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NORM INEQUALITIES FOR POSITIVE ELEMENTARY OPERATORS AND ENTANGLEMENT OF STATES

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Abstract

We present norm inequalities for positive elementary operators via Cauchy-Schwarz inequality and Minkowskis inequality techniques. Norm inequalities are presented in Euclidean algebras linked to Minkowski's light cones. Lastly, we explore the applications in quantum theory particularly in entanglement of states.

Keywords: Norm inequalities, Positivity, Elementary Operators, State entanglements.

1 Introduction

Let H be an infinite dimensional separable complex Hilbert space and $\mathcal{M}$ the C*-algebra of all bounded linear operators on H. An operator $\phi : \mathcal{M} \to \mathcal{M}$ defined by $\phi_{\bar{X},\bar{Y}}$ is called an elementary operator if it is expressible as $\phi_{\bar{X},\bar{Y}}(S) = \sum_{i=1}^{n} X_i S Y_i, \forall X_i, Y_i$ fixed in $\mathcal{M}$, $\forall\, S \in \mathcal{M}$ where $\bar{X}$, $\bar{Y}$ are n-tuples of of the operators. For all $X \in \mathcal{M}$ and A, B fixed in $\mathcal{M}$, examples of the elementary operators are: the left multiplication operator, $L_A(X) = AX$, the right multiplication operator, $R_B(X) = XB$, the generalized derivation (implemented by $A,\ B$), $\delta_{A,B} = L_A - R_B$, the inner derivation (implemented by A), $\delta_A = AX - XA$, the basic elementary operator(implemented by $A,\ B$),

$M_{A,\,B}(X) = AXB$, and the Jordan elementary operator(Two-sided multiplication operator)(implemented by $A,\ B$), $\mathcal{U}_{A,\,B}(X) = AXB + BXA$.
Consider $\mathcal{M}_n(\mathbb{C})$, the space of all $n \times n-$complex matrices. For $S \in \mathcal{M}_n(\mathbb{C})$ there exists a unitary matrix U and positive semidefinite matrix P such that S=UP, and this is a polar decomposition of S where P is unique and $P = |S| = (S^*S)^{\frac{1}{2}}$. U is also unique and has an inverse. Clearly, $S^*S = P^2$ and $SS^* = UP^2U$. It is a fact that S is normal if $S^*S = SS^*$ and self adjoint if $S = S^*$. Moreover, S is normal if $UP = PU$ and self commutator if $S^*S - SS^*$. Now $S^*S - SS^* = 0$ if and only if $UP - PU = 0$. We note that $S \in \mathcal{M}$ is said to be positive if $S \geq 0$ implies $\phi(S) \geq 0\ \forall S \in \mathcal{M}$. In this paper, we give norm inequalities for positive elementary operators and their applications in entanglement of states. For details on norms of elementary operators see [**??**-**??**] and the references therein.

2 Preliminaries

We introduce basic definitions in this section that are useful in the sequel.

Definition 2.1. *Let $\mathbb{R}^n$ be an $n-$dimensional Euclidean space with inner products and $\mathbb{R}$ the set of real numbers. We define Cauchy-Schwarz inequality by $(\sum_{i=1}^n X_iY_i)^2 \leq (\sum_{i=1}^n X_i^2)(\sum_{i=1}^n Y_i^2), \forall\, X_i, Y_i \in \mathbb{R}^n (i = 1 \ldots n)$. Moreover, let x_i, y_i be complex numbers, we define Minkowski's inequality by $(\sum_{i=1}^n |x_i + y_i|^p)^{\frac{1}{p}} \leq (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}} + (\sum_{i=1}^n |y_i|^p)^{\frac{1}{p}}$, where $\frac{1}{p} + \frac{1}{q} = 1$.*

Remark 2.2. *For triangle inequality we have $\|\eta + \xi\|_p \leq \|\eta\|_p + \|\xi\|_p$, where $1 \leq p \leq \infty$ and η and ξ are members of $L^p(\Omega)$ where Ω is a measure space.*

If η and ξ are linearly dependent i.e. $\eta = \alpha\xi$ for $\alpha > 0$ then the norm is $\|\eta\|_p = (\int |\eta| d\alpha)^{\frac{1}{p}}$, if $p < \infty$. We attain the essential supremum $\|\eta\|_\infty = ess\sup_{x\in\Omega} |\eta(x)|$ if $p = \infty$. Generally, $\|\eta\|_p = \sup_{\|\xi\|_q=1} \int |\eta\xi| d\alpha$, for $\frac{1}{p} + \frac{1}{q} = 1$.

