FINITE ELEMENT APPROACH TO THE SOLUTION OF FOURTH ORDER BEAM EQUATION:

$$u_{tt} + c^2 u_{xxxx} = f(x,t)$$

Oduor Michael E. Okoya*

T. J. O. Aminer*

Hagai Amakobe James**

Nthiiri Joyce Kagendo***

Abstract

Finite element method is a class of mathematical tool which approximates solutions to initial and boundary value problems. Finite element, basic functions, stiffness matrices,systems of ordinary differential equations and hence approximate solutions of partial differential equations which involves rendering the partial differential equation into system of ordinary differential equations.

The ordinary differential equations are then numerically integrated.

We present a finite element approach in solving fourth order linear beam equation: $u_{tt} + c^2 u_{xxxx} = f(x,t)$, which arises in model studies of building structures wave theory.

In physical application of waves in building structures, coefficient c^2, has the meaning of flexural rigidity per linear mass density and $f(x,t)$ external forcing term. In this paper, we give a solution to the beam equation with $c^2 = 139$ and $f(x,t) = 100$.

Keywords: beam equation, finite element,approximation functions, stiffness matrix

* Department of Mathematics and Actuarial Science, JaramogiOgingaOdinga University of Science and Technology,Bondo, Kenya

** ACK Milimani Girls Secondary School, Musanda-Kenya

*** Department of Mathematics, MasindeMuliro University of Science and Technology, Kakamega, Kenya

1. Introduction

A beam is a structural element or member designed to support loads applied at various points along the element. Beams are one of the components used in structural engineering and can be in 1-dimension, 2-dimension or 3-dimension. They can be horizontal, vertical and also at angles. We have analyzed a uniform elastic beam, simply supported length L and subjected to vertical forces acting in the principal plane of a symmetrical cross-section as shown below.

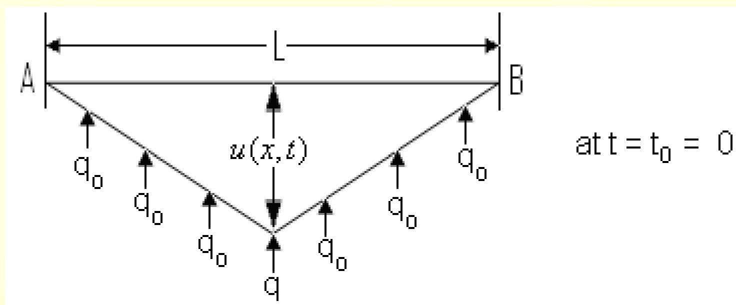

Figure 1: A uniform elastic beam.

q – Centre concentrated load.

q_0 – Uniform distributed load intensity.

L – The length AB.

$u(x,t)$ – deflection at (x,t). where x is a one dimension spatial variable point at time t.

The flexture of the uniform elastic length L whose ends are simply supported can be modeled by the equation:

$$\frac{\partial^2 u}{\partial t^2} + c^2 \frac{\partial^4 u}{\partial x^4} = f(x,t) \qquad (1.1)$$

subject to boundary conditions:

$$u(0,t) = u(L,t) = 0 \qquad (1.1a)$$

$$u_{xx}(0,t) = u_{xx}(L,t) = 0 \qquad (1.1b)$$

According to [12] ,finite element method has been used to approximate the deflection *u(x)* of a simply supported beam. ZaferAhsan[17] applied Laplace transform method, Singh[12] used Rayleigh method and Osongo[15] used direct integration method to the beam equation.

W.T.Thomson[14] employed mode-summation method which gave results in series form with no immediate practical evaluation.

Since all the above mentioned methods presented their solutions in non closed form or single degree of freedom or which lacked immediate practical evaluation, there was a need to look for an alternative numerical approach to the two dimensional one degree of freedom beam equation $u_{tt} + c^2 u_{xxxx} = f(x,t)$.

