NORM INEQUALITIES OF NORM-ATTAINABLE OPERATORS AND THEIR ORTHOGONAL EXTENSIONS

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Abstract
We present new norm inequalities of matrices of norm-attainable operators and characterize the maps that act on matrices of these operators. Moreover, we characterize completely bounded norms, give their orthogonal extensions and extensions via norm-convergence in $NA(H)$-classes.

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1 Introduction
The algebra of norm-attainable operators, $NA(H)$, is one of the subclasses under consideration in this note. Consider an infinite dimensional complex Hilbert space $H$ and $B(H)$ the algebra of all bounded linear operators on $H$. A lot of results have been obtained in the study of the properties of several classes and matrices of operators acting on Hilbert spaces. Considering norm-attainable operators, there are nice results on them especially on the necessary and sufficient conditions for norm-attainability [2]. Recently, characterizations on this subclass of operators has been done in [4]. For more details on norm-attainable operators see [4-18]. In this paper, we study some important properties of norm-attainable operators and characterize the maps that act on matrices of these operators. See details on completely bounded maps in [1 and 19] and the references therein. Lastly, we give the orthogonal extensions of these norms.

2 Preliminaries and Notations
In this section we give some basic definitions and notations that we shall use in the sequel.
Definition 2.1. Let $A, S \in B(H)$, $A$ is said to be positive if $(Ax, x) \geq 0$, $\forall \ x \in H$ and normal if $AA^* = A^*A$. $S$ is an isometry (co-isometry) if $S^*S = SS^* = I$ where $I$ is an identity operator in $B(H)$.

From this stage and in the sequel, we denote a bounded linear operator, a positive operator and a norm-attainable operator in $B(H)$ by $A$, $A_P$ and $A_N$ respectively. We also denote by $\dim N(A_N)$, the dimension of the null space of $A_N$. We also denote the algebra of all norm-attainable operators by $NA(H)$.

Definition 2.2. Let $\theta : B(H) \to B(H)$ be a bounded linear operator. $\theta$ is said to be completely bounded if $\sup \|\theta_n : \forall n \in \mathbb{N}\| < \infty$ and this supremum is called the completely bounded norm denoted by $\| \cdot \|_{CB}$.

We denote by $CB[NA(H), NA(H)]$ the algebra of all completely bounded operators from $NA(H)$ to $NA(H)$. Clearly, this algebra is complete with respect to the completely bounded norm. HS-norm denotes the Hilbert-Schmidt norm.

Definition 2.3. Let $H$ be a Hilbert space, $B(H)$ the algebra of all bounded linear operators on $H$ and let $N(H) \subseteq B(H)$ be a subalgebra. Let $M_{n,m}[B(H)]$ be a $n \times m$ matrix algebra with entries from $B(H)$. Then the inclusion, $M_{n,m}[NA(H)] \subseteq M_{n,m}[B(H)]$ endows this subalgebra with a collection of matrix norms and we call $N(AH)$ together with this collection of matrix norms on $M_{n,m}[NA(H)]$ an operator algebra. When $m = n$, we have

$$M_{n,m}[NA(H)] = M_{n,n}[NA(H)] = M_n[NA(H)].$$

3 Norm-attainable Operators

In this section, we study norm-attainable operators and some of their properties.

Definition 3.1. An operator $A_N \in NA(H)$ is said to be norm-attainable if there exists a unit vector $x \in H$, $\|A_Nx\| = \|A_N\|$.

Lemma 3.2. If $A_N, B_N \in NA(H)$ are norm-attainable then $A_N + B_N$, $A_N - B_N$ and $\lambda A_N$, $\lambda \in \mathbb{C}$ are norm-attainable.

Proof. By direct summing of the operators $A_N$ and $B_N$ with a large enough rank-one projection, the sum and the difference of the two operators are norm-attainable. The case of coefficient $\lambda$ is easy to see. \hfill \Box

Lemma 3.3. For a norm-attainable operator $A_N \in NA(H)$, $A_N$ is norm-attainable if its adjoint, $A_N^*$, is norm-attainable.
We show that a nonzero $A_N \in NA(H)$ is norm-attainable implies that $A_N^*$ is norm-attainable. Let $A_N \in NA(H)$ be norm-attainable, then there exists a unit vector $x \in H$ such that $\|A_Nx\| = \|A_N\|$. That is $A_N^*A_Nx = \|A_N\|^2x$. Let $\xi = \frac{A_Nx}{\|A_N\|}$, then $\xi$ is a unit vector and hence $\|A_N^*\| = \|A_N\| = \|A_N^*\|$.