Definition 2.3. *Let $\mathbb{R}^n$ be an Euclidean space and $\mathbb{R}$ the set of all real numbers. Let $X, Y \in \mathbb{R}^n$, where $n \leq 4$. If $n = 4$, we define the Minkowski's form by $[X, Y] = X_1Y_1 - X_2Y_2 - X_3Y_3 - X_4Y_4$. The set defined as follows: $\Pi = \{X \in \mathbb{R}^4, [X, X] > 0$ and $X_1 > 0\}$ is called a light cone.*

Remark 2.4. *We can extend Definition **??** as follows: For $(t, X)(u, Y) \in \mathbb{R} \times \mathbb{R}^n$, we have $[(t, X)(u, Y) = tu - \langle X, Y\rangle]$, where $\langle X, Y\rangle$ is the usual inner product of $\mathbb{R}^n$. A class of cones is thus obtained denoted by:*
$\Pi^C_{n+1} = \{(t, X) \in \mathbb{R} \times \mathbb{R}^n, t > \sqrt{\langle X, X\rangle}\}$.

3 Norm Inequalities

The following result which is analogous to [**??**, Theorem 6] is very important in this work.

Theorem 3.1. *Let $\phi_{\bar{X},\bar{Y}}(S) = \sum_{i=1}^n X_iSY_i$ be an elementary operator from $\Pi^C_{n+1}(H) \to \Pi^C_{n+1}(H)$. ϕ is $n+1-$positive if and only if there exisits $R_1 \ldots R_r$ and $T_1 \ldots T_k$ in $\{X_1 \ldots X_n\}$ with $r + k \leq n$ such that $(T_1 \ldots T_k)$ is $n + 1-$contractive locally linear combination of $R_1 \ldots R_r$ and $\phi(S) \sum_{i=1}^n R_iSR_i^* + \sum_{i=1}^n T_i^*ST_i \forall S \in \Pi_{n+1}(H)$*

Proof. Since $S^*S = P^2$ and $SS^* = UP^2U$, by ther proof of [**??**, Lemma 5] we obtain the result. □

Proposition 3.2. *Let S_i and $K_i \in \Pi^C_{n+1}(H)$ be positive operators then $(\sum_{i=1}^n S_iK_i)^2 \leq (\sum_{i=1}^n S_i^2)(\sum_{i=1}^n K_i^2)$ and $\|(\sum_{i=1}^n S_iK_i)^2\| \leq \|\sum_{i=1}^n S_i^2\|\|\sum_{i=1}^n K_i^2\|$.*

Proof. Consider the set of all quadratic polynomials V i.e.

$$(S_1V+K_1)^2+\ldots+(S_nV+K_n)^2 = \left(\sum_{i=1}^n (S)_i^2\right).V^2+2\left(\sum_{i=1}^n (S_i.K_i)\right).V+\sum_{i=1}^n (K_i^2).$$

Clearly, it's non-negative hence its discriminant is obviously less or equal to zero i.e. $\left(\sum_{i=1}^{n} S_i.K_i\right)^2 - \sum_{i=1}^{n} S_i^2 \sum_{i=1}^{n} K_i^2 \leq 0$. For the case of $n-$copies of H i.e. H^n, we have $\sum_{i=1}^{n}\sum_{j=1}^{n}(S_iK_j - S_jK_i)^2 = \sum_{i=1}^{n} S_i^2 \sum_{j=1}^{n} K_j^2 + \sum_{j=1}^{n} S_j^2 \sum_{i=1}^{n} K_i^2 - 2\sum_{i=1}^{n} S_iK_i \sum_{i=1}^{n} S_jK_j$, which gives after identical collections $\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(S_iK_j - S_jK_i)^2 = \sum_{i=1}^{n} S_i^2 \sum_{i=1}^{n} K_i^2 - \sum_{i=1}^{n}(S_iK_i)^2$. Thus, $\sum_{i=1}^{n} S_i^2 \sum_{i=1}^{n} K_i^2 - (\sum_{i=1}^{n} S_iK_i)^2 \geq 0$. By submultiplicative operator norm, we obtain $\left\|\left(\sum_{i=1}^{n} S_iK_i\right)^2\right\| \leq \|\sum_{i=1}^{n} S_i^2\| \, \|\sum_{i=1}^{n} K_i^2.\|$ □

Remark 3.3. *We obtain the same results if we consider Schatten $p-$classes and the Hilbert-Schmidt norm on S_i, K_i in $\Pi_{n+1}^{C} H^n$.*

Theorem 3.4. *Let S and $K \in \Pi_{n+1}^{C}(H)$ be positive operators then* $\|S + K\|^2 \leq \|S\|^2 + \|K\|^2$.

