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A LOGISTIC NONLINEAR BLACK-SCHOLES-MERTON PARTIAL DIFFERENTIAL EQUATION: EUROPEAN OPTION

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Abstract

Nonlinear Black-Scholes equations provide more accurate values by taking into account more realistic assumptions, such as transaction costs, illiquid markets, risks from an unprotected portfolio or large investor's preferences, which may have an impact on the stock price, the volatility, the drift and the option price itself. Most modern models are represented by nonlinear variations of the well-known Black-Scholes Equation. On the other hand, asset security prices may naturally not shoot up indefinitely (exponentially) leading to the use of Verhulst's Logistic equation. The objective of this study was to derive a Logistic Nonlinear Black Scholes Merton Partial Differential equation by incorporating the Logistic geometric Brownian motion. The methodology involves, analysis of the geometric Brownian motion, review of logistic models, process and lemma, stochastic volatility models and the derivation of the linear and nonlinear Black-Scholes-Merton partial differential equation. Illiquid markets have also been analyzed alongside stochastic differential equations.
The result of this study may enhance reliable decision making based on a rational prediction of the future asset prices given that in reality the stock market may depict a nonlinear pattern.

Keywords: Non-Linear; Black Scholes; Brownian Motion; Logistic Brownian Motion; Illiquid Markets.

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1. Introduction

Logistic Geometric Brownian Motion Model

In relaxing one of the assumptions of the Black-Scholes-Merton partial differential equation and using the Walrasian law and the excess demand function $ED(S(t)) = Q_D(S(t)) - Q_S(S(t))$, where $ED(S(t))$ represents the excess demand, $Q_D(S(t))$ and $Q_S(S(t))$ are the quantities demanded and supplied respectively, the price of an asset follows a logistic geometric Brownian motion given by equation;

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$$dS - \mu S(S^* - S)dt + \sigma S(S^* - S)dZ$$

$$\frac{1}{S}\frac{dS}{(S^* - S)} = \mu dt + \sigma dZ \qquad (1)$$

Where S^* is the Walrasian market equilibrium price, S is the stock price at any given time t, μ is the drift rate and σ is the volatility of the stock price at any given time t. Here, volatility σ is constant, [37].

We use the Logistic Geometric Brownian Motion in equation (1) and a choice of portfolio in equation $\Pi = -C + \frac{\partial C}{\partial S}S$ and the change in portfolio equation $\delta\Pi = -\delta C + \frac{\partial C}{\partial S}\delta S$ to derive to derive the Logistic Black-Scholes-Merton Partial differential equation give as,[37]

$$\frac{\partial C}{\partial t} + rS(S^* - S)\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 (S^* - S)^2 \frac{\partial^2 C}{\partial S^2} = rC$$

We intend to derive a Logistic nonlinear Black-Scholes-Merton partial differential equations for the European call option with volatility depending on different factors, such as stock price, time, the option price and the respective derivatives. We therefore intend to take into account several effects that are not included in the linear models as well as the recent Nonlinear models. In particular we shall emphasize the logistic Brownian Motion.

The methodology will involve, analysis of the geometric Brownian motion, review of logistic models, $It\hat{o}'s$ process and lemma, stochastic volatility models and the derivation of the linear and nonlinear Black-Scholes-Merton partial differential equation. Illiquid markets shall also be analysed. An analysis of stochastic differential equations will also be done extensively. Finally, an attempt will be made to derive the Logistic Non-linear Black-Scholes-Merton Partial Differential Equation.

2. The Logistic Nonlinear Black-Scholes-Merton Partial Differential Equation

In this section we use the Geometric Brownian motion equation given by equation 1 above and the random walk in discrete time given by

$$d\delta = \mu S\delta t + \sigma S\varepsilon\sqrt{\delta t} + O\left(\delta t^{\frac{3}{2}}\right) \qquad (2)$$

with the assumption that the portfolio is revised every δt where δt is a finite and a fixed time step and that the hedged portfolio has an expected return equal to that from a risk free bank deposit, which is the same as the valuation policy in discrete hedging with no transaction costs.