2 preliminaries

Basing on [6,7,10,11,12], Finite element method involves discretization, development of local mass, stiffness and force matrices and then assembly to global load, stiffness and force matrix using finite element method thereafter approximate resultant ordinary differential equations.

Consider the BVP

$$\mu \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 u}{\partial x^2} \right) = \mu f(x,t) \qquad (2.1)$$

subject to boundary conditions,

(i) u(0,t)=u(L,t)=0

(ii) u_{xx} (0,t)=u_{xx}(L,t)=0 .

The function of approximation is given by:

$$\hat{u}(x,t) = \sum_{i=1}^{5} \alpha_i(t) \phi_i(x) \qquad (2.2)$$

where

$\phi_i(x)$ is the shape function or basis function

and $\alpha_i(t)$ the Fourier coefficients

then , solution is assumed to be in form;

$$\hat{u}(x,t) = \sum_{i=1}^{5} \alpha_i(t) \sin\left(\frac{i\pi x}{L}\right) . \qquad (2.3)$$

and

$$residual = r(x,t) = \mu \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 u}{\partial x^2} \right) - \mu f(x,t) = 0 \qquad (2.4)$$

Multiplying r(x,t) by test function v(x) and varying the integral to weak form and integrate the product of r(x,t) and v(x) and equate to zero;

$$\int_{x_{e-1}}^{x_e}\left[\mu v\frac{\partial^2 u}{\partial t^2}+v\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 u}{\partial x^2}\right)-v\mu f(x,t)\right]dx=0 \qquad (2.5)$$

If assumed approximate solution is;

$$\hat{u}(x,t)=\sum_{i=1}^{5}\phi_i(x)\alpha_i(t) \qquad (2.6)$$

then;

$$\frac{\partial \hat{u}}{\partial t}(x,t)=\sum_{i=1}^{5}\phi_i(x)\dot{\alpha}_i(t) \qquad (2.7)$$

$$\frac{\partial^2 \hat{u}}{\partial t^2}=\sum_{i=1}^{5}\phi_i(x)\ddot{\alpha}_i(t) \qquad (2.8)$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 \hat{u}}{\partial x^2}\right)=\sum_{i=1}^{5}\phi_i^{(iv)}(x)\alpha_i(t) \qquad (2.9)$$

Substituting equations (2.8) and (2.9) into (2.5) we obtain:

$$\int_{x_{e-1}}^{x_e}\left[\mu v\sum_{i=1}^{5}\phi_i(x)\ddot{\alpha}_i(t)+vEI\sum_{i=1}^{5}\phi_i^{(iv)}(x)\alpha_i(t)-v\mu f(x,t)\right]dx=0 \qquad (2.10)$$

Equation (2.10) can be divided into three major parts (I,II and III):

$$I=\int_{x_{e-1}}^{x_e}\mu v\sum_{i=1}^{5}v(x)\phi_i(x)\ddot{\alpha}_i(t)dx \qquad (2.11)$$

this gives mass matrix $C_{ij}=\sum_{i=1}^{5}\mu\ \ddot{\alpha}_i(t)\int_{x_{e-1}}^{x_e}\phi_i(x)\phi_j(x)dx$

in matrix form we have:

$$C_{ij}=\frac{\mu L}{2\pi}\begin{bmatrix}-0.5 & 0.4714 & 0.5 & 0.3771 & 0.1667\\ 0.4714 & 0 & 0.8485 & 0.6667 & 0.3367\\ 0.5 & 0.8485 & 0.1667 & 0.8081 & 0.5\\ 0.3771 & 0.6667 & 0.8081 & 0 & 0.6285\\ 0.1667 & 0.3367 & 0.5 & 0.6285 & -0.1\end{bmatrix}\begin{bmatrix}\ddot{\alpha}_1\\ \ddot{\alpha}_2\\ \ddot{\alpha}_3\\ \ddot{\alpha}_4\\ \ddot{\alpha}_5\end{bmatrix}$$

$$II = \int_{x_{e-1}}^{x_e} \sum_{i=1}^{5} v(x)\phi_i^{iv}\alpha_i(t)dx \qquad (2.12)$$

gives the stiffness matrix $K_{ij} = \sum_{i=1}^{5} EI \int_{X_{e-1}}^{X_e} v(x)\frac{d^2}{dx^2}\left(\alpha_i(t)\frac{d^2u}{dx^2}\right)dx$.