**Theorem 3.4.** Let $S'$ and $S''$ be isometries or co-isometries in a unit ball $NA(H)_1$. For a norm-attainable operator $A_N \in NA(H)_1$ we have $A_N = \frac{S' + S''}{2}$.

**Proof.** Let $A_N = S[T]$ be the polar decomposition of $A_N$. Now, since $A_N \in NA(H)_1$, $T$ is a contraction (in fact, a positive contraction) in $NA(H)_1$, therefore, $I - T^2$ is also a positive contraction in $NA(H)_1$. Let $K$ and $K'$ respectively be defined by $K = T + i\sqrt{I - T^2}$ and $K' = T - i\sqrt{I - T^2}$. It is clear that $K^* = K'$ and hence, $KK^* = K^*K = I$ and $K'K'^* = K'^*K' = I$, so $K$ and $K'$ are unitaries and $T = \frac{K + K'}{2}$. If $\text{dim}N(A_N) < \text{dim}N(A_N^*)$, then $S$ can be taken to be an isometry and therefore, putting $S' = S[K]$ and $S'' = S[K']$, then $S'$ and $S''$ are isometries and

$$A_N = S[T] = S[\frac{K + K'}{2}] = \frac{S' + S''}{2} \quad (1)$$

If $\text{dim}N(A_N) > \text{dim}N(A_N^*)$, then $S$ can be taken to be co-isometry and therefore, $S'$ and $S''$ in Equation (1) can be taken as co-isometries.

**Remark 3.5.** If $\text{dim}N(A_P) = \text{dim}N(A_P^*)$, then $S$ can be taken to be a unitary and therefore, $S'$ and $S''$ in Equation (1) can be taken as unitaries.

**Theorem 3.6.** An operator $A \in NA(H)$ is normal if it is norm-attainable.

**Proof.** Assuming $A \in NA(H)$ is normal, we show that it is norm-attainable. Since $A$ is normal, then $AA^* = A^*A$. By Lemma 3.3 its adjoint is norm-attainable. Consider a unit vector $x \in H$. Now $A(A^*x) = A^*(Ax)$. This implies that $\|Ax\|^2 = \langle Ax, Ax \rangle = \langle A^*Ax, x \rangle \leq \|A(A^*x)\| \leq \|A^*\|\|A\|\|x\| = \|A\|^2\|x\| = \|A\|^2$. Taking a positive square root on both sides yields the required results. The reverse inequality is trivial and hence this completes the proof.
4 Norm inequalities in $NA(H)$-Classes

Theorem 4.1. Consider a $C^*$-algebra $B(H)$, $NA(H)$ a subalgebra of $B(H)$ and a map $\theta$, such that $\theta : NA(H) \to B(H)$. Let $T_{N;j,k} \in M_n[NA(H)]$ be a norm-attainable operator. For $n$-tuples of $\theta$, whereby $\theta_n : M_n[NA(H)] \to M_n[B(H)]$, we define $\theta_n[T_{N;j,k}] = [\theta(T_{N;j,k})]$, $\forall T_{N;j,k} \in M_n[NA(H)]$. Moreover, $||\theta|| \leq ||\theta||_{CB}$ holds.

Proof. For simplicity, we take $T_{N;j,k} = T_{j,k}$ throughout the proof. Now, when $n = 1$, then by definition of $\theta_n$, $\theta_1$ and $\theta$ are coincidental [1] hence, $||\theta|| = ||\theta_1||$. We therefore give proofs when $n = 2$ and when $n = 3$. We use an analogous technique to the one used in [1]. For $n = 2$, let $T_{j,k} \in M_2[NA(H)]$, $j, k = 1, 2$, then for $\theta_2 : M_2[NA(H)] \to M_2[B(H)]$, we have, $\theta_2 \left[ \begin{array}{cc} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{array} \right] = \theta(T_{1,1}) \theta(T_{1,2}) \quad \theta(T_{2,1}) \theta(T_{2,2})$ and

