operators. This discussion enabled them to extend the approximate symmetry of BJO into other spaces. However, our study concentrated on BJO with an extension into non-symmetric approximate BJO as a pre-condition to achieve our establishment of the orthogonality conditions in Banach spaces. We also, upon relaxation of some conditions on approximate BJO, generalized BJO whenever the conditions on symmetry are met.

Johnstone et al [59], characterized orthogonality of self-adjoint operators and their translates into other operators. They went ahead to explain numerous consequences of BJO in Hermitian matrices and the applications of such to quantum information theories, where matrices are positive semidefinite of trace one. They extended this study to the determination of the types of operators which are BJO to diagonal matrices. The foregoing assertions restrict the orthogonality results to when the operators involved are self-adjoint and semidefinite. In our work, we established BJO in Banach spaces for a range of other classes of operators and determine under what conditions such operators are BJO. We also established the BJO results in the operator norms in the Banach spaces.

Arambasic and Rajna [6], discussed BJO in Hilbert C^*-modules. They introduced the notion of orthogonality which they called strong BJ-orthogonality. Our work however, characterized BJO in general Banach space setting. We also delved into the notion of strong orthogonality but in a general Banach space setting. Paul and Sain in [97], obtained a complete characterization of approximate BJO of operators which was an advancement on the study of approximate BJO of compact linear operators. In our study, we also investigated classes of operators like unitary, normal, Hermitian and their compositions in establishing BJO. We restricted our study to well defined spaces.

Shoja et al, in [123], studied orthogonality via the semi-inner-product. They achieved their results through the concept of the norming sequence and strict convexity of operators. The foregoing result pegged BJO to strict convexity of compact operators. Our study however, considered other classes of operators such as normal operators, unitary operators and operator projections to establish BJO in Banach spaces.

Bose et al in [21], established BJO by finding the symmetries for l_p-spaces onto themselves. They then found all the isometries of these spaces. They established the BJO results by use of filters and ultrafilters as tools to achieve the BJO results. These assertions by Bose [22] and others characterized BJO in terms of the symmetries, most specifically, the left symmetry. However, in Banach spaces, we based our study on the premise that BJO is not symmetric, both right and left. However, this study by Bose and others in [21] is a motivation to our study and as a result, we also delved into the study of spaces where BJO is also right symmetric, and even both left and right symmetric.

In [21], the authors characterized BJO in Sch$\ddot{a}$ffer unitary dilations of the operators T and the commutant of the operator ST. We found it interesting, in our work, to investigate the outcome on BJO if the operators S and T are not commuting. We also found it interesting to investigate the outcome of BJO on other classes of operators and most specifically, the behavior of BJO on unitary operators.

In [94], the authors established BJO whenever the operators are unitary dilations or isometries, that is, U_T and U_A respectively and showed that U_T is BJ-orthogonal to U_A. In our work, we considered the basic operators such as unitary operators, normal operators, perturbation in operator theory among other operators in establishing BJO in Banach spaces. In [95], the authors established BJO whenever an operator undergoes self perturbation for an integer for any positive integer. For our work, this form of perturbation created an interest and led us to check if BJO still holds whenever the self perturbation is taken for any negative integers or even when the integer is strictly above zero.

In [109], the authors characterized BJO via ρ-dilations. They established this result from the premise of norming sequences of both the operators A and T provided the operators are dilations. Our work however, concentrated on different classes of the operators A and T and their perturbations with other classes of operators to establish BJO in Banach spaces. We also investigated operator sequences for BJO as opposed to norming sequences as a tool to establish our results.

In [22], the authors characterized BJO and its positive symmetry on the closure of the convex hull of norm sequences. However, our work sought to establish BJO on numerical ranges of operators and their sequences, which has a close connection with the convex hull of operator sequences. Also, still in [22], the authors established BJO with both left and right symmetry as a property in terms of sequences and most specifically, the zero sequence. On the other hand, we established BJO from the premise of non-symmetry of BJO. However, we also investigated the special classes of operators and operator sequences when symmetry holds as a condition of BJO. We also made a contribution in such classes by relaxing the conditions of sequences to establish BJO in Banach spaces.

In [23], the authors characterized BJO of bounded linear operators between Hilbert spaces and Banach spaces through the cartesian decomposition of operators. The authors further showed that it is possible to extend the Bhatia-Semrl theorem into the case of orthogonality in bounded linear operators between infinite dimensional spaces. They also focussed on orthogonality of bounded linear operators alongside the orthogonality in positive operators in both real and complex Banach spaces. The authors finally studied some related properties in orthogonality between different notions of orthogonality and explored different applications for such relationship between the different notions of orthogonality. Our work however, established BJO conditions as an isolated notion of orthogonality and concentrated on SRB-spaces as our target space of operation. We also established the orthogonality conditions in the classes of operators such as the approximate semi-orthogonal operators.

Turnsek, in [127], studied operators preserving James' orthogonality and characterized isometries and co-isometries in bounded linear operators in terms of orthogonality. Due to this study, the author obtained conjugate linear mappings and surjective maps that preserve James orthogonality in any directions. They proved the orthogonality results via the numerical range approach where they restricted their approach to the interior points of self-adjoint unitary operators where such operators attain their norms on unit vectors. They also established the orthogonality conditions through the orthogonal projections of subspaces of operators and asserted that the orthogonality due to James is only preserved provided the numerical range of the operators involved are ellipsoids with zero as their interior points. However, our study established the orthogonality results on the basis of orthogonality due to Birkhoff-James as a notion. We also restricted our study to polar decompositions of operators and the index conjugations of elements in the closure of SRB-spaces.

The authors in [9], investigated BJO conditions on operators in Hilbert C^*-modules where they obtained the necessary conditions on given elements in modules such that the elements achieve the orthogonality property of symmetry. They established these orthogonality conditions through the minimality of the inner product spaces and projections of one-dimensional Hilbert spaces. The authors also described the conditions under which orthogonality, and most specifically, BJO is left or right additive through the invertibility of nonzero elements of subspaces. The authors also studied strong BJO and proved its symmetric relation in Hilbert modules of subspaces through the isomorphism of the C^*-algebras to the space of all complex numbers. On the other hand, our work utilized the involution property of the C^*-algebras to establish the BJO conditions for operators in Banach spaces. We also established this orthogonality via the subclass of approximately semi orthogonal operators in SRB-classes.

Dehghani and Zamani in [40], introduced a new concept of orthogonality called the approximate ρ_*-orthogonality. This notion of orthogonality, they claimed, preserves mapping between normed spaces. They also proved that approximate ρ_*-orthogonality is homogenous. The authors also showed that for a norm which is G$\hat{a}$teaux differentiable, the approximate ρ_*-orthogonality coincides to the approximate BJO in normed spaces. They finally showed that every approximate ρ_*-orthogonality mapping is a multiplicity by a scalar of almost isometry. These properties that preserve approximate BJO were later verified by the authors of the work in [83]. In our work, we characterized approximate orthogonality in Hilbert spaces and in infinite Banach spaces. We also established the orthogonality conditions in the spaces of Hilbert C^*-modules. We finally established these conditions in projections different from the normal orthogonal projections in Banach spaces.

In [103], the authors, when they studied orthogonality to matrix subspaces, obtained a necessity for an operator to be BJO to any subspace of a complex Hilbert space under an inner product. They achieved this orthogonality result basing their argument on the existence of a density operator of complex rank. The authors constructed the proof of their finding through the concept of subdifferential of composition maps. Through this study, the authors proved an expression for a distance formula between an operator from any unital C^*-subalgebras of all complex Hilbert spaces. In our work, we established the orthogonality conditions on finite dimensional Hilbert spaces and on positive semidefinite operators. We also used the concept of tensor norms to achieve our establishment of BJO conditions on Banach spaces.

Eskandari et al in [46], characterized conditions in triangle inequality that lead to some equivalence relations in terms of states of C^*-algebras. They extended this study into establishing equivalent statements to the parallelogram identity for some vectors in the Hilbert C^*-modules. The authors also established when the equality conditions hold in triangle inequality for the cases of all operators that are adjointable on a Hilbert C^*-modules and as a result, gave conditions that are necessary and sufficient for any two vectors in a pre-Hilbert C^*-module that the inner product of the two vectors has a negative real part, in the case Pythagorean identity. Towards the tail end of the study, the authors introduced the notion of Pythagoras orthogonality as well as characterizing its orthogonality properties. Finally, the authors went a notch higher to present some examples that illustrate the existence of a relationship between BJO, Pythagoras orthogonality as well as the connection to the usual orthogonality in the class of Hilbert C^*-modules. In our work, we used the Triangle inequality as a methodology in advancing our establishment of orthogonality conditions in Banach spaces. We also established the BJO conditions in C^*-algebras and proved the orthogonality conditions for any two operators provided that their tracial norms are strongly BJO.

Ali Zamani in [131] generalized the concept of BJO of operators in Hilbert spaces through the concept of the semi-inner product. The author introduced a new relation in orthogonality called the T-Birkhoff-James orthogonality, for T being an operator in a complex Hilbert space and that the operator T is bounded linear operator in the Hilbert space. In fact, for this particular case, any two operators are BJO in Hilbert spaces if the operators are bounded with respect to the semi norm induced by any positive operator. The authors additionally extended the Bhatia-Semrl property and showed that any two operators are T-Birkhoff-James orthogonal provided the existence of a sequence of T-unit vectors such that the sequence converges. Finally, the author introduced the notion of T-Birkhoff-James orthogonality of operators in semi-Hilbertian spaces where they showed that even for this case of orthogonality, the property of homogeneity still holds. The author then extended the concept of T-Birkhoff-James orthogonality into developing some formulae for T-distance of operators to the class of some constants of other cases of semi-Hilbertian spaces. In our work, we established BJO conditions for operators in complex planes. We also proved the homogeneity property of the BJO and used this particular property in establishing BJO conditions for other subclasses of operators on Banach spaces.

