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Zablon Muga
Jaramogi Oginga Odinga University of Science and Technology

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Sum Construction of Automorphic Symmetric Balanced Incomplete Block Designs

C.W. Otulo[1*], Z. M. Muga[2], N. B. Okelo[3], J.O. Nyakinda[4]

[1,2,3,4]School of Mathematics and Actuarial Science, Jaramogi Oginga Odinga University of Science and Technology, P.O. Box 210-40601, Bondo-Kenya

[*]*Corresponding Author: owandera@tum.ac.ke*

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Abstract- In this study Sum construction method of automorphic symmetric balanced incomplete block designs has been presented in details. Efficiency of a test design used in the Sum construction of automorphic symmetric balanced incomplete block designs has been determined alongside its existence. The process involved the application of sum construction to give new designs of parameters D (v, b, $\lambda_1+\lambda_2$) and an application of Bruck Ryser Chwola theorem extensively. A test design used in the sum construction method has been found to be existing with an efficiency of 77.78%. The sum constructed automorphic symmetric balanced incomplete block designs provide more information per block than the parent designs.

Keywords—Symmetric Balanced Incomplete Block Design (SBIBD), Automorphism and Automorphic Symmetric Balanced Incomplete Block Design (AUSBIBD)

I. INTRODUCTION

A theoretical BIBD concept was initially developed in the field of statistical and experimental design but later applied and expanded to other fields of science and art [1]. The application areas include; coding theory, forestry, medical innovations, network enhancement, engineering and cryptography among others [2]. The theory of experimental design came into existence through the determination of [3] in the early 1930s. The work of [4] provided the three principles on designs of experiments namely; randomization, replication and blocking. In [5] in his latest review on the construction of BIBD showed that the conditions for the existence of BIBD are only necessary but not sufficient, this argument implies that even if the conditions are fulfilled a BIBD may still fail to exist. There are many sets of possible parameters for which the existence question has not been settled [6]. It is usually very difficult to decide on the method of construction of a BIBD because the conditions that are necessary and sufficient for existence of any BIBD with given parameters are not fully known [7]. A part from randomized complete block designs, BIBD's are among the most studied structures in design theory under combinatorial mathematics. BIBD's were introduced by [8], who defined a BIBD as an arrangement of ***b*** blocks in a manner that exactly ***k*** distinct objects are contained in every block with every object occurring in exactly ***r*** different blocks, and every pair of distinct objects occurring together in exactly λ blocks. Many authors [9] and researchers [10] have paid attention to the construction of BIBD's leading to several techniques available for the construction of BIBDs. These include the trade-off method, symmetric repeated difference method, variety cutting and construction from finite permutation geometries among others. A study by [11] on construction of BIBD's referred to the variables *(v, b, k, r λ)* as the necessary parameters of any BIBD. The author in [9] while studying on the duals of BIBD revealed that BIBD's and their duals are variance balanced like the randomized complete block designs. He further added that the designs which are variance balanced are among the binary block designs with block size ***k*** less than number of treatments. In this study, a method described as sum construction of AUSBIBD's has been presented in details.

II. LITERAURE REVIEW

Design and experiment as a subject was founded by a profound statistician in [5**]** who gave the three principles on designs of experiments namely; randomization, replication and blocking.

A study by [8] on a mathematical review of automorphisms for BIBD and applications, revealed the mathematical concepts required for construction of t-designs via their isomorphisms and automorphisms. A description of BIBD, SBIBD and an example of construction of a BIBD was described that study. The study mentioned Sum construction method of SBIBD's but failed to describe the details on sum construction, instead it elaborated much on t-designs. The conclusion for the study showed that BIBD are very important in research and when properly developed, BIBD can give rise to appropriate results in statistical analysis. The study however did not provide any practical application to the concept of sum construction. An area the current study has employed as key success by treating a data set to a classical ANOVA to reveal the efficiency of sum constructed design. Basing on fast algorithm format for a class of SBIBD suggested by [9] we have further improved on this work by providing an algorithm to aid the sum construction of AUSBIBD.