$$\left\| \theta_2 \left[ \begin{array}{cc} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{array} \right] \right\| = \left\| \begin{array}{cc} \theta(T_{1,1}) & \theta(T_{1,2}) \\ \theta(T_{2,1}) & \theta(T_{2,2}) \end{array} \right\|$$

$$= \left( \sum_{j=1}^{2} \sum_{k=1}^{2} \left\| \theta(T_{j,k}) \right\|^2 \right)^{\frac{1}{2}} \quad \text{by HS-norm}$$

$$\geq \left( \left\| \theta(T_{1,1}) \right\|^2 + \left\| \theta(T_{1,2}) \right\|^2 + \left\| \theta(T_{2,1}) \right\|^2 + \left\| \theta(T_{2,2}) \right\|^2 \right)^{\frac{1}{2}}$$

$$\geq \left\| \theta_1(T_{1,1}) \right\|$$

$$= \left\| \theta_1(T_{1,1}) \right\|.$$

Therefore,

$$||\theta_2|| = \sup\{|\theta_2([T_{j,k}])| : T_{j,k} \in M_2[NA(H)]\} \geq \sup\{|\theta_1(T_{1,1})|| = ||\theta_1||.$$

and hence $||\theta_2|| \geq ||\theta_1||$.

When $n = 3$, $\theta_3 \left[ \begin{array}{ccc} T_{1,1} & T_{1,2} & T_{1,3} \\ T_{2,1} & T_{2,2} & T_{2,3} \\ T_{3,1} & T_{3,2} & T_{3,3} \end{array} \right] = \left[ \begin{array}{ccc} \theta(T_{1,1}) & \theta(T_{1,2}) & \theta(T_{1,3}) \\ \theta(T_{2,1}) & \theta(T_{2,2}) & \theta(T_{2,3}) \\ \theta(T_{3,1}) & \theta(T_{3,2}) & \theta(T_{3,3}) \end{array} \right]$ which
implies that
\[\left\| \theta_3 \right\| \left[ \begin{array}{ccc} T_{1,1} & T_{1,2} & T_{1,3} \\ T_{2,1} & T_{2,2} & T_{2,3} \\ T_{3,1} & T_{3,2} & T_{3,3} \end{array} \right] = \left\| \begin{array}{ccc} \theta(T_{1,1}) & \theta(T_{1,2}) & \theta(T_{1,3}) \\ \theta(T_{2,1}) & \theta(T_{2,2}) & \theta(T_{2,3}) \\ \theta(T_{3,1}) & \theta(T_{3,2}) & \theta(T_{3,3}) \end{array} \right\| \]
\[= \left[ \sum_{j=1}^{3} \sum_{k=1}^{3} \left\| \theta(T_{j,k}) \right\|^2 \right]^{\frac{1}{2}} \]
\[= \left[ \left( \left\| \theta(T_{1,1}) \right\|^2 + \left\| \theta(T_{1,2}) \right\|^2 + \left\| \theta(T_{1,3}) \right\|^2 + \left\| \theta(T_{2,1}) \right\|^2 + \left\| \theta(T_{2,2}) \right\|^2 + \left\| \theta(T_{2,3}) \right\|^2 + \left\| \theta(T_{3,1}) \right\|^2 + \left\| \theta(T_{3,2}) \right\|^2 \right) \right]^{\frac{1}{2}} \]
\[\geq \left[ \left\| \theta(T_{1,1}) \right\|^2 + \left\| \theta(T_{1,2}) \right\|^2 + \left\| \theta(T_{2,1}) \right\|^2 + \left\| \theta(T_{2,2}) \right\|^2 \right]^{\frac{1}{2}} \]
\[= \left[ \sum_{j=1}^{2} \sum_{k=1}^{2} \left\| \theta(T_{j,k}) \right\|^2 \right]^{\frac{1}{2}} \]
\[= \left\| \theta(T_{j,k}) \right\|. \]