In matrix form the stiffness matrix is;

$$K_{ij} = \frac{EI\pi^3}{2L^3}\begin{bmatrix} -0.5 & 1.885 & 4.5 & 6.033 & 4.167 \\ 1.885 & 0 & 30.54 & 42.67 & 33.67 \\ 4.5 & 30.54 & 13.5 & 116.37 & 112.5 \\ 6.033 & 42.67 & 116.37 & 0 & 251.41 \\ 4.167 & 33.67 & 112.5 & 251.41 & -62.5 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix}$$

$$III = \int_{X_{e-1}}^{X_e} \mu v f(x,t)dx \qquad (2.13)$$

we obtain the nodal force vector

$$F_j = \int_{x_{e-1}}^{x_e} v(x)\mu f(x,t)dx$$

In matrix form wehave

$$F = \frac{\mu L}{\pi}\begin{bmatrix} -0.7071.f(x,t) \\ 0 \\ 0.2357f(x,t) \\ 0.25f(x,t) \\ 0.14142f(x,t) \end{bmatrix} \qquad (2.14)$$

The equilibrium equation is;

$$\frac{\mu L}{2\pi}\begin{bmatrix} -0.5 & 0.4714 & 0.5 & 0.3771 & 0.1667 \\ 0.4714 & 0 & 0.8485 & 0.6667 & 0.3367 \\ 0.5 & 0.8485 & 0.1667 & 0.8081 & 0.5 \\ 0.3771 & 0.6667 & 0.8081 & 0 & 0.6285 \\ 0.1667 & 0.3367 & 0.5 & 0.6285 & -0.1 \end{bmatrix}\begin{bmatrix} \ddot{\alpha}_1 \\ \ddot{\alpha}_2 \\ \ddot{\alpha}_3 \\ \ddot{\alpha}_4 \\ \ddot{\alpha}_5 \end{bmatrix} + \frac{EI\pi^3}{2L^3}\begin{bmatrix} -0.5 & 1.885 & 4.5 & 6.033 & 4.167 \\ 1.885 & 0 & 30.54 & 42.67 & 33.67 \\ 4.5 & 30.54 & 13.5 & 116.37 & 112.5 \\ 6.033 & 42.67 & 116.37 & 0 & 251.41 \\ 4.167 & 33.67 & 112.5 & 251.41 & -62.5 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} =$$

$$\frac{\mu L}{\pi}\begin{bmatrix} -0.7071.f(x,t) \\ 0 \\ 0.2357f(x,t) \\ 0.25f(x,t) \\ 0.14142f(x,t) \end{bmatrix}. \qquad (2.15)$$

A similar equation with ten subdivisions(case b) with have an equilibrium equation of the form