Suppose the price of an asset follows a Logistic Geometric Brownian motion given by equation (1) given as

$$dS = \mu S(S^* - S)dt + \sigma S(S^* - S)dZ$$

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then over a sufficiently small time interval δt the change in stock price is given by

$$d\delta = \mu S(S^* - S)\delta t + \sigma S(S^* - S)\varepsilon\sqrt{\delta t} + O\left(\delta t^{\frac{3}{2}}\right) \tag{3}$$

Where ε is a random drawing from a normal distribution table.
We set up a hedged portfolio Π as

$$\Pi = C(S,t) - \Delta S \tag{4}$$

Where $\Delta = \frac{\partial C}{\partial S}(S,t)$. Henceforth we suppress dependance of Π, C, Δ on t over δt After a given time δt therefore the portfolio becomes

$$\Pi + \delta\Pi = C(S + \delta S, t + \delta t) - \Delta(S + \delta S) \tag{5}$$

from which it follows that

$$\delta\Pi = C(S + \delta S, t + \delta t) - \Delta(S + \delta S) - C(S,t) + \Delta S \tag{6}$$

Expanding this in Taylor's series we obtain

$$\delta\Pi = C\frac{\partial C}{\partial t}\delta t + \frac{\partial C}{\partial S}\delta S + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(\delta S)^2 + \ldots - \Delta\delta S$$
$$= \sqrt{\delta t}\sigma S(S^* - S)\varepsilon\left(\frac{\partial C}{\partial S} - \Delta\right) + \delta t\left(\frac{\partial C}{\partial t} + \mu S(S^* - S)\left(\frac{\partial C}{\partial S} - \Delta\right) + \frac{1}{2}\sigma^2 S^2 (S^* - S)^2 \varepsilon^2 \frac{\partial^2 C}{\partial S^2}\right) + \ldots$$

Which has not accounted for the inevitable transaction costs that will be incurred on rehedging. The costs are

$$kS|C|.$$

The quantity C of the underlying asset that are bought is given by the change in the delta from a given time step to the next:

$$C \approx \frac{\partial C}{\partial S}(S + \delta S, t + \delta t) - \frac{\partial C}{\partial S}(S,t)$$

Which can be approximated by

$$C = \frac{\partial C}{\partial S} + \frac{\partial^2 C}{\partial S^2}\delta S + \frac{\partial^2 C}{\partial S \partial t}\delta t + \ldots - \frac{\partial C}{\partial S}$$

Where all derivatives are now evaluated at(S,t). After two terms canceling we get the approximation

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$$C \approx \frac{\partial^2 C}{\partial S^2}\delta S \approx \frac{\partial^2 C}{\partial S^2} S\varepsilon\sigma\sqrt{\delta t}$$

Subtracting the cost from the change in portfolio value gives a total change of $\delta\Pi = d\Pi - kS\,|\,c\,|$ which is

$$\delta\Pi = \sqrt{\delta t}\sigma S(S^* - S)\varepsilon\left(\frac{\partial C}{\partial S} - \Delta\right) + \delta t\left(\frac{\partial C}{\partial t} + \mu S(S^* - S)\left(\frac{\partial C}{\partial S} - \Delta\right) + \frac{1}{2}\sigma^2 S^2 (S^* - S)^2 \varepsilon^2 \frac{\partial^2 C}{\partial S^2}\right) +$$
$$- k\sigma S^2 |\varepsilon| \sqrt{\delta t}\left|\frac{\partial^2 C}{\partial S^2}\right| + \ldots$$

The mean of this is

$$E[\delta\Pi] = \delta t\left(\frac{\partial C}{\partial t} + \mu S(S^* - S)\left(\frac{\partial C}{\partial S} - \Delta\right) + \frac{1}{2}\sigma^2 S^2 (S^* - S)^2 \varepsilon^2 \frac{\partial^2 C}{\partial S^2}\right) + -k\sigma S^2 \sqrt{\frac{2\delta t}{\pi}}\left|\frac{\partial^2 C}{\partial S^2}\right| + \ldots$$