This implies that
\[\left\| \theta_3 \right\| = \sup\{ \left\| \theta_3[(T_{j,k})] : [T_{j,k}] \in M_3[NA(H)] \} \]
\[\geq \sup\{ \left\| \theta_2[(T_{j,k})] : [T_{j,k}] \in M_2[NA(H)] \} \]
\[= \left\| \theta_2 \right\|\]

and therefore, \( \left\| \theta_3 \right\| \geq \left\| \theta_2 \right\| \). Lastly, consider \( \theta_{n+1} : M_{n+1}[NA(H)] \rightarrow M_{n+1}[B(H)] \) defined by \( \theta_{n+1}[(T_{j,k})] = [\theta(T_{j,k})] \) for all \( j, k = 1, \ldots, n + 1 \). We obtain,
\[\left\| \theta_{n+1}[(T_{j,k})] \right\| = \left\| \theta(T_{j,k}) \right\| \]
\[= \left[ \sum_{j=1}^{n+1} \sum_{k=1}^{n+1} \|\theta(T_{j,k})\|^2 \right]^{\frac{1}{2}} \]
\[\geq \left[ \sum_{j=1}^{n} \sum_{k=1}^{n} \|\theta(T_{j,k})\|^2 \right]^{\frac{1}{2}} \]
\[= \left\| \theta_n[(T_{j,k})] \right\|\]

So, \( \left\| \theta_{n+1} \right\| \geq \left\| \theta_n \right\| \) by taking supremum on both sides of the inequality above. By complete boundedness of the norm of \( \theta \), \( \left\| \theta \right\|_{CB} = \sup\{ \left\| \theta_n \right\| : n \in \mathbb{N} \} \) which implies that \( \left\| \theta \right\|_{CB} \geq \left\| \theta_n \right\| \ \forall \ n \in \mathbb{N} \). Therefore, \( \left\| \theta \right\| \leq \left\| \theta \right\|_{CB} \) which completes the proof. \( \square \)
5 Orthogonal Extensions

In this section, we give norm-attainable operator-valued orthogonal extensions of matrix inequalities. Orthogonal extensions are known in different settings, but we include simple proofs, since more elaborate ones have appeared in literature.

Definition 5.1. Two operators $T$ and $P$ in $NA(H)$ are said to be orthogonal if $\langle T, P \rangle = 0$. Operators $T_j, T_k$, $(j, k = 0, 1, ...)$ are said to have orthogonal extensions if $\langle T_j, T_k \rangle = 0$.

Proposition 5.2. Let $(M_{j;k})$ be a positive definite $n \times n$ matrix and $T_j, (1 \leq j \leq n)$ are elements of $NA(H)$, then $\sum_{j=1}^{n} \sum_{k=1}^{n} \langle T_j, T_k \rangle \geq 0$.

Proof. Consider an orthonormal basis $\{e_1, e_2, ..., e_m\}$ of the subspace of $NA(H)$ generated by the elements $T_j$, and we write $T_j = \sum_{r=1}^{m} T_j(r) e_r$. From Theorem 4.1 for each $r$ we have positivity attained and taking summation over $r$ completes the proof.

Theorem 5.3. The matrix $[T]_{B}^{i;j}$ of an orthogonal operator $T \in NA(H)$ in an arbitrary orthonormal basis $B$ is orthogonal. Conversely, if in an orthonormal basis $B$ the matrix of a linear operator $T \in NA(H)$ is orthogonal, then $T$ is orthogonal.

Proof. Consider an orthonormal basis $B = e_1, ..., e_n$ in $NA(H)$. Let $T$ be a linear operator. Then $[T]_{B}^{i;j} = ([T(e_1)]_{B}^{i;j}, ..., [T(e_n)]_{B}^{i;j})$. Recall that for an orthonormal basis, the inner product in $NA(H)$ is equal to the standard scalar product in $\mathbb{R}^n$ of their respective coordinate column (or row) operators. Suppose that $T$ is orthogonal, the set of Operators $T(e_1), ..., T(e_n)$ is orthonormal (since $e_1, ..., e_n$ is orthonormal). Hence the columns $[T(e_1)]_{B}^{i;j}, ..., [T(e_n)]_{B}^{i;j}$ are orthonormal, hence $[T]_{B}^{i;j}$ is an orthogonal matrix. Conversely, assume that the matrix $[T]_{B}^{i;j}$ is orthogonal. $\implies T(e_1), ..., T(e_n)$ is an orthonormal set, i.e., $T$ preserves all pairwise scalar products of the elements of the basis $B$. It follows then that $T$ preserves all inner products of vectors of $NA(H)$, i.e., $T$ is an orthogonal operator.