$$\frac{\mu L}{2\pi}\begin{bmatrix} 0.2939 & 0.0393 & 0.0561 & 0.0697 & 0.0793 & 0.0844 & 0.085 & 0.0813 & 0.0735 & 0.0624 & 0.0490 \\ 0.0393 & -0.2378 & 0.1090 & 0.1354 & 0.1541 & 0.1643 & 0.1657 & 0.1585 & 0.1437 & 0.1225 & 0.0965 \\ 0.0561 & 0.1090 & -0.1585 & 0.1934 & 0.2204 & 0.2354 & 0.2378 & 0.2281 & 0.2075 & 0.1778 & 0.1414 \\ 0.0697 & 0.1354 & 0.1934 & -0.0735 & 0.2747 & 0.2939 & 0.2978 & 0.2868 & 0.2622 & 0.2264 & 0.1823 \\ 0.0793 & 0.1541 & 0.2204 & 0.2747 & 0 & 0.3371 & 0.3429 & 0.3319 & 0.3057 & 0.2667 & 0.2179 \\ 0.0844 & 0.1643 & 0.2354 & 0.2939 & 0.3371 & 0.0490 & 0.3713 & 0.3618 & 0.3364 & 0.2972 & 0.2476 \\ 0.085 & 0.1657 & 0.2378 & 0.2978 & 0.3429 & 0.3713 & 0.0679 & 0.3757 & 0.3533 & 0.3173 & 0.2705 \\ 0.0813 & 0.1585 & 0.2281 & 0.2868 & 0.3319 & 0.3618 & 0.3757 & 0.0594 & 0.3567 & 0.3266 & 0.2860 \\ 0.0735 & 0.1437 & 0.2075 & 0.2622 & 0.3057 & 0.3364 & 0.3533 & 0.3567 & 0.3417 & 0.3253 & 0.2939 \\ 0.0624 & 0.1225 & 0.1778 & 0.2264 & 0.2667 & 0.2972 & 0.3173 & 0.3266 & 0.3253 & 0 & 0.2943 \\ 0.0490 & 0.0965 & 0.1414 & 0.1823 & 0.2179 & 0.2476 & 0.2705 & 0.2860 & 0.2939 & 0.2943 & -0.0267 \end{bmatrix}\begin{bmatrix} \ddot{\alpha}_1 \\ \ddot{\alpha}_2 \\ \ddot{\alpha}_3 \\ \ddot{\alpha}_4 \\ \ddot{\alpha}_5 \\ \ddot{\alpha}_6 \\ \ddot{\alpha}_7 \\ \ddot{\alpha}_8 \\ \ddot{\alpha}_9 \\ \ddot{\alpha}_{10} \\ \ddot{\alpha}_{11} \end{bmatrix}$$

$$
+\frac{EI\pi^3}{2L^3}\begin{bmatrix}
-0.2939 & 0.1572 & 0.5049 & 1.1152 & 1.9825 & 3.0384 & 4.165 & 5.2032 & 5.951 & 6.239 & 5.927 \\
0.1572 & -3.804 & 3.924 & 8.666 & 15.41 & 23.66 & 32.48 & 40.58 & 46.56 & 48.99 & 46.72 \\
0.5049 & 3.924 & -12.84 & 27.85 & 49.59 & 76.27 & 104.9 & 131.4 & 151.25 & 160.1 & 154.0 \\
1.1152 & 8.666 & 27.849 & -18.81 & 109.9 & 169.3 & 233.5 & 293.7 & 339.9 & 362.3 & 352.9 \\
1.9825 & 15.41 & 49.59 & 109.9 & -21.46 & 303.4 & 420 & 531.1 & 619 & 666.7 & 659.3 \\
3.0384 & 23.66 & 76.27 & 169.3 & 303.4 & -63.4 & 654.9 & 833.7 & 980.9 & 1070.1 & 1078.5 \\
4.165 & 32.48 & 104.9 & 233.5 & 420 & 654.9 & -163 & 1178 & 1402 & 1554 & 1603 \\
5.2032 & 40.58 & 131.4 & 293.7 & 531.1 & 833.7 & 1178 & -243.5 & 1848.6 & 2090 & 2214 \\
5.951 & 46.56 & 151.3 & 339.9 & 619.1 & 980.9 & 1402 & 1848.6 & 312 & 2639.6 & 2880 \\
6.239 & 48.99 & 160.1 & 362.3 & 666.7 & 1070.1 & 1554 & 2090 & 2639.6 & 0 & 3561 \\
5.927 & 46.27 & 154.0 & 352.9 & 659.3 & 1078.5 & 1603 & 2214 & 2880 & 3561 & -391.2
\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\\ \alpha_5\\ \alpha_6\\ \alpha_7\\ \alpha_8\\ \alpha_9\\ \alpha_{10}\\ \alpha_{11}\end{bmatrix} =
$$

$$
\frac{\mu f(x,t)L}{\pi}\begin{bmatrix}-0.9511\\ -0.4045\\ -0.1959\\ 0.7725\\ 0\\ 0.0515\\ 0.0840\\ 0.1011\\ 0.1057\\ 0.1\\ -0.0865\end{bmatrix} . \qquad (2.16)
$$

From [5, 6], stability depends only on mass and stiffness of the system and is independent of the number of subdivisions of the beam.