Because $E[\varepsilon] = 0, E[\varepsilon^2] = 1$ and $E[|\varepsilon|] = \sqrt{\frac{2}{\pi}}$

We also find that

$$E\left[(\delta\Pi)^2\right] = \delta t\left[\sigma^2 S^2 (S^* - S)^2 \varepsilon^2 \left(\frac{\partial C}{\partial S} - \Delta\right)^2 - 2k\sigma S^2 (S^* - S)^2 \left|\frac{\partial^2 C}{\partial S^2}\right| \sigma S(S^* - S)\left(\frac{\partial C}{\partial S} - \Delta\right)\varepsilon\,|\,\varepsilon\,|\right.$$
$$\left. + k^2\sigma^2 S^4 (S^* - S)^4 \left(\frac{\partial^2 C}{\partial S^2}\right)^2 \varepsilon^2 + \ldots\right]$$
$$= \delta t\left(\sigma^2 S^2 (S^* - S)^2 \varepsilon^2 \left(\frac{\partial C}{\partial S} - \Delta\right)^2 + k^2\sigma^2 S^4 (S^* - S)^4 \left(\frac{\partial^2 C}{\partial S^2}\right)^2\right) + \ldots$$

since $E[\varepsilon\,|\,\varepsilon\,|] = 0$

The variance of the portfolio change is therefore

$$\mathrm{var}[\delta\Pi] = E\left[(\delta\Pi)^2\right] - (E[\delta\Pi])^2$$
$$= \delta t\left(\sigma^2 S^2 (S^* - S)^2 \varepsilon^2 \left(\frac{\partial C}{\partial S} - \Delta\right)^2 + \left(1 - \frac{2}{\pi}\right) k^2 \sigma^2 S^4 (S^* - S)^4 \left(\frac{\partial^2 C}{\partial S^2}\right)^2\right) + \ldots$$

to leading order. For finite hedging period δt and finite cost k this cannot generally be made to vanish. However, the variance, or risk, can be minimized by choosing

$$\Delta = \frac{\partial C}{\partial S}$$

With this choice,

$$E[\delta\Pi] = \delta t\left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \left(S^* - S\right)^2 \frac{\partial^2 C}{\partial S^2} - k\sigma S^2 \left(S^* - S\right)^2 \sqrt{\frac{2\delta t}{\pi}}\left|\frac{\partial^2 C}{\partial S^2}\right|\right)$$

to leading order. This quantity is an expectation allowing for the expected amount of transaction costs. We now set this quantity equal to the amount that would have been earned by a risk free account:

$$\delta t\left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \left(S^* - S\right)^2 \frac{\partial^2 C}{\partial S^2} - k\sigma S^2 \left(S^* - S\right)^2 \sqrt{\frac{2\delta t}{\pi}}\left|\frac{\partial^2 C}{\partial S^2}\right|\right)$$

$$= r\Pi\delta t = r\left(C - S\frac{\partial C}{\partial S}\right)\delta t$$

On dividing by δ and rearranging we obtain the Logistic nonlinear Black Scholes Merton partial differential equation given by

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \left(S^* - S\right)^2 \frac{\partial^2 C}{\partial S^2} - k\sigma S^2 \left(S^* - S\right)^2 \sqrt{\frac{2}{\pi\delta t}}\left|\frac{\partial^2 C}{\partial S^2}\right| + rS\frac{\partial C}{\partial S} - rC = 0$$

3. Conclusions and Recommendations

In this article we have managed to derive a Logistic nonlinear Black Scholes Merton Partial differential equation based on the model with transaction costs. This comes as an advancement in the study of the nonlinear Black Scholes Merton Partial differential equation and in its application in the prediction of future asset prices where transaction costs are considered together with the logistic geometric Brownian motion unlike in previous studies where the Geometric Brownian motion has been used.

We recommend that interested scholars solve the differential equation in order to enhance prediction of future asset prices based on the model derived.

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