Now it is easy to see that without loss of generality, fixing an orthonormal basis in $NA(H)$, gives a $1 \leftrightarrow 1$ correspondence between orthogonal matrices and orthogonal operators. At this point we consider extensions via norm convergence.

6 Extensions via Norm-convergence

Norm inequalities can be extended via norm-convergence. We note that a sequence $\{T_j\}$ of operators converges strongly to $T$ if $\lim_{j \to \infty} T_j x = T$, for all
$x \in H$. Norm-convergence implies strong convergence while boundedness of $\Sigma_{j=1}^{n}T_{j}^{*}T_{j}$ is actually equivalent to strong convergence of $\Sigma_{j=1}^{n}T_{j}^{*}T_{j}$ since any norm-bounded, increasing sequence of self-adjoint operators converges strongly. It is interesting to show that either norm-convergence or strong convergence is inherited by the transformed sequence.

Considering weaker conditions, the norm of the limit of a weakly convergent sequence of operators in $NA(H)$ may be strictly less than the norms of the terms in the sequence, corresponding to a loss of energy in oscillations, at a singularity, or by escape to infinity in the weak limit. Therefore, the expansion of any positive functional $\phi$ in any orthonormal basis contains coefficients that wander off to infinity. Hence, we note that if the norms of a weakly convergent sequence converge to the norm of the weak limit, then the sequence is strongly convergent.

It is known that the boundedness of the pointwise values of a family of linear functionals $\phi$ implies the boundedness of their norms. We prove that a weakly convergent sequence is bounded, hence gives a necessary and sufficient condition for weak convergence of operators in $NA(H)$.

**Theorem 6.1.** Let $T_{j}$ be a sequence of operators in $NA(H)$ and $G$ a dense subset of $NA(H)$. Then $T_{j}$ converges weakly to $T$ if and only if $\|T_{j}\| \leq \lambda$ for some constant $\lambda$. Moreover, $\langle T_{j}, P \rangle \to \langle T, P \rangle$ as $j \to \infty$ for all $P \in G$.

**Proof.** Suppose that $T_{j}$ is a weakly convergent sequence. By the definition of bounded linear functionals $\phi_{n}$ given by $\phi_{n}(T) = \langle T, T_{j} \rangle$. Then $\|\phi_{n}\| = \|T_{j}\|$. But $\phi_{n}(T)$ is convergent for each $T \in NA(H)$, it is a bounded sequence, and by uniform boundedness theorem $\{\|\phi_{n}\|\}$ is bounded. Hence both conditions are necessary and satisfied. Next, we prove the reverse inclusion. Suppose that $T_{j}$ satisfies the two conditions. If $A \in NA(H)$, then for any $\beta > 0$ there is a $P \in G$ such that $\|A - P\| < \beta$, and there is an $M$ such that $|\langle T_{j} - T, P \rangle| < \beta$, for $j \geq M$. From the first condition in the theorem, and by Cauchy-Schwarz inequality it follows that for $j \geq M$

$$\left|\langle T_{j} - T, A \rangle\right| \leq \left|\langle T_{j} - T, P \rangle\right| + \left|\langle T_{j} - T, A - P \rangle\right| \leq \beta + \|T_{j} - T\| + \|A - P\| = (1 + \lambda + \|T\|)\beta.$$  

Thus, it follows that $T_{j} \to T$ which completes the proof.

**Example 6.2.** Suppose that $e_{i}$, $(i = 1, 2, \ldots)$ is an orthonormal basis of $NA(H)$. Then a sequence $T_{j}$ is weakly convergent to $T$ if and only if it is bounded and its coordinates converge, i.e. $\langle T_{j}, e_{i} \rangle \to \langle T, e_{i} \rangle$ for each $i = 1, 2, \ldots$. Thus, the boundedness of the sequence is sufficient to ensure weak convergence in $NA(H)$. 
References