Proof. By triangle inequality and Theorem **??** positivity is taken care of. Lastly, by invoking Proposition **??**, the proof follows immediately. □

Example 3.5. *Consider $S, K \in \Pi_{n+1}^{C}(H)$ and let $\Pi_{n+1}^{C}(H)$ be a probability space where S, K are treated as random variables. If the inner product is defined as $\langle S, K\rangle = E(SK)$, where $E(SK)$ is the expectation of S and K, then $|Cov(SK)|^2 \leq Var(S)Var(K)$.*

Indeed, by Cauchy-Shwarz inequality we have, $|E(SK)|^2 \leq E(S^2)E(K^2)$. Now if $\tau = E(S)$ and $\delta = E(K)$, then clearly we have

$$\begin{aligned}
|Cov(SK)|^2 &= |E((S-\tau)(K-\sigma))|^2 \\
&= |\langle S-\tau, K-\sigma\rangle|^2 \\
&\leq \langle S-\tau, S-\sigma\rangle\langle K-\tau, K-\sigma\rangle \\
&= E(S-\tau)^2 E(K-\sigma)^2 \\
&= Var(S)Var(K).
\end{aligned}$$

Here Var denotes variance and Cov denotes covariance in the usual probability sense.

Theorem 3.6. *Let $\phi_{\bar{S},\bar{K}}(T)$ be an elementary operator on $\prod_{n+1}^{C}(H^n)$ and $S_i, K_i \in \prod_{n+1}^{C}(H^n)$ be positive operators, then $\|A+B\|_p \leq \|A\|_p + \|B\|_p$ where A and B are summable representations of S_i and K_i in the form $A = (\sum_{i=1}^{n} |S_i|^p)^{\frac{1}{p}}$ and $B = (\sum_{i=1}^{n} |K_i|^p)^{\frac{1}{p}}$.*

Proof. We shall use Minkowski's inequality in this proof. By $p-$norm and the convexity over $\mathbb{R}^+$, since S_i and K_i are positive it's easy to see that $\|A+B\|^p \leq 2^{p-1}(\|A\|^p + \|B\|^p)$. Next we consider two cases. *Case 1*: $\|A+B\|_p = 0$ then by Minkowski's inequality the result holds. If it is less than zero then we have *case 2* and so by triangle inequality and by Holder's inequality(HI) we obtain,

$$\begin{aligned}
\|A+B\|_p^p &= \int |A+B|^p d\tau \\
&= \int (|A+B|)|A+B|^{p-1} d\tau \\
&\leq \int (|A|+|B|)|A+B|^{p-1} d\tau \\
&= \int |A||A+B|^{p-1} d\tau + \int |B||A+B|^{p-1} d\tau \\
&\leq_{HI} \left(\left(\int |A|^p d\tau \right)^{\frac{1}{p}} + \left(\int |B|^p d\tau \right)^{\frac{1}{p}} \right) \left(\int |A+B|^{p-1\frac{p}{p-1}} d\tau \right)^{1-\frac{1}{p}} \\
&= (\|A\|_p + \|B\|_p) \frac{\|A+B\|_p^p}{\|A+B\|_p}.
\end{aligned}$$

By multiplying both sides by $\frac{\|A+B\|_p}{\|A+B\|_p^p}$ we obtain the result. □

4 Norms in Light Cones of positive operators

Let $\phi_{X,Y}(S)$ be an elementary operator on the algebra of light cones $\Pi_{n+1}^{LC}(H)$ and $S \in \Pi_{n+1}^{LC}(H)$. Let X, Y be fixed in $\Pi_{n+1}^{LC}(H)$. We denote the composition operation in $\Pi_{n+1}^{LC}(H)$ by $(t, X) * (u, Y) = tu + \langle X, Y\rangle + tY + uX$, where the unity is $(1, 0)$ in $\Pi_{n+1}^{LC}(H)$. An associated norm with respect to spectral theory is given by $\|X\|_\alpha = max\{|\alpha|, \alpha \in \sigma(X)\}$.