In [10], the authors considered three aspects of orthogonality in Hilbert C^*-modules in relation to some C^*-algebras. They considered the BJO, the strong BJO and finally the orthogonality in relation to an $\mathcal{A}$-valued inner product spaces. They characterized orthogonality in some classes of Hilbert C^*-modules where there were possibilities of any of the two concepts of orthogonality coinciding. Moreover, the authors characterized the subclasses in normed spaces equipped with various orthogonalities where a given type of orthogonality is an implication of the other. More specifically, the authors established orthogonality conditions in classes of

Hilbert C^*-modules where BJO implies strong BJO and in the cases where strong BJO is an implication of orthogonality with respect to the IPS. In our work however, we established the BJO conditions in restriction to the SRB-classes. We also established the conditions of BJO independently of the strong BJO in Banach spaces.

In [32], in the study of operators reversing orthogonality in normed spaces, the author dealt with linear operators on normed spaces and considered those properties under which orthogonality is reversed. The author considered all those operators in normed spaces where BJO preserving property is upheld. The author, In particular, asserted that operators which are nonzero and linear at the same time, preserve orthogonality and are also injective. In this study, the author appreciated the non-symmetry of orthogonality but asserted that it would be interesting if they considered the problem of reversing BJO. For all operators that obey linear similarity, the author established that the reversing and preserving BJO properties have an equivalence relation. However, the author also established that there exists spaces that fail to admit nontrivial mappings which reverse BJO. Such spaces where the property of reversing BJO fail to hold were established in this work of which the author called them the two-dimensional normed spaces, more specifically, the Minkowski planes. In our work, however, we established the orthogonality conditions in the SRB-spaces. We also dealt with conditions that preserve orthogonality as opposed to such conditions that reverse BJO.

2.4 Relationship between the set of Norm-Attainable Vectors and the Set of Norm-Attainable Operators

It is a well known fact that BJO can be used to ascertain norm-attainability [114]. However, a clear relationship between $\mathfrak{NA}_v$ and $\mathfrak{NA}_{op}$ is not known [57]. There is very little in the literature regarding this relationship. Therefore, in this study we embarked on the task of determining this relationship.

Chapter 3

RESEARCH METHODOLOGY

3.1 Introduction

To solve problems on properties of bounded linear operators on Banach spaces like spectrum, compactness, norms, numerical ranges and orthogonality, certain methods and techniques are useful. The methodology involved the use of known inequalities such as Cauchy-Schwarz inequality, Triangle inequality and Minkowski's inequality. We also used technical approaches of Tensor products and Direct sum decomposition. Finally, we used some fundamental principles of some known results that have been advanced beforehand to establish our results. We discuss such methods, tools and techniques in this chapter.

3.2 Known Inequalities

In this section, we discuss the known inequalities.

3.2.1 Cauchy-Schwarz inequality

It is also referred to sometimes as Cauchy-Bunyakovsky-Schwarz inequality. It can be loosely defined as an upper bound in a space X, called the inner product space [5], between $u, v \in X$ in terms of the products of the norms of the vectors $u, v \in X$. The statement of the inequality is;

$$|\langle u, v\rangle|^2 \leq \langle u, u\rangle.\langle v, v\rangle \tag{3.2.1}$$

for all u, v in the inner product space X, and $\langle ., .\rangle$ is the inner product. One of the spaces that the inner product gives rise to is the Euclidean l_2 norm which is sometimes referred to as the induced norm [2]. Because of this, we can equally state the Cauchy-Schwarz inequality(CSI) as $\|u\| = \sqrt{\langle u, u\rangle}$ where $\langle u, u\rangle > 0$ [12]. In a more familiar way, taking the positive square root of Equation 3.2.1, we generate another more familiar statement of the CSI as;

$$|\langle u, v\rangle| \leq \|u\|\|v\|. \tag{3.2.2}$$

It is important to note that equality holds in 3.2.2 if and only if the vectors u and v are linearly dependent [19]. We have utilized CSI in chapter four of our work to prove the result in Theorem 4.10 of our first objective which relies on states in a C^*- algebra. It has come handy to help us to prove that if $\mathfrak{C}$ is a C^*- algebra, then $A \perp_{BJ}^{s} B$ in $\mathfrak{C}$ if there exists a state ξ in $\mathfrak{C}$ such that $\xi(AA^*) = \|A\|$ and $\xi(BB^*) = 0$ for $A, B \in \mathfrak{C}$.

3.2.2 Triangle inequality

Let x and y be two vectors in the inner product space X. Then, $\|x+y\| \leq \|x\| + \|y\|$, for all $x, y \in X$ [24]. Our work uses the Triangle inequality alongside Minkowski's inequality to prove a result of Theorem 4.6 of our first objective which seeks to establish Birkhoff-James orthogonality conditions for operators on Banach spaces via Schatten p- norms.

3.2.3 Minkowski's inequality for sequences

Minkowski's inequality was named after a German mathematician called Hermann Mikowski. It is an inequality which establishes that the L^P are actually normed spaces. To the statement of the inequality, let $1 \leq p < \infty$ and let a and b be members of $L^P(X)$ where X is a measure space. Then $a + b \in L^P(X)$ and we have the inequality;

$$\|a + b\| \leq \|a\|_p + \|b\|_p, \tag{3.2.3}$$

with equality if $1 < p < \infty$ if and only if a and b are positively linearly dependent [26]. It is also important to note that Minkowski's inequality is the triangle inequality in the class $L^P(X)$ [28]. We can generalize this inequality to sequences and vectors as;

$$(\sum_{k=1}^{n} |i_k + j_k|^p)^{\frac{1}{p}} \leq (\sum_{k=1}^{n} |X_k|^p)^{\frac{1}{p}}.$$

We have employed the techniques of Minkowski's inequality and the triangle inequality with a good effect to prove a result of Theorem 4.6 of our first objective which shows that if $P, Q \in L(\overline{\mathcal{B}_1}, \overline{\mathcal{B}_2})$ then, $P \perp_{BJ} Q$ if P has a polar decomposition $P = U|A|$ and $tr|P|^{p-1}U^*Q = 0$ in the Schatten p-norm.

3.3 Technical Approaches

These include some of the techniques that we have used to advance the proofs of our results. They include the following techniques;

3.3.1 Polar decomposition of operators

Polar decomposition of a linear transformation on a finite dimensional Hilbert space $\mathcal{H}$ is a factorization of the linear transformation into a product of hermitian operator and an orthogonal transformation [36]. Therefore, a polar decomposition of an operator Q acting on a Hilbert space has a general representation of $Q = UP$ where U is a partial isometric operator and P is a positive operator [35]. In our work, we have used polar decomposition of operators to develop the proof of Theorem 4.6 which seeks to establish BJO conditions for operators on Banach spaces in our first objective. This result establishes BJO condition in Schatten p-norms.

3.3.2 Direct sum decomposition

In this study, we have used direct sum decomposition to determine norm-attainability of operators on Banach spaces via BJO as our second objective. In particular, we used direct sum decomposition to construct the proof of Lemma 4.20 regarding the set of all norm-attainable operators in finite dimensional Hilbert spaces.

3.3.3 Tensor product

Given W a linear bounded operator on a Hilbert space $\mathcal{H}_1$ and another bounded linear operator R on another Hilbert space $\mathcal{H}_2$, then there exists a unique linear bounded operator T on $\mathcal{H}_1 \otimes \mathcal{H}_2$ such that $T(r_1 \otimes r_2) = Wr_1 \otimes Rr_2$ for all $r_1 \in \mathcal{H}_1$ and $r_2 \in \mathcal{H}_2$, called a tensor product of operators W and R denoted by $W \otimes R$ [37]. We note that $W \otimes R$ is finite dimensional and its dimension is the product of dimension of W and R [38]. The tensor product is associative, that is, given vector spaces P, R, W, there is an isomorphism $(P \otimes R) \otimes W \cong P \otimes (R \otimes W)$, which assigns $(p \otimes r) \otimes w$ to $p \otimes (r \otimes w)$. It is also commutative in the sense that given two vector spaces R and W, we have that $R \otimes W \cong w \otimes r$, for all $r \in R$ and for all $w \in W$ [45]. We can also state that up to isomorphism, the tensor product is unique. In our work, we have effectively utilized the tensor product to establish BJO conditions for operators on Banach spaces as our first objective. This evident in the statement and the proof thereof of Theorem 4.16 of our first objective.

3.4 Fundamental Principles

In this section, we discuss some of the known principles that we use in our results and their proofs. These include some of the established theorems developed beforehand.

3.4.1 Hahn-Banach Theorem

Hahn-Banach theorem is a very important tool that we use in our work especially in dual spaces. For a normed space, complex or real, a linear subspace of the normed space and an element of the dual of the subspace, there exists elements in the dual space that extend it [29]. The theorem preserves the norms in the extension of the subspaces. In our work, we use the Hahn-Banach theorem to construct a proof on the investigation of the relationship between the set of all norm-attainable vectors and the set of all norm-attainable operators via BJO on Banach spaces as our third objective. The use of this theorem is evident in the proof of Proposition 4.27 of our third objective.

3.4.2 Bhatia-$\check{S}$emrl Criterion

Bhatia-$\check{S}$emrl Criterion provides an establishment of BJO of operators in Hilbert spaces. Two operators in a Hilbert space qualify to be BJO given the existence of an element in the set of all norm-attainable operators such that their inner product is zero [47]. The set of all operators obeying the BSP has an equivalence in the set of all norm-attainable operators [1]. In our work, we used this property to develop a proof of the establishment of

BJO conditions for operators in Banach spaces as our first objective. We have used this property in good effect to advance the proof of Proposition 4.12 on orthogonality by the tensor products.