3 NUMERICAL SOLUTIONS

The approximate solution of Equation (2.15) is given by

$$
\begin{bmatrix}
-1.471m-0.5n & 0.9428m+1.885n & 0.1515m+4.5n & 0.0875m+6.033n & -0.0437m+4.167n \\
0.9428m+1.885n & -1.3199m & 1.0303m+30.54n & 0.1482m+42.67n & 0.0067m+33.67n \\
0.1515m+4.5n & 1.0303m+30.54n & -1.3232m+13.5n & 0.95495m+116.37n & 0.1919m+112.5n \\
0.0875m+6.033n & 0.1482m+42.67n & 0.9495m+116.57n & -1.4366m & 1.257m+251.4n \\
-0.0033m+4.167n & 0.0067m+33.67n & 0.0348m+112.5n & 0.857m+251.4n & -0.8285m-62.5n
\end{bmatrix}\begin{bmatrix}\alpha_1(t)\\ \alpha_2(t)\\ \alpha_3(t)\\ \alpha_4(t)\\ \alpha_5(t)\end{bmatrix}
$$

$$
=\begin{bmatrix}-0.7071\times\omega\times f(x,t)\\ 0\\ \dfrac{\omega\times 0.7071\times f(x,t)}{3}\\ \dfrac{\omega\times f(x,t)}{4}\\ \dfrac{\omega\times 0.7071\times f(x,t)}{5}\end{bmatrix} \qquad (3.1)
$$

$m=-34373.01, n=762.99, \omega=27.4984$

See([6],[12]).

Substituting the values above in thematrix (3.1) with boundary conditions (1.1a),

$$\alpha_1(t) = \alpha_5(t) = 0$$

we obtain;

$$\begin{bmatrix} 45368.94 & -12112.79 & 27462.70 \\ -12112.79 & 55782.73 & 55964.64 \\ 27462.70 & 56117.24 & 49380.27 \end{bmatrix} \begin{bmatrix} \alpha_2(t) \\ \alpha_3(t) \\ \alpha_4(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 648.14 \\ 687.46 \end{bmatrix}$$

Using mat lab we obtain solutions as:

$$\alpha_2(t) = 0.0012739\,s, \alpha_3(t) = 0.0097102\,s, \alpha_4(t) = 0.0021783\,s. \qquad (3.2)$$

From (2.6), we obtain the deflection equation as :

$$\hat{u}(x,t) = 0.0012739 \sin 0.5\pi x + 0.0097102 \sin 0.75\pi x + 0.0021783 \sin \pi x \qquad (3.3)$$