Definition 4.1. *Let $\Pi_{n+1}^{LC}(H)$ be the algebra of all light cones in H. Considering symmetry in $\mathbb{R}$ and for the algebra $sym(m, \mathbb{R})$, we have an operator defined by: $\phi_{X,Y}(S) = X*(Y*S)+Y*(X*S)-X*Y*S, \ \ \forall S \in \Pi_{n+1}^{LC}(H)$ and X, Y fixed in $\Pi_{n+1}^{LC}(H)$ and the norm is given by $\|\phi_{X,Y}(S)\|_\infty \geq \frac{1}{2}\|X\|_\infty\|Y\|_\infty$ as a lower estimate[**??**].*

Theorem 4.2. *Let $\phi_{n+1}^{LC}(H)$ be the algebra of the light cones in H with dimensions ≥ 3. Then we have $\|\phi_{X,Y}(S)\|_\alpha \geq (\sqrt{2}-1)\|X\|_\infty\|Y\|_\infty, \ \ \forall\ X, Y$ fixed in $\Pi_{n+1}^{LC}(H)$ and for all positive operators $S \in \Pi_{n+1}^{LC}(H)$.*

Proof. By definition of supremum norm we have $\|\phi_{X,Y}(S)\|_\infty$ standing for $\sup_{\|S\|_\infty \leq 1} \|\phi_{X,Y}(S)\|_\alpha$. By [**??**, Lemma 1], we have $\|\phi_{X,Y}(S)\|$ represented as $\|\phi_{X,Y}(S)\| \geq max\{|2\|u\| - 1|, |2\|v\| - 1|\}$ and by the proof in [**??**, Theorem 3], result follows analogously. □

Remark 4.3. *There are interesting cases of the lower norm estimate where the coefficient of the norms have been obtained for Jordan elementary operators, for example, $1, \frac{1}{2}, \frac{2}{3}, (\sqrt{2}-1)$ among others in different algebras. The result obtained here is an assertion of the Stacho and Zalar [**??**] lower norm estimate in the case of $\Pi_{n+1}^{LC}(H)$.*

5 Applications to entanglements of states

Entanglement is a basic physical resource to realize various quantum information and quantum communication tasks such as quantum cryptography, teleportation, dense coding and key distribution. Composite quantum systems are systems that naturally decompose into two or more subsystems, where each subsystem itself is a proper quantum system. Most frequently, the individual subsystems are characterized by their mutual distance that is larger than the size of a subsystem. A typical example is a string of ions, where each ion is a subsystem, and the entire string is the composite system. Formally, the Hilbert space H associated with a composite, or multipartite system, is given by the tensor product of the spaces corresponding to each of the subsystems. By definition, these states are systems where the second electron is in the opposite configuration to the first. If one of the electrons is measured to be up along a given direction, then the other will definitely be down without the need for measurement. A pair of electrons in this state are said to be *entangled*. None of these two states are product states and for every product state there is a direction along which you will measure the spin of the first electron to be 1 with certainty and a direction along which you will measure the spin of the second electron to be 1 with 100 percent certainty. All we need to do to produce an entangled state is to bring two electrons in close enough proximity so that their magnetic fields interact and then leave them alone. Eventually, the electrons will be entangled, after maybe having emitted a photon. This is due to the fact that the electrons in opposite configuration is the lowest energy state of a two electron system. We can also think about as the first electron being the magnet that prepares the second (or vice versa). In this regard, we can treat an elementary operator $\phi_{\bar{X},\bar{Y}}(S)$ as a quantum system of the sum of the basic ones $X_1SY_1, ..., X_nSY_n$ as subsystems. To illustrate this, suppose molecule A_j consists of N_j atoms ($j = 1, 2$) and these two molecules, A_1 and A_2, are combined to form molecule A_3 which has N_3 atoms. Thus, one wishes to obtain or approximate the ground state energy and wave function, E_{A_3} and ψ_{A_3} respectively, of the system A_3 from the simpler ground state energy and wave function E_{A_j} and ψ_{A_j} of the constitutive systems A_j, $j = 1, 2$. For example, the formula cited by Goldstein and Levy given below

$$C_3H_8 = 2C_2H_6 - CH_4$$

(which reads, "One propane equals two ethanes minus one methane") suggests the obvious expression for the ground state energy $E_{C_3H_8}$ of propane in terms of the ground state energies $E_{C_2H_6}$ and E_{CH_4} of ethane and methane: $C_3H_8 = 2C_2H_6 - CH_4$ and this later formula, whilst not correct, is known to be very accurate, the relative error between the two sides of the equation above is less than 0.01 percent. This precision shows the significance of entanglement in quantum systems.

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