3.4.3 Bolzano-Weierstrass Theorem

This is a theorem on boundedness of sequences. For this theorem, every bounded sequence has an equivalent convergent subsequence in the real space since every limit exists for all bounded sequences in the space [52]. Every convergent sequence is bounded and not conversely. This theorem is beneficial when dealing with sequentially compact sets in the real space [49]. Bolzano-Weierstrass theorem has been used in our second objective where we set out to determine norm-attainability of operators on Banach spaces via BJO. It came in handy to offer a proof to Lemma 4.20 on BJO and norm-attainable operators.

3.4.4 Hausdorff-Toeplitz Theorem

This is a theorem on Numerical ranges, sometimes called the field of values, of operators. For this theorem, the numerical range of any operator is convex [48]. In our work, we have used Hausdorff-Toeplitz Theorem to determine norm-attainability of operators on Banach spaces via BJO as our second objective. We have used the idea of this theorem to construct the proof of Proposition 4.17 on reflexive smooth Banach spaces.

3.4.5 Separation Hyperplane Theorem

Separation Hyperplane Theorem is a general proposition about disjoint convex sets in a finite dimensional Euclidean space that traces its origin to Hermann Minkowski. For this case, let A and B be two disjoint nonempty convex subsets of the Euclidean space $\mathbb{R}^n$. Then there exists a nonzero vector v and a real number c such that $\langle x, v\rangle \geq c$ and $\langle y, v\rangle \leq c$, for all $x \in A$ and for all $y \in B$ whic is an implication that the space $\langle ., v\rangle = c$ where the normal vector v separates A and B. In our work, we used the theorem to aid in the proof of Proposition 4.19 on Birkhoff-James orthogonality and norm-attainability as our second objective.

Chapter 4

RESULTS AND DISCUSSION

4.1 Introduction

In this chapter, we characterize Birkhoff-James orthogonality in Banach spaces. In particular, we establish BJO conditions for operators in Banach spaces.
We give results on orthogonality in infinite cases. We delve into approximate orthogonality, strong orthogonality and orthogonality in tensor products of Banach spaces.

4.2 BJO Conditions for Operators on Banach Spaces

In this section, we establish BJO conditions for operators on Banach spaces. In Hilbert spaces, we define approximate orthogonality by $a \perp^{\epsilon} b$

if and only if $|\langle a,b\rangle| \leq \epsilon\|a\|\|b\|, \forall \epsilon \in [0,1)$. We characterize approximate orthogonality in Hilbert spaces and in infinite Banach spaces. We begin with a result on homogeneity of BJO.

Proposition 4.1. *Let $\overline{\mathcal{B}}$ be a smooth reflexive Banach space. Then for $A, B \in \overline{\mathcal{B}}, A \perp_{BJ}^{\epsilon} B$ if $\forall \lambda \in \mathbb{C}$,*

$$\|A + \lambda B\| \geq \|A\|^2 - 2\epsilon\|A\|\|\lambda B\|, \forall \epsilon \in [0,1). \tag{4.2.1}$$

Moreover, $A \perp_{BJ}^{\epsilon} B$ implies that $\mu A \perp_{BJ}^{\epsilon} \omega B$ for any $\mu, \omega \in \mathbb{C}$.

Proof. The case of $\mu = 0$ is obvious and so we omit. Let $\mu \neq 0$, then we have that

$$\begin{aligned} \|\mu A + \lambda\omega B\|^2 &= |\mu|^2\|A + \lambda\frac{\omega}{\mu}B\|^2 \\ &\geq |\mu|^2(\|A\|^2 - 2\epsilon\|A\|\|\lambda\frac{\omega}{\mu}B\|) \\ &= \|\mu A\|^2 - 2\epsilon\|\mu A\|\|\lambda\omega B\|. \end{aligned}$$

So,

$$\|\mu A + \lambda\omega B\|^2 \geq \|\mu A\|^2 - 2\epsilon\|\mu A\|\|\lambda\omega B\|. \tag{4.2.2}$$

Hence, $\perp_{BJ}^{\epsilon}$ is homogenous. If we put $\mu = 1$ and $\omega = 1$ in Inequality 4.2.2, we obtain the inequality 4.2.1 □

Next, we give the homogeneity in a more general context.

Proposition 4.2. *Let $\overline{\mathcal{B}}$ be a SRB-space . Then for $A, B \in \overline{\mathcal{B}}, A \perp_{BJ}^{\epsilon} B$ if $\forall \lambda \in \mathbb{C}$,*

$$\|A + \lambda B\| \geq \sqrt{1-\epsilon^2}\|A\|, \forall \epsilon \in [0,1). \tag{4.2.3}$$

Moreover, $A \perp_{BJ}^{\epsilon} B$ *implies that* $\mu A \perp_{BJ}^{\epsilon} \omega B$ *for any* $\mu, \omega \in \mathbb{C}$.

Proof. The case of $\mu = 0$ is obvious and therefore we leave it out. Let $\mu \neq 0$, then we have that;

$$\begin{aligned} \|\mu A + \lambda\omega B\|^2 &= |\mu|^2\|A + \lambda\frac{\omega}{\mu}B\|^2 \\ &\geq |\mu|^2(\|A\|^2 - \sqrt{1-\epsilon^2}\|A\|\|\lambda\frac{\omega}{\mu}B\|) \\ &= \|\mu A\|^2 - \sqrt{1-\epsilon^2}\|\mu A\|\|\lambda\omega B\|. \end{aligned}$$

So,

$$\|\mu A + \lambda\omega B\|^2 \geq \|\mu A\|^2 - \sqrt{1-\epsilon^2}\|\mu A\|\|\lambda\omega B\|. \tag{4.2.4}$$

Hence, $\perp_{BJ}^{\epsilon}$ is homogenous. If we put $\mu = 1$ and $\omega = 1$ in inequality 4.2.4, we obtain the inequality 4.2.3. □

Proposition 4.3. *Let* $\overline{\mathcal{B}}$ *be a SRB-space. Then, for* $A, B \in \overline{\mathcal{B}}, A \perp_{BJ}^{\epsilon} B$ *if* $\forall\lambda \in \mathbb{C}$,

$$\|A + \lambda B\| \geq \sqrt{1-\epsilon^n}\|A\|, \forall\epsilon \in [0,1), \forall n \in \mathbb{N}. \tag{4.2.5}$$

Moreover, $A \perp_{BJ}^{\epsilon} B$ *implies that* $\mu A \perp_{BJ}^{\epsilon} \omega B$ *for any* $\mu, \omega \in \mathbb{C}$.

Proof. The case of $\mu = 0$ is obvious and therefore we leave it out. Let $\mu \neq 0$, then we have that;

$$\begin{aligned} \|\mu A + \lambda\omega B\|^2 &= |\mu|^2\|A + \lambda\frac{\omega}{\mu}B\|^2 \\ &\geq |\mu|^2(\|A\|^2 - \sqrt{1-\epsilon^n}\|A\|\|\lambda\frac{\omega}{\mu}B\|) \\ &= \|\mu A\|^2 - \sqrt{1-\epsilon^n}\|\mu A\|\|\lambda\omega B\|. \end{aligned}$$

So,

$$\|\mu A + \lambda\omega B\|^2 \geq \|\mu A\|^2 - \sqrt{1-\epsilon^n}\|\mu A\|\|\lambda\omega B\|. \tag{4.2.6}$$

Hence, $\perp^{\epsilon}_{BJ}$ is homogenous. If we put $\mu = 1$ and $\omega = 1$ in inequality 4.2.6, we obtain the inequality 4.2.5. $\square$

Lemma 4.4. *Let $\overline{\mathcal{B}}$ be a SRB-space and $A, B \in \overline{\mathcal{B}}$. Then $A \perp^{\epsilon}_{BJ} B$ on $\overline{\mathcal{B}}$ if A and B are approximately semi-orthogonal.*

Proof. Let A and B be approximately semi-orthogonal, i.e, $\langle B, A\rangle = \epsilon\|A\|\|B\|$. Let $l \in [0,1]$ for some $\theta \in \{-\pi, \pi\}$. We have that $\langle B, A\rangle = l\epsilon\|A\|\|B\|e^{i\theta}$. Let $\lambda \in \mathbb{C}$ be given arbitrarily, then;

$$\begin{aligned}\|A+\lambda B\|\|A\| &\geq |\langle A+\lambda B, A\rangle| \\ &= |\|A\|^2 + \lambda\langle B, A\rangle| \\ &= |\|A\|^2 + l\epsilon\|A\|\|B\|\lambda e^{i\theta}|.\end{aligned}$$

Therefore;

$$\begin{aligned}\|A+\lambda B\| &\geq |\|A\| + l\epsilon\|B\|\lambda e^{i\theta}| \\ &= |\|A\| + l\epsilon\|B\|\mathbb{R}e(\lambda e^{i\theta}) + il\epsilon\|B\|\lim \lambda e^{i\theta}|.\end{aligned}$$

Squaring both sides with the right hand side having real and imaginary parts and from [31], we have that $\|A+\lambda B\|^2 \geq \|A\|^2 - 2\epsilon\|A\|\|B\|$, an implication that $A \perp^{\epsilon}_{BJ} B$. $\square$

Theorem 4.5. *Let $\overline{\mathcal{B}_1}$ and $\overline{\mathcal{B}_2}$ be non-zero infinite dimension and SRB-*

spaces and $A, B \in L(\overline{\mathcal{B}_1}, \overline{\mathcal{B}_2})$. *Then*

$$\|A + \lambda B\| \geq \|A\|^2 - 2\epsilon\|A\|\|\lambda B\|, \forall \epsilon \in [0, 1) \tag{4.2.7}$$

Proof. We prove this by contradiction. Let Inequality 4.2.7 fail to hold. This implies that there exists $\lambda \in \mathbb{R}$ such that $0 < \|A + \lambda B\|^2 < \|A\|^2 - 2\epsilon\|A\|\|\lambda B\|$. Suppose that $\lambda < 0$. Then for any λ, we have that $\|A + \lambda B\|^2 > 0$. By [112], we have that $z \in L(\overline{\mathcal{B}_1}, \overline{\mathcal{B}_2})$ such that $\|z\| = 1$ for any $z = \frac{A+\lambda B}{\|A+\lambda B\|}$. From [96], we have that $\|A+\lambda B\| = \|A\|^2 - 2\epsilon\|A\|\|\lambda B\|$ which contradicts the Inequality 4.2.7. Hence, our earlier supposition does not hold and as a result, Inequality 4.2.7 holds and thus $A \perp_{BJ}^{\epsilon} B$. □

The next result employs the technique of polar decomposition with Schatten p-norms in Banach spaces of trace class.