The approximate solution of Equation (2.16) in matrix form is given by

$$\begin{bmatrix}
0.549m - 0.294n & 0.157n - 0.271m & 0.003m + 0.505n & 0.054m + 1.12n & 0.004m + 1.98n & 0.005m + 3.04n & 0.004m + 4.17n & 0.004m + 5.20n & 0.003m + 5.95n & 0.002m + 6.24n & 0.036m + 5.93n \\
0.316m + 0.157n & -0.624m - 3.8n & 0.320m + 3.92n & 0.008m + 8.67n & 0.009m + 15.4n & 0.009m + 23.7n & 0.009m + 32.5n & 0.008m + 40.6n & 0.006m + 40.6n & 0.005m + 49n & 0.071m + 46.7n \\
0.003m + 0.505n & 0.320m + 3.92n & -0.62m - 12.8n & 0.325m + 27.9n & 0.012m + 49.5n & 0.013m + 76.3n & 0.012m + 105n & 0.011m + 131n & 0.009m + 151n & 0.067m + 160n & 0.105m + 154n \\
0.004m + 1.12n & 0.008m + 8.66n & 0.325m + 27.9n & -0.615m - 19n & 0.329m + 110n & 0.015m + 169n & 0.015m + 233n & 0.014m + 294n & 0.011m + 340n & 0.373m + 362n & 0.138m + 353n \\
0.005m + 1.98n & 0.009m + 15.4n & 0.012m + 49.6n & 0.329m + 110n & -0.612m - 22n & 0.331m + 303n & 0.017m + 420n & 0.015m + 531n & 0.013m + 619n & 0.01m + 667n & 0.169m + 659n \\
0.005 + 3.04n & 0.009m + 32.7n & 0.013m + 76.3n & 0.005m + 169n & 0.07m + 303n & 0.252m - 63.5n & -0.11m + 655n & 0.016m + 834n & 0.014m + 981n & 0.01m + 1070n & 0.198m + 1079n \\
0.004m + 4.17n & 0.009m + 32.5n & 0.012m + 105n & 0.015m + 234n & 0.017m + 420n & 0.332m + 655n & -0.6m - 163n & 0.33m + 1178n & 0.014m + 1402n & 0.01m + 1555n & 0.224m + 1603n \\
0.004m + 5.2n & 0.008m + 40.6n & 0.011m + 131n & 0.014m + 294n & 0.015m + 531n & 0.016m + 834n & 0.33m + 1178n & -0.6m - 244n & 0.327m + 1849n & 0.01m + 2090n & 0.245m + 2214n \\
0.003m + 6n & 0.006m + 40.6n & 0.009m + 151n & 0.01m + 340n & 0.013m + 619n & 0.014m + 981n & 0.014m + 1402n & 0.02m + 1846n & 0.001m - 312n & 0.015m + 2040n & 0.263m + 2880n \\
0.002m + 6.24n & 0.005m + 49n & 0.007m + 160n & 0.008m + 362n & 0.009m + 667n & 0.01m + 1070n & 0.01m + 1555n & 0.01m + 2099n & 0.324m + 2640n & -0.620m & 0.589m + 2880n \\
0.001m + 5.93n & 0.003m + 46.7n & 0.004m + 154n & 0.005m + 353n & 0.006m + 659n & 0.007m + 1079n & 0.007m + 1603n & 0.008m + 2214n & 0.008m + 2880n & 0.268m + 3561n & -0.2m - 391n
\end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \\ \alpha_9 \\ \alpha_{10} \\ \alpha_{11} \end{bmatrix} = \begin{bmatrix} -\omega.f(x,t).0.9511 \\ -\omega.f(x,t).0.4045 \\ -\omega.f(x,t).0.1959 \\ \omega.f(x,t).0.7725 \\ 0 \\ \omega.f(x,t).0.0515 \\ \omega.f(x,t).0.0840 \\ \omega.f(x,t).0.1011 \\ \omega.f(x,t).0.1057 \\ \omega.f(x,t).0.1 \\ -\omega f(x,t).0.0865 \end{bmatrix}$$

$m = -34373.01, n = 762.99, \omega = 27.4984$

and

$\alpha_2 = 0.699109s, \alpha_3 = 1.228128s, \alpha_4 = -0.420568s, \alpha_5 = 0.013978s, \alpha_6 = -0.012808s, \alpha_7 = -0.010774s, \alpha_8 = -0.010725s$
$\alpha_9 = -0.012832s, \alpha_{10} = -0.013529s$

From (2.6) deflection equation is given by:

$$\hat{u}(x,t) = 0.699109\sin 0.5\pi x + 1.228128\sin 0.75\pi x - 0.420568\sin \pi x + 0.0139778\sin 1.25\pi x - 0.012808\sin 1.5\pi x$$

$$-0.010774\sin 1.75\pi x - 0.010725\sin 2\pi x - 0.012832\sin 2.25\pi x - 0.013529\sin 2.5\pi x \qquad (3.4)$$

Graphical output of Equation (3.3)