Theorem 4.6. *Let* $P, Q \in L(\overline{\mathcal{B}_1}, \overline{\mathcal{B}_2})$. *Then* $P \perp_{BJ} Q$ *if* P *has a polar decomposition* $P = U|A|$ *and* $tr|P|^{p-1}U^*Q = 0$ *in the Schatten p-norm.*

Proof. Let $tr|P|^{p-1}UQ = 0$. Then for all $\lambda \in \mathbb{C}$, we have $tr|P|^p = tr|P|^{p-1}(|A| + \lambda U^*Q)$. From Minkowski's inequality, we obtain,

$$\begin{aligned} tr|P|^p &\leq |\||P|^{p-1}\|_k\||P| + \lambda U^*Q\|_p| \\ &= \||P|^{p-1}\|_q\|P + \lambda Q\|_p. \end{aligned}$$

But $(tr|P|^p)^{1-\frac{1}{k}} = (tr|P|^p)^{\frac{1}{p}} = \|P\|_p$ since k is the index conjugate of p, that is $\frac{1}{p} + \frac{1}{q} = 1$. So, $\|P\|_p \leq \|P + \lambda Q\|_p$, that is , $\|P + \lambda Q\| \geq \|P\|_q$ □

Remark 4.7. These results hold true for sequence matrices as seen from the work of Bhatia and Semrl [16].

At this point, we move to BJO in C^*-algebras. It is known in [66] that, C^*-algebras are Hilbert C^*-modules. Hence, inner product is given as $\langle B, A\rangle = B^*A$ for all $A, B \in \mathfrak{C}$. Let $\mathfrak{C}$ be a C^*-algebra. Then for $A, B \in \mathfrak{C}$ A is strongly BJ-orthogonal to B denoted by $A \perp^s_{BJ} B$ if for all $C \in \mathfrak{C}, \|A + BC\| \geq \|A\|$. If A and B are mutually strongly BJ-orthogonal, that is, $A \perp^s_{BJ} B$ and $B \perp^s_{BJ} A$, then it is denoted by $A \perp^{ms}_{BJ} B$. We begin with the following proposition:

Proposition 4.8. *Strong BJO is intrinsically orthogonal in a C^*-algebra $\mathfrak{C}$.*

Proof. Let $\mathfrak{C}_0$ and $\mathfrak{C}$ be two C^*-algebras such that $\mathfrak{C}_0 \subset \mathfrak{C}$. Let $A, B \in \mathfrak{C}_0$. Since $\mathfrak{C}_0$ is a C^*-module and is itself a C^*-algebra, then from [63], $A \perp^s_{BJ} B$ if and only if $A \perp^s_{BJ} B\langle B, A\rangle$ if and only if $A \perp^s_{BJ} BB^*A$ □

Lemma 4.9. *Let $\mathfrak{C}$ be a C^*-algebra, then $A \perp^s_{BJ} B$ if and only if $|A^*|$ is strongly BJO to $|B^*|$.*

Proof. Since $|A| = \sqrt{A^*A}$, we have that $|B^*||B^*| = (BB^*)^{\frac{1}{2}}(BB^*)^{\frac{1}{2}} = BB^*$. But we have from Proposition 4.8 that $A \perp^s_{BJ} B$ if and only if $A \perp^s_{BJ} BB^*A$. So, we obtain $A \perp^s_{BJ} B$ if and only if $A \perp^s_{BJ} |B^*|$. Consider a partial isometry $A = |A^*|E$ for some $E \in \mathfrak{C}_0 \subset \mathfrak{C}$ and also consider $|A^*| = AE^*$. Let $|A^*| \perp^s_{BJ} B$, then for $C \in \mathfrak{C}_0 \subset \mathfrak{C}$ we obtain from [112] that;

$$\begin{aligned}\|A\| = \||A^*|E\| \leq \||A^*|\| &\leq \||A^*| + BCE^*\| \\ &= \|(A + BC)E^*\| \\ &\leq \|A + BC\|.\end{aligned}$$

Therefore, $\|A + BC\| \geq \|A\|$, for all $C \in \mathfrak{C}_0$. So, $A \perp_{BJ}^{s} B$.
Conversely, if $A \perp_{BJ}^{s} B$, then for all $C \in \mathfrak{C}$, we have that;

$$\begin{aligned} \|A\| = \|AE^*\| \leq \|A\| &\leq \|A + BCE\| \\ &= \|(|A^*| + BC)E\| \\ &\leq \||A^*| + BC\|. \end{aligned}$$

Hence $A \perp_{BJ}^{s} |B^*|$. Also $A \perp_{BJ}^{s} B$ if $|A^*| \perp_{BJ}^{s} B$ □

Next, we prove a result with regards to states in $\mathfrak{C}$.

Theorem 4.10. *Let $\mathfrak{C}$ be a C^*-algebra. Then $A \perp_{BJ}^{s} B$ in $\mathfrak{C}$ if there is a state ξ in $\mathfrak{C}$ such that $\xi(AA^*) = \|A\|$ and $\xi(BB^*) = 0$ for $A, B \in \mathfrak{C}$.*

Proof. Since ξ is a state in $\mathfrak{C}$, then for any $C \in \mathfrak{C}$, we have by Cauchy-Schwarz inequality that $|\xi\langle C^*, C^*\rangle|^2 \leq \xi(C^*C) = 0$. Let $\|A\| = 1$. We have;

$$\|A+BC\|^2 = \|(A+BC)(A+BC)^*\| \geq |\xi(AA^*+AC^*B^*+BCA^*+BCC^*B^*)| \tag{4.2.8}$$

From the involution property of C^*- algebras, we have that $\xi(AC^*B^*) = \overline{\xi(BCA^*)} = 0, \xi(AA^*) = 1$ and $\xi(BCA^*) = \xi(BCC^*B^*) = 0$.
Hence, from Inequality 4.2.8, we have that $\|A + BC\|^2 \geq 1 = \|A\|^2$. Taking positive square root on both sides, we have, $\|A + BC\| \geq \|A\|$ that is, $A \perp_{BJ}^{s} B$. □

Corollary 4.11. *Let $A, B \in \mathfrak{C}$. For any projection $P \in \mathfrak{C}$ such that $PA = A$ and $PB = 0$, we have that $B \perp_{BJ}^{s} A$.*

Proof. With the conditions of the Corollary 4.11, we have $C \in \mathfrak{C}$ such that $\|A+BC\| \geq \|P(A+BC)\| = \|PA\| = 1 = \|A\|$. That is, $\|A+BC\| \geq \|A\|$. Hence, $A \perp_{BJ}^{s} B$ in $\mathfrak{C}$. □

Next, we give results of BJO in relation to tensor products of operators. We begin with the following proposition;

Proposition 4.12. *Let $\overline{\mathcal{B}_1}$ and $\overline{\mathcal{B}_2}$ be SRB-spaces with tensor norm. Then $A_1 \otimes B_1 \perp_{BJ} A_2 \otimes B_2$ on $\overline{B_1} \otimes^{\pi} \overline{B_2}$ where $A_1, A_2 \in \overline{B_1}, B_1, B_2 \in \overline{B_2}, A_1 \perp_{BJ} B_2$ in $L(\overline{B_1}, \overline{B_2})$ and $\|.\|_{\pi}, \|.\|_{INJ}$ are cross norms and injective norm respectively.*

Proof. By Bhatia and Semrl property, $A_1 \perp_{BJ} A_2$ and so we have $\psi \in \overline{B_1^*}$ with $\|\psi\| = 1$. This implies that $|\psi(A_1)| = \|A_1\|$ since ψ is a functional and $\psi(A_2) = 0$. Next, consider $\chi \in \overline{B^*}$ with $\|\chi\| = 1$. This similarly implies that $\chi(B_1) = \|B_1\|$. Let $z \in \mathbb{C}$ be given, then we obtain that

$$\begin{aligned}
\|(A_1 \otimes B_1) + z(A_2 \otimes B_2)\|_{\pi} &\geq \|(A_1 \otimes B_1) + z(A_2 \otimes B_2)\|_{INJ} \\
&= \sup\{|\psi'(A_1)\chi'(B_1) \\
&+ z\psi'(A_2)\chi'(B_2)| : \psi' \in B_{\overline{B_1}}, \chi' \in B_{\overline{B_2}}\} \\
&\geq |\psi(A_1)\chi(B_1) + z\psi(A_2)\chi(B_2)| \\
&= \|A_1\|\|B_1\| \\
&= \|A_1 \otimes B_1\|.
\end{aligned}$$

Therefore, $\|(A_1 \otimes B_1) + z\psi(A_1 \otimes B_2)\| \geq \|A_1 \otimes B_1\|$. Hence $A_1 \otimes B_1 \perp_{BJ} A_2 \otimes B_2$ on $\overline{B_1} \otimes^{\pi} \overline{B_2}$. □

Remark 4.13. It is known from [112] that in reflexive and smooth Banach spaces, $\overline{\mathcal{B}_1}$ and $\overline{\mathcal{B}_2}, A \perp B$ if and only if there exists $\phi \in \overline{B_1^*}$ such that $|\phi(A_1)| = \|\phi\|\|A\|$ with $\phi(B_1) = 0$, $\|A_1 \otimes B_1\|_\pi = \|A_1\|\|B_1\|$ as a cross norm and

$$\|W\|_{INJ} = \sup\left\{|\sum_{i=1}^{n}\phi(A_1)\psi(B_1)|\right\}, W = \sum A_i \otimes B_i \in \overline{B_1} \otimes \overline{B_2}.$$

$B_{\overline{B_1^*}}$ is the closed unit ball of $\overline{B_1^*}$.

Remark 4.14. The converse of Proposition 4.12 is not true in general, that is, $A_2 \otimes B_2 \not\perp_{BJ} A_1 \otimes B_1$ on $\overline{B_2} \otimes \overline{B_1}$ since BJO is antisymmetric.

At this point, we consider two special cases.

Lemma 4.15. *Let $\Omega = C_{\mathbb{C}}[0,1]$ be the set of all complex functions on $[0,1]$. Let $T \in \Omega$ be an identity function and 1 be a constant function. Then $T \otimes 1 \perp_{BJ} 1 \otimes T$ in Ω.*

Proof. From Bhatia and Semrl property and Proposition 4.12, we have that

$$\|T \otimes 1 + z(1 \otimes T)\|_{INJ} = \|T.1 + z(1.T)\| \sup_{p,q\in[0,1]} |p + zq| \geq 1 = \|T \otimes 1\|.$$

□

Theorem 4.16. *Let $\circledcirc$ be the space of all operators acting on the l^2- space and $A, B, C \in \circledcirc$. Then $A\otimes I \perp_{BJ} B\otimes C$ on $\circledcirc\otimes\circledcirc$ where I is the identity operator in $\circledcirc$.*

Proof. Let $\circledcirc \otimes \circledcirc \subseteq L(l^2 \overline{\otimes} l^2)$ in which $\overline{\otimes}$ is a tensor product in Hilbert spaces. Also, consider $z \in \mathbb{C}$ then we have;

$$\begin{aligned} \|A \otimes I + z(B \otimes C)\| &= \sup_{l \in (l^2 \overline{\otimes} l^2), \|l\|=1} \|(A \otimes I)l + z(B \otimes C)l\| \\ &\geq \|(A \otimes I)(i_1 \otimes i_2) + z(B \otimes C)(i_1 \otimes i_2)\| \\ &= 1 + |z|^2 \geq 1 = \|A \otimes I\|. \end{aligned}$$

So, $\|A \otimes I + z(B \otimes C)\| \geq \|A \otimes I\|$. Hence, $A \otimes I \perp_{BJ} (B \otimes C)$ in $\circledcirc \otimes \circledcirc$. □

4.3 BJO and Norm-attainability

In this section, we characterize norm-attainability via BJO. We give conditions under which operators attain their norms via BJO.
We begin with the following proposition;

Proposition 4.17. *Let $\overline{\mathcal{B}}$ be a SRB-space and A_1, A_2 in $\overline{\mathcal{B}}$ be such that $A_1 \perp_{BJ} A_2$. Let $\lambda \in \mathbb{R}$, then $A_1, A_2 \in \mathfrak{NA}_{op}$.*

Proof. We first prove the necessity. Since $\overline{\mathcal{B}}$ is smooth, then we have $\lambda \in \mathbb{R}$. We need to show that there exists a unit vector $x \in \mathcal{B}$ such that $\|A_1x\| = \|A_1\|$ and $\mathbb{R}e\langle A_1x, A_2x\rangle = 0$ if and only if $\|A_1 + \lambda A_2\| \geq \|A_1\|$. Consider a smooth unit point $x \in \mathcal{B}$ such that $\|A_1 + \lambda A_2\|^2 \geq \|(A_1 + \lambda A_2)x\|^2 = \|A_1\|^2$.
For the proof of sufficiency, suppose that A_1 is a positive semidefinite matrix operator and there exists a unit vector $w \in \mathcal{B}$ such that $A_1w = \|A_1\|w$

and $\mathbb{R}e\langle A_1w, A_2w\rangle = 0$. From [16], A_1 can be expressed as a singular value decomposition as $A_1 = UA_1^{\mu}V$, we have that $\|A_1^{\mu} + \lambda U^{\star}A_2V^{\star}\| \geq \|A_1^{\mu}\|\forall\lambda \in \mathbb{R}$ and $A_1^{\mu} = \|A_1^{\mu}\|w, \mathbb{R}e\langle A_w^{\mu}, U^{\star}A_2V^{\star}w\rangle = 0$. So, for $x = V^{*}w$ we obtain $\|A_1x\| = \|A_1\|$ and $\mathbb{R}e\langle A_1x, A_2x\rangle = 0$. But, A_1 is positive semidefinite.

From Gateaux derivative criterion, let a quadratic form be defined by $\mathfrak{f} : \mathcal{B} \longrightarrow \overline{\mathcal{B}}$ by $u \mapsto \langle V, A_2V\rangle$. Then from Hausdorff-Toeplitz theorem, we have that the set $c = \{\langle V, A_2V\rangle : \|V\| = 1, A_1V = \|A_1\|V\}$ which is the image of $\{V : \|V\| = 1, A_1V = \|A_1\|V\}$ under $\mathfrak{f}$. From the numerical range property of convexity, C is convex.

So, $C^{\circ} = \{\mathbb{R}e\langle V, A_2V\rangle : \|V\| = 1, A_1V = \|A_1\|V\}$ is convex.

From $Conv(C^{\circ})$ we find from [16], that there exists a unit vector w such that $A_1w = \|A_1\|w$ and $\mathbb{R}e\langle w, A_2w\rangle = 0$ which implies that $\mathbb{R}e\langle A_1w, A_2w\rangle = 0$ and hence $A_1, A_2 \in \mathfrak{N}\mathfrak{A}_{op}$. □

Remark 4.18. The result in Proposition 4.17 considers λ when it is real. In the next result, we consider λ when it is complex.

Proposition 4.19. *Let $\overline{\mathcal{B}}$ be a SRB and let $A_1, A_2 \in \overline{\mathcal{B}}$ be such that $A_1 \perp_{BJ} A_2$. Let $\lambda \in \mathbb{C}$, then $A_1, A_2 \in \mathfrak{N}\mathfrak{A}_{op}$.*

Proof. Following the same argument as in Proposition 4.17, let $A_1 \perp_{BJ} A_2$ hold. This suffices if there exists a unit vector $x \in \mathcal{B}$ such that;

$$\|A_1 + \lambda A_2\| \geq \|A_1\|, \forall\lambda \in \mathbb{C}. \tag{4.3.1}$$

Now let $l, \theta \in \mathbb{R}$ be given, then Inequality 4.3.1 becomes;

$$\|A_1 + le^{i\theta}A_2\| \geq \|A_1\|, \forall l, \theta \in \mathbb{R}. \tag{4.3.2}$$

Since A_2 is dependent on θ and fixing θ we obtain from Inequality 4.3.2, that $\|A_1 + lA_2\| \geq \|A_1\|, \forall l \in \mathbb{R}$.
Suppose that A_1 is positive semidefinite by Proposition 4.17, then

$$A_1 w_0 = \|A_1\| w_0 \text{ and } \mathbb{R}e\ e^{i\theta}\langle A_1 w_0, A_2 w_0\rangle = 0. \tag{4.3.3}$$

Define the set $C' = \{\langle A_2^* A_1 w, w\rangle : \|w\| = 1, A_1 w = \|A_1\| w\}$. This is a complex subset of $\mathbb{C}$. In fact, it is a numerical range of $A_2^* A$. Let $0 \notin C'$. Now, since C' is the numerical range of complex numbers, then there exists $v \in \mathbb{C}$ such that;

$$\mathbb{R}e\overline{v}\langle A_2^* A_1 w, w\rangle > 0, \tag{4.3.4}$$

from Separation Hyperplane Theorem [85] and the fact that $A_1 w = \|A_1\| w$.
Let $V = |V|e^{i\theta_0}$ and substituting in strict Inequality 4.3.4, we obtain $\mathbb{R}e\ e^{-i\theta_0}\langle A_2^* A_1 w, w\rangle > 0$, $\forall w \in \mathcal{B}$ with $\|w\| = 1, A_1 w = \|A_1\| w$. This contradicts Condition 4.3.3 since $0 \in \mathbb{C}$. That is $A_1 w = \|A_1 w\|$ and $\langle A_1 w, A_2 w\rangle = 0$. Hence, $A_1, A_2 \in \mathfrak{N}\mathfrak{A}_{op}$. □

In this next lemma, we consider Hilbert spaces in infinite dimensions.