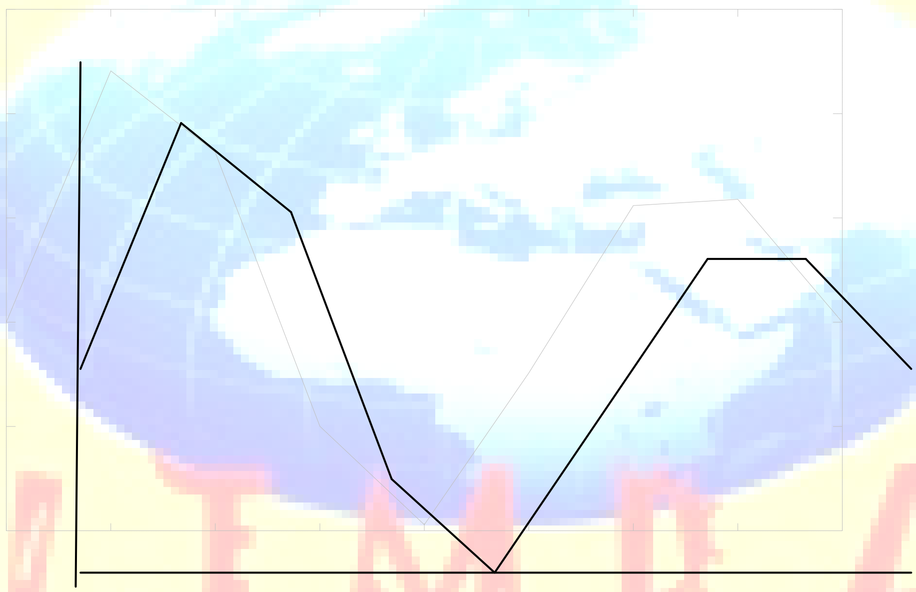

Figure 2: Deflections $\hat{u}(x,t)$ *against, length* $(x)(m)$

Graphical output of Equation (3.4)

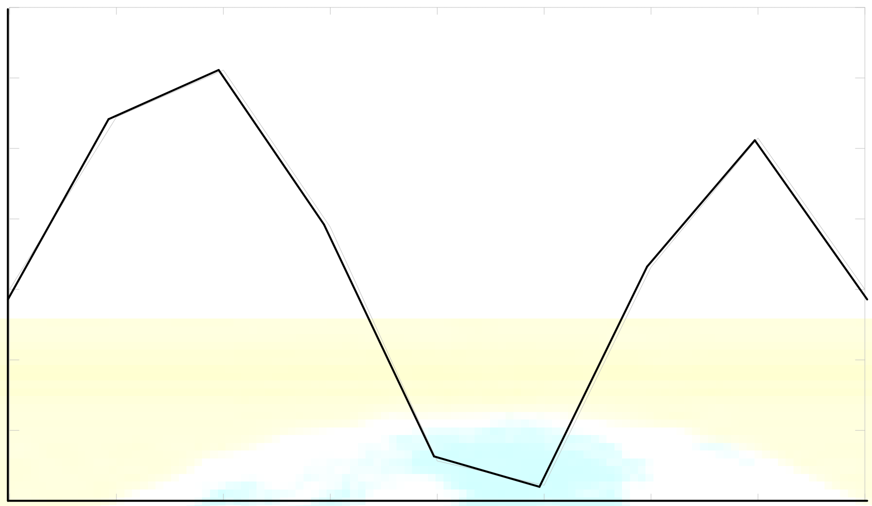

Figure 3: *Deflections* $\hat{u}(x,t)$ *against length(x)(m)* .

4 CONCLUSIONS

We have solved the beam equation usingthefinite element method and the graphical outputs (fig 2) and (fig 3) show that the solution obtained satisfy boundary conditions hence the solutions within $0 \le u(x,t) \le L$ are assumed consistent with exact values. Deflections in (fig 2) and fig (3) portray beam deflection of a similar pattern. The deflections in the graphical output smoothens as the number of subdivisions increases.

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