Lemma 4.20. *Let $\mathcal{H}$ be a finite dimensional Hilbert space and $A_1, A_2 \in B(\mathcal{H})$. We have that $A_1, A_2 \in \mathfrak{N}\mathfrak{A}_{op}$ if and only if $A_1 \perp_{BJ} A_2$.*

Proof. Suppose that $A_1 \perp_{BJ} A_2$ holds. Let $J : \mathcal{H} \longrightarrow \mathcal{H}$ and J° be on $\mathcal{H} \bigoplus \mathcal{H}$ given by

$$J^\circ = \begin{pmatrix} 0 & J \\ 0 & 0 \end{pmatrix}.$$

Since J° acts on the direct sum of $\mathcal{H}$, then $\|J^\circ\| = \|J\|$. Taking similar representations of A_1 and A_2 as A_1° and A_2° on $\mathcal{H}\bigoplus\mathcal{H}$ respectively, we have that $\|A_1^\circ + \lambda A_2^\circ\| \geq \|A_1^\circ\|, \forall\lambda \in \mathbb{C}$. Let $x_n \bigoplus y_n$ be a sequence of unit vectors in $\mathcal{H}\bigoplus\mathcal{H}$, then

$$\lim_{n\to\infty} \|A_1^\circ(x_n \oplus y_n)\| = \|A_1^\circ\|, \tag{4.3.5}$$

and

$$\lim_{n\to\infty} \langle A_1^\circ(x_n \oplus y_n), A_2^\circ(x_n \oplus y_n)\rangle = 0.$$

From Equation 4.3.5, we find that

$$\lim_{n\to\infty} \|A_1 x_n\| = \|A_1\|.$$

But

$$\|A_1\| = \lim_{n\to\infty} \|A_1 x_n\| \leq \|A_1\| \lim \inf_{n\to\infty} \|x_n\|$$

which implies that

$$\lim \inf_{n\to\infty} \|x_n\| \geq 1.$$

But $\|x_n\| \leq 1, \forall n \in \mathbb{N}$. So we have

$$\lim \sup_{n\to\infty} \|x_n\| \leq 1$$

and therefore

$$\lim_{n\to\infty} \|x_n\| = 1.$$

By Bolzano-Weierstrass Theorem, we have the sequence x_n of unit vectors in $\mathcal{H}$ such that

$$\lim_{n\to\infty} \|A_1 x_n\| = \|A_1\|$$

and

$$\lim_{n\to\infty} \langle A_1 x_n, A_2 x_n\rangle = 0.$$

Hence, $A_1, A_2 \in \mathfrak{NA}_{op}$.

Conversely, let $A_1, A_2 \in \mathfrak{NA}_{op}$. This implies that there exists a sequence of unit vectors in $\mathcal{H}$ such that $\forall n \in \mathbb{N}$;

$$\begin{aligned} \|A_1 + \lambda A_2\|^2 &\geq \|(A_1 + \lambda A_2)x_n\|^2 \\ &= \|A_1 x_n\|^2 + |\lambda|^2 \|A_2 x_n\|^2 + 2\mathbb{R}e\overline{\lambda}\langle A_1 x_n, A_2 x_n\rangle \\ &\geq \|A_1 x_n\|^2 + 2\mathbb{R}e\overline{\lambda}\langle A_1 x_n, A_2 x_n\rangle. \end{aligned}$$

As $n \to \infty$, we have that $\|A_1 + \lambda A_2\|^2 \geq \|A_1\|^2$. Taking positive square root on both sides, we have that $\|A_1 + \lambda A_2\| \geq \|A_1\|$.

Hence, $A_1 \perp_{BJ} A_2$. □

Theorem 4.21. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. Then $A_1, A_2 \in \mathfrak{NA}_{op}$ if and only if $A_1 \perp_{BJ} A_2$.*

Proof. Let $A_1, A_2 \in B(\mathcal{H}_1, \mathcal{H}_2)$ and suppose that $A_1 \perp_{BJ} A_2$. From Lemma 4.20, there exists a sequence of unit vectors x_n such that

$$\lim_{n\to\infty} \|A_1 x_n\| = \|A_1\|$$

and

$$\lim_{n\to\infty} \langle A_1 x_n, A_2 x_n\rangle = 0.$$

But by Bolzano-Weierstrass Theorem, x_n must have a convergent subsequence converging to a unit vector x. Hence, $A_1, A_2 \in \mathfrak{NA}_{op}$.

Conversely, let $A_1, A_2 \in \mathfrak{NA}_{op}$. Then there exists $x_n \in \mathcal{H}$ such that $\|A_1 + \lambda A_2\| \geq \|A_1\|, \forall \lambda \in \mathbb{C}$. From the converse of Lemma 4.20, the proof is complete. □

Corollary 4.22. *Let $\mathcal{H}$ be a complex Hilbert space and $A_1, A_2 \in B(\mathcal{H})$. Then $A_1, A_2 \in \mathcal{NA}_{op}$ if and only if $A_1 \perp_{BJ} A_2$.*

Proof. Let $\|A_1\| = 1$. Then by Corollary 3.12 of [81], we have that $A_1 \perp_{BJ} A_2$ if and only if $0 \in C^a = Conv(\{W^\star(A_2x) : w^\star(A_1x) = \|A_1\|\})$. Since C^a is non-void and convex, we have that $A_1 \perp_{BJ} A_2$ if and only if $0 \in C^b = Conv(\{\langle A_2x, y\rangle : x, y \in S_{\mathcal{H}}, \langle A_1x, y\rangle = \|A_1\| = 1\})$. Indeed, $\{\langle A_2x, y\rangle : x, y \in S_{\mathcal{H}}, \langle A_1x, y\rangle = 1\} = \{\langle A_2x, A_1x\rangle : x \in S_{\mathcal{H}}, \|A_1x\| = 1\}$ □

Corollary 4.23. *Let $\mathcal{H}$ be a complex Hilbert space and $A_1, A_2 \in B(\mathcal{H})$. Then there exists a sequence $x_n \in S_{\mathcal{H}}$ with $\lim_{n\to\infty} \|A_1x_n\| = \|A_1\|$ and $\lim_{n\to\infty}\langle A_1x_n, A_2x_n\rangle = 0$ if and only if $A_1 \perp_{BJ} A_2$.*

Proof. This proof follows directly from Corollary 4.22 by taking $n \to \infty$ for the sequence x_n. Indeed, for all $n \in \mathbb{N}$, $A_1 \perp_{BJ} A_2$ if and only if

$$0 \in C^d = Conv(\{\lim_{n\to\infty}\langle A_2x_n, y_n\rangle : x_n, y_n \in S_{\mathcal{H}}, \lim_{n\to\infty}\langle A_1x_n, y_n\rangle = \|A_1\|\}).$$

But C^d is non-void and convex and so for all $n \in \mathbb{N}$, we have that $\{\lim\langle A_2x, y_n\rangle : x_n, y_n \in S_{\mathcal{H}}, \lim_{n\to\infty}\langle A_1x_n, y_n\rangle = \|A_1\|\} = \{\lim\langle A_2x_n, A_1x_n\rangle : x_n \in S_{\mathcal{H}}, \lim_{n\to\infty} \|A_1x_n\| = \|A_1\|\}$. □

Next, we prove that $A_1, A_2 \in \mathfrak{NA}_{op}$ via BJ-ϵ-approximate orthogonality.

Proposition 4.24. *Let $\mathcal{B}$ be a SB-space. Then $A_1 \perp_{BJ}^{\epsilon} A_2$ if and only if there exists a sequence of vectors in $S_{\mathcal{B}}$ such that* $\lim_{n\to\infty} \|A_1 x_n\| = \|A_1\|$ *and* $\lim_{n\to\infty} \|A_2 x_n\| \leq \epsilon^2 \|A_2\|$.

Proof. Suppose that the sequence in $S_{\mathcal{B}}$ is an orthonormal sequence. Then for every $A_1, A_2 \in \mathcal{B}$, there exists $\mathcal{P}$ in the $\text{Span}\{A_1, A_2\}$ such that $A_1 \perp_{BJ} \mathcal{P}$. This statement is true since $A_1 \perp_{BJ}^{\epsilon} A_2$ from the statement of the problem. Moreover, $\|P - A_2\| \leq \epsilon^2 \|A_2\|$. This implies that there exists an orthonormal sequence of unit vectors x_n such that

$$\lim_{n\to\infty} \|A_1 x_n\| = \|A_1\|$$

and

$$\lim_{n\to\infty} \|P x_n\| = 0.$$

Therefore, we have that $\|P x_n - A_2 x_n\| \leq \|P - A_2\| < \epsilon^2 \|A_2\|$ which implies that

$$\lim_{n\to\infty} \|A_1 x_n\| = \|A_1\|$$

and

$$\lim_{n\to\infty} \|A_2 x_n\| \leq \epsilon^2 \|A_2\|.$$

Conversely, let there be a sequence $x_n \in S_{\mathcal{B}}$ such that

$$\lim_{n\to\infty} \|A_1 x_n\| = \|A_1\|$$

and

$$\lim_{n\to\infty} \|A_2 x_n\| \leq \epsilon^2 \|A_2\|.$$

We prove that $A_1 \perp_{BJ}^{\epsilon} A_2$. Now let $\lambda \in \mathbb{R}$, then for $A_1, A_2 \in \mathcal{B}$, we have that

$$\begin{aligned} \|(A_1 + \lambda A_2)x_n\|^2 &\geq |\|A_1 x_n\| - |\lambda|\|A_2 x_n\||^2 \\ &\geq \|A_1 x_n\|^2 - 2\|A_1 x_n\||\lambda|\|A_2 x_n\|. \end{aligned}$$

Taking limits as $n \to \infty$, we have $\|A_1 + \lambda A_2\|^2 \geq \|A_1\|^2 - 2\epsilon^2 \|A_1\|\|\lambda A_2\|$ which implies that $A_1 \perp_{BJ}^{\epsilon} A_2$. □

Theorem 4.25. *Let $\mathcal{B}$ be a SB-space then $A_1 \perp_{BJ}^{\epsilon} A_2$ if there exists a sequence $v_n \in S_{\mathcal{B}}$ and another sequence ϵ_n of real numbers such that for all $m \in \mathbb{N}$,*

$$\|A_1 v_n + \lambda A_2 v_n\|^2 \geq (1 - \epsilon_n^m)\|A_1 v_n\|^2 - n\epsilon\sqrt{1 - \epsilon_n^m}\|A_1 v_n\|\|\lambda A_2\|, \forall \lambda \geq 0. \tag{4.3.6}$$

Proof. From the argument of the first part of the proof of Proposition 4.24, we have an orthonormal sequence $\nu_n \in S_{\mathcal{B}}$ such that for $A_1, A_2 \in \mathcal{B}$, there exists P_1 in the Span$\{A_1, A_2\}$ such that $A_1 \perp_{BJ} P_1$. This implies that $A_1 \perp_{BJ} A_2$. Since we have ν_n being orthonormal, let there be a sequence ϵ_n of real numbers such that $\epsilon_n \to 0$, $\lim_{n\to\infty} \|A_1 \nu_n\| = \|A_1\|$ and $P_1 \nu_n \in A_1 \nu_n|_{+\epsilon_n^m}, \forall m \in \mathbb{N}$. This is an implication that $\forall \lambda \geq 0$ we obtain;

$$
\begin{aligned}
\|A_1\nu_n + \lambda A_2\nu_n\|^2 &= \|(A_1\nu_n + \lambda P_1\nu_n) + \lambda(A_2\nu_n - P_1\nu_n)\|^2 \\
&\geq |\|A_1\nu_n + \lambda P_1\nu_n\| - |\lambda|\|A_2\nu_n - P_1\nu_n\||^2 \\
&\geq \|A_1\nu_n + \lambda P_1\nu_n\|^2 - 2\|A_1\nu_n + \lambda P_1\nu_n\||\lambda|\|P_1\nu_n - A_2\nu_n\| \\
&\geq (1 - \epsilon_n^m)\|A_1\nu_n\|^2 - m\epsilon\sqrt{1 - \epsilon_n^m}\|A_1\nu_n\|\|\lambda A_2\|.
\end{aligned}
$$

$\forall m \in \mathbb{N}$. To prove sufficiency, let Inequality 4.3.6 hold. Without loss of generality, picking the sequence ν_n, then we have that $\forall \lambda \geq 0$,

$$
\begin{aligned}
\|A_1 + \lambda A_2\|^2 &\geq \|(A_1 + \lambda A_2)\nu_n\|^2 \\
&\geq (1 - \epsilon_n^m)\|A_1\nu_n\| - m\epsilon\sqrt{1 - \epsilon_n^m}\|A_1\nu_n\|\|\lambda A_2\|.
\end{aligned}
$$

Taking limits as $n \to \infty$, we obtain $\|A_1 + \lambda A_2\| \geq \|A_1\|^2 - m\epsilon\|A_1\|\|\lambda A_2\|$. Hence, $A_1 \perp_{BJ} A_2$. □

Corollary 4.26. *Let $\mathcal{B}$ be a SB-space then $A_1 \perp_{BJ}^{\epsilon} A_2$ if there exists a sequence $w_n \in S_{\mathcal{B}}$ and another sequence χ_n^m of real numbers such that for all $m \in \mathbb{N}$ and for all $\lambda \leq 0$,*

$$
\|A_1 w_n + \lambda A_2 w_n\|^2 \geq (1 - \chi_n^m)\|A_1 w_n\|^2 - n\epsilon\sqrt{1 - \chi_n^m}\|A_1 w_n\|\|\lambda A_2\|.
$$

Proof. Again, from the argument of the first part of the proof of Proposition 4.24, we have an orthonormal sequence $w_n \in S_{\mathcal{B}}$ such that for $A_1, A_2 \in \mathcal{B}$, there exists P_2 in the Span$\{A_1, A_2\}$ such that $A_1 \perp_{BJ} P_2$. This implies that $A_1 \perp_{BJ} A_2$. Since we have w_n being orthonormal, let there be a sequence χ_n of real numbers such that $\chi_n \to 0$, and $\lim_{n\to\infty} \|A_1 w_n\| = \|A_1\|$ and $P_2 w_n \in A_1 w_n|_{-\chi_n^m}, \forall m \in \mathbb{N}$. This is an implication that $\forall \lambda \leq 0$ we obtain;

$$\begin{aligned}
\|A_1 w_n + \lambda A_2 w_n\|^2 &= \|(A_1 w_n + \lambda P_2 w_n) + \lambda(A_2 w_n - P_2 w_n)\|^2 \\
&\leq |\|A_1 w_n + \lambda P_2 w_n\| - |\lambda|\|A_2 w_n - P_2 w_n\||^2 \\
&\leq \|A_1 w_n + \lambda P_2 w_n\|^2 - 2\|A_1 w_n + \lambda P_2 w_n\||\lambda|\|P_2 w_n - A_2 w_n\| \\
&\leq (1-\chi_n^m)\|A_1 w_n\|^2 - m\epsilon\sqrt{1-\chi_n^m}\|A_1 w_n\|\|\lambda A_2\|, \forall m \in \mathbb{N}
\end{aligned}$$

To prove sufficiency, let Inequality 4.3.6 hold. Without loss of generality, pick the sequence w_n, then we have that $\forall \lambda \leq 0$,

$$\begin{aligned}
\|A_1 + \lambda A_2\|^2 &\leq \|(A_1 + \lambda A_2) w_n\|^2 \\
&\leq (1-\chi_n^m)\|A_1 w_n\| - m\epsilon\sqrt{1-\chi_n^m}\|A_1 w_n\|\|\lambda A_2\|.
\end{aligned}$$

Taking limits as $n \to \infty$ we obtain $\|A_1 + \lambda \mathcal{A}_2\| \geq \|A_1\|^2 - m\epsilon\|A_1\|\|\lambda A_2\|$. Hence $A_1 \perp_{BJ} A_2$. □

4.4 Relationship between $\mathfrak{NA}_v$ and $\mathfrak{NA}_{op}$

In this section, we characterize two sets namely, the set of norm-attainable operators and the set of norm-attainable vectors independently. Then we establish the relationship between these two sets via BJO. We denote the set of all norm-attainable vectors by $\mathfrak{NA}_v$ and the set of all norm-attainable operators by $\mathfrak{NA}_{op}$. The Birkhoff-James orthogonality set (BJOS) is denoted by $\mathcal{O}_{BJOS}^{(\epsilon:x,y)}$ and $\mathcal{O}_{BJOS}^{(\epsilon:A,B)}$.
We begin with the following proposition;

Proposition 4.27. *Let $\mathcal{B}$ be a Banach space of all members of $\mathfrak{NA}_v$ and*

$\mathcal{O}^{(\epsilon:x,y)}_{BJOS}$ of elements of $\mathcal{B}$ then $\mathcal{O}^{(\epsilon:x,y)}_{BJOS}$ is non-void.

Proof. Since $\mathcal{O}^{(\epsilon:x,y)}_{BJOS}$ is an $\epsilon-$approximately BJO, then for any $x, y \in \mathfrak{B}$, we have that $\epsilon \in [0, 1)$. It is sufficient to show that $\mathcal{O}^{(\epsilon:x,y)}_{BJOS} \neq \emptyset$. By Hahn-Banach theorem we have that for any $y \in \mathfrak{B}$ which is nonzero, there exists a linear functional ϕ on $\mathfrak{B}$ to the set of complex numbers such that $\phi(y) = \|\phi\|\|y\|$. It follows that $\|\phi\|\|y\| = |\phi(\overline{x} + y)| \leq \|\phi\|\|\overline{x} + y\|, \forall \overline{x} \in Ker(\phi)$. So,

$$y \perp_{BJ} \overline{x}, \forall \overline{x} \in Ker(\phi). \tag{4.4.1}$$

Now consider a scalar $\alpha = \frac{\phi(x)}{\|\phi\|\|y\|}$. We obtain $\phi(x - \alpha y)$ and so $x - \alpha y \in Ker(\phi)$. From Equation 4.4.1 we have that $y \perp_{BJ} (x - \alpha y)$ and hence $\alpha \in \mathcal{O}^{(\epsilon:x,y)}_{BJOS}$. Therefore, $\mathcal{O}^{(\epsilon:x,y)}_{BJOS}$ is non-void. □

Proposition 4.28. *For any $\zeta \neq 0$ in $\mathbb{C}, \mathcal{O}^{(\epsilon:x,y)}_{BJOS} = \zeta^{-1}\mathcal{O}^{(\epsilon:x,y)}_{BJOS}$.*

Proof. It is known that BJ ϵ-orthogonality is homogenous. So, from the definition of BJO, we have that $\mathcal{O}^{(\epsilon:x,y)}_{BJOS} = \{\alpha \in \mathbb{C} : y \perp^{\epsilon}_{BJ} (x - (\zeta\alpha)y)\}$. □

Remark 4.29. Next, we carry out characterization in terms of equivalent norms in $\mathcal{B}$. Consider two usual operator norms $\|.\|$ and $\|.\|'$ in $\mathcal{B}$. We obtain the equivalence relation between $\mathcal{O}^{(\epsilon:x,y)}_{BJOS}$ and $\mathcal{O}^{(\epsilon^o:x,y)}_{BJOS}$ in the next lemma.

Lemma 4.30. *Let $\mathcal{B}$ be a Banach space of all operators in $\mathfrak{NA}_v$. Let $\|.\|$ and $\|.\|'$ be equivalent norms in $\mathcal{B}$. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be positive real numbers such that $\mathcal{P}_1\|B\| \leq \|B\|' \leq \mathcal{P}_2\|B\|$, for all $B \in \mathcal{B}$. Let $x, y \in \mathcal{B}$ with $y \neq 0$ and $\epsilon \in [0, 1)$. Then $\mathcal{O}^{(\epsilon:x,y)}_{BJOS} \subseteq \mathcal{O}^{(\epsilon^o:x,y)}_{BJOS}$ where $\epsilon^o = \sqrt{\frac{1-\mathcal{P}_1(1-\epsilon^n)}{\mathcal{P}_2}}$, for all $n \in \mathbb{N}$.*

Proof. let $\alpha \in \mathcal{O}_{BJOS}^{(\epsilon:x,y)}$. Then it follows from Proposition 4.28 that

$$\|x - \lambda y\| \geq \sqrt{1-\epsilon^n}\|y\||\alpha - \lambda|, \forall \lambda \in \mathbb{C}, \forall n \in \mathbb{N}. \tag{4.4.2}$$

or,

$$\|x - \lambda y\| \geq \sqrt{1-\epsilon^n \frac{\mathcal{P}_1}{\mathcal{P}_2}}\|y\||\alpha - \lambda|, \forall \lambda \in \mathbb{C}, \forall n \in \mathbb{N}. \tag{4.4.3}$$

which implies that

$$\|x - \lambda y\| \geq \sqrt{1-(\epsilon^o)^n}\|y\||\alpha - \lambda|, \forall \lambda \in \mathbb{C}, \forall n \in \mathbb{N}. \tag{4.4.4}$$

From Inequalities 4.4.2 and 4.4.4 we have an implication that $\mathcal{O}_{BJOS}^{(\epsilon:x,y)} \subseteq \mathcal{O}_{BJOS}^{(\epsilon^o:x,y)}$. □

Proposition 4.31. *Let $\overline{\mathcal{B}}$ be a SB-space of all members of $\mathfrak{N}\mathfrak{A}_{op}$ and $\mathcal{O}_{BJOS}^{(\epsilon:A_1,A_2)}$ be of elements of $\overline{\mathcal{B}}$. Then $\mathcal{O}_{BJOS}^{(\epsilon:A_1,A_2)}$ is non-void.*

Proof. This proof is a straight forward analogy of the proof of Proposition 4.27. This follows by replacing x with A_1 and y with A_2. □

At this point, we give the relationship between $\mathfrak{N}\mathfrak{A}_{op}$ and $\mathfrak{N}\mathfrak{A}_v$ via $\mathcal{O}_{BJOS}^{(\epsilon:x,y)}$ and $\mathcal{O}_{BJOS}^{(\epsilon:A_1,A_2)}$.

Theorem 4.32. *Let $\overline{\mathcal{B}}$ be a SB-space and $A \in \overline{\mathcal{B}}$. Let $\mathcal{O}_{BJOS}^{(\epsilon:x,y)} \subseteq \mathcal{O}_{BJOS}^{(\epsilon:A_1,A_2)}$. If $x \in \mathcal{B}$ is a left symmetric point, then Ax is a left symmetric point of $\overline{\mathcal{B}}$ and $\mathfrak{N}\mathfrak{A}_v$ is isometrically equivalent to $\mathfrak{N}\mathfrak{A}_{op}$.*

Proof. Let $Ax \perp_{BJ} y$ for some $y \in \mathcal{B}$. Trivially, if $y = 0$, then $y \perp_{BJ} Ax$. So, suppose that $y \neq 0$.
Now, $S_A = S_{\mathcal{B}}$ since A is isometric hence there exists a nonzero vector

$\eta \in \mathcal{B}$ such that $y = A\eta$. Since $Ax \perp_{BJ} A\eta$ and $x \in S_A$, it follows that $x \perp \eta$, an implication that $\mathcal{O}_{BJOS}^{(\epsilon:x,y)} \subseteq \mathcal{O}_{BJOS}^{(\epsilon:A_1,A_2)}$ for all $A_1, A_2 \in \overline{\mathcal{B}}$.

Now, since $\overline{\mathcal{B}}$ is smooth, then we have that $\eta \perp_{BJ} x$ as x is a left symmetric point in $\mathcal{B}$. Now, $\mathcal{B}$ is also smooth and $\frac{\eta}{\|\eta\|} \in S_A$. By [57], we have that $A(\frac{\eta}{\|\eta\|}) \perp_{BJ} Ax$.

But BJO is homogenous hence $A\eta \perp_{BJ} Ax$. Since A is isometric operator in $\mathfrak{NA}_{op}$ and x is a left isometric point in $\mathfrak{NA}_v$, then $\mathfrak{NA}_{op}$ is isometrically equivalent to $\mathfrak{NA}_v$ □

Remark 4.33. Theorem 4.32 holds where $\mathcal{B}$ and $\overline{\mathcal{B}}$ are both reflexive and smooth and $A \in \overline{\mathcal{B}}$ is compact.

Chapter 5

CONCLUSION AND RECOMMENDATIONS

5.1 Introduction

In this last chapter, we draw the conclusion and make recommendations based on our objectives of study and the results obtained therein.

5.2 Conclusion

Our first objective of this study was to establish BJO conditions for operators on Banach spaces. We have been able to show that if $\overline{\mathcal{B}}$ is a smooth reflexive Banach space, then for two operators $A, B \in \overline{\mathcal{B}}$, A is approximately Birkhoff-James orthogonal to the operator B, written $A \perp_{BJ}^{\epsilon} B$ if $\forall \lambda \in \mathbb{C}$, $\|A + \lambda B\| \geq \|A\|^2 - 2\epsilon\|A\|\|\lambda B\|, \forall \epsilon \in [0, 1)$. Moreover, $A \perp_{BJ}^{\epsilon} B$ implies that $\mu A \perp_{BJ}^{\epsilon} \omega B$ for any $\mu, \omega \in \mathbb{C}$ to equally show that BJO is homogenous. We have also been able to show that if $\overline{\mathcal{B}}$ is a SRB-space and $A, B \in \overline{\mathcal{B}}$, then $A \perp_{BJ}^{\epsilon} B$ on $\overline{\mathcal{B}}$ if A and B are approximately semi-orthogonal. Finally, we have shown that if $\mathfrak{C}$ is a C^*-algebra, then $A \perp_{BJ}^{s} B$ in $\mathfrak{C}$ if there is a state ξ in $\mathfrak{C}$ such that $\xi(AA^*) = \|A\|$ and $\xi(BB^*) = 0$ for $A, B \in \mathfrak{C}$.

Our second objective of the study was to determine norm-attainability of operators on Banach spaces via BJO. For this objective, we have been able to show that if $\overline{\mathcal{B}}$ is a SRB-space and A_1, A_2 in $\overline{\mathcal{B}}$ are such that $A_1 \perp_{BJ} A_2$, then $A_1, A_2 \in \mathfrak{NA}_{op}$. We have also been able to show that given $\mathcal{H}$, a finite dimensional Hilbert space and $A_1, A_2 \in B(\mathcal{H})$, then we have that $A_1, A_2 \in \mathfrak{NA}_{op}$ if and only if $A_1 \perp_{BJ} A_2$. We have finally determined that if $\mathcal{B}$ is a SB-space, then $A_1 \perp_{BJ}^{\epsilon} A_2$ if there exists a sequence $v_n \in S_{\mathcal{B}}$ and another sequence ϵ_n of real numbers such that for all $m \in \mathbb{N}$, $\|A_1 v_n + \lambda A_2 v_n\|^2 \geq (1-\epsilon_n^m)\|A_1 v_n\|^2 - n\epsilon\sqrt{1-\epsilon_n^m}\|A_1 v_n\|\|\lambda A_2\|, \forall \lambda \geq 0$.

Our third and final objective required us to investigate the relationship between the set of norm-attainable vectors and the set of norm-attainable operators via BJO on Banach spaces. For this objective, we have been able to prove that if $\mathcal{P}_1$ and $\mathcal{P}_2$ are positive real numbers such that $\mathcal{P}_1\|B\| \leq \|B\|' \leq \mathcal{P}_2\|B\|$, for all $B \in \mathcal{B}$. Let $x, y \in \mathcal{B}$ with $y \neq 0$ and $\epsilon \in [0, 1)$. Then $\mathcal{O}_{BJOS}^{(\epsilon:x,y)} \subseteq \mathcal{O}_{BJOS}^{(\epsilon^o:x,y)}$ where $\epsilon^o = \sqrt{\frac{1-\mathcal{P}_1(1-\epsilon^n)}{\mathcal{P}_2}}$, for all $n \in \mathbb{N}$. Finally, we proved that given $\overline{\mathcal{B}}$, a SB-space and $A \in \overline{\mathcal{B}}$. Let $\mathcal{O}_{BJOS}^{(\epsilon:x,y)} \subseteq \mathcal{O}_{BJOS}^{(\epsilon:A_1,A_2)}$. If $x \in \mathcal{B}$ is a left symmetric point, then Ax is a left symmetric point of $\overline{\mathcal{B}}$ and $\mathfrak{NA}_v$ is isometrically equivalent to $\mathfrak{NA}_{op}$.

5.3 Recommendations

Studies on structural and geometrical properties of Banach spaces have been carried out over decades with several interesting results obtained. More specifically, characterization of orthogonality in different notions has been carried out by quite a number of scholars beforehand. However, we recommend that further studies can still be carried out to:

(i). Establish mutually strong BJO conditions for operators on Banach spaces.

(ii). Determine norm-attainability of operators on Banach spaces via mutually strong BJO.

(iii). Investigate the relationship between the set of norm-attainable vectors and the set of norm-attainable operators via mutually strong BJO on Banach spaces.

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