In a study by [7] on the construction of SBIBD, block designs are many subsets with similar properties. These properties must satisfy some conditions which are important to certain application in the field of study of experimental design, software testing, algebraic geometry, and cryptography. His study widely captured the balanced incomplete block design (BIBD) concluding that when all the conditions pertaining to design are satisfied, then the symmetry remain uninterfered with. According to a study by [1] on the construction methods of optimum chemical balanced weight design, pair wise efficiency and variance balanced block designs were used to obtain optimum chemical balanced weight design. The study focused on the incidence matrices of known SBIBD and conditions under which a design would become an A-optimal design. In the conclusion the study noted that, pair wise balanced designs have the same efficiency as variance balanced designs. However, such designs require a larger number of replication in order to be attained.

III. METHODOLOGY

The theorems that have led to the sum construction have been summarized in this section.

3.1 Sum Construction method

Given two designs that are on the same point set most preferably SBIBD's, sum construction involves systematic addition of a fixed value to the treatments in any block of a given design to form a collection of all the blocks. By fixing some parameters, a new design is obtained through a method of construction known to this study as sum construction. In this work, we have concentrated on the sum construction of AUSBIBD. The theorems below have been fully employed in this work

Theorem 3.1

If a (v, k, λ_1) -BIBD (X_1 , A_1) and a (v, k, λ_2)-BIBD (X_1,A_2) exist on a set X then (v, k, ($\lambda_1 + \lambda_2$)) exists on the same set.

Proof

Let $A = A_1 \cup A_2$ represent the union containing the sets A_1 and A_2 then A is a multi-set of non-empty subsets of X, clearly $|X|$=v, furthermore since every block in A_1contains k points and we have that every blocks in A_2 also contain k points. Then it follows that A too has k points

If (x_1, y) ∈ X are chosen such that x≠y, and that the pair containing (x, y) treatments of λ_1 blocks in set A_1 and the pair containing (x, y) are treatments of λ_2 blocks in set A_2 then the pair (x, y) is impliedly contained in $\lambda_1 + \lambda_2$ blocks in the set A. This is proved true for any arbitrarily selected points x, y ∈ X, hence (X, A) is a (v, k, $\lambda_1 + \lambda_2$)-SBIBD.

Corollary 3.2

If a design (X, A) with parameters (v, k, λ) is a BIBD existing on the set X, then for every positive integer **p** ≥ 1 a BIBD (X, A^*) whose parameters are (v, k **p**λ) exist on X

Proof

Let be a positive integer such that p ≥1 and the set A^* be the union of the multisets of A with itself up to **p** times i.e $A^* = A \cup A \cup A \dots \dots . . \cup A$.

By theorem 3.1, the design (X, A^*) is a BIBD whose parameters are $(v, k, \lambda + \lambda \dots \dots . . +\lambda) = (v, k\, p\lambda)$. This implies the sum construction is carried on the multi set union **p** times.

Theorem 3.3

Let N denote the incidence matrix for a SBIBD whose parameters are (v, k, λ) -design, then NN^T = $r\lambda J(r - \lambda)I$, where J is a r × v matrix having all elements being one and $\hat{j}$ is a v × b matrix having all elements being one. More over any design whose incidence matrix is known to satisfy these conditions also satisfy $r(k-1)$ = $\lambda(v-1)$ and $rv = bk$, whenever $v > k$, the incidence matrix is that of a SBIBD whose parameters are (v, k, λ).

IV. RESULTS AND DISCUSSION

The results on considered cases of the sum construction are explained in this section.

A. 4.1 Cases of Sum construction method

In a bid to show examples on the described method, we give the results on sum construction by considering a BIBD with parameters: v=12, r= 11, b=22, λ=2 and k= 6 then letting $v = (0,1,2, \dots ,10, \infty)$

We fix our λ's for the second design and use the sum construction technique to generate up new designs, that is;

Case 1: λ_2=2

The following are the resulting designs which are automorphic in nature

(1, 3,4,5,9, ∞)	(0, 2, 6, 7, 8, 10)
(3, 5,6,7,0, ∞)	(2, 4, 8, 9, 10, 1)
(5, 7,8,9,2, ∞)	(4, 6, 10, 0, 1, 3)
(7, 9, 10, 0, 4, ∞)	(6, 8, 1, 2, 3, 5)
(9, 0,1,2,6, ∞)	(8, 10, 3, 4, 5, 7)
(0, 2,3,4,8, ∞)	(10, 1, 5, 6, 7, 9)
(2, 4, 5,6,10, ∞)	(1, 3, 7, 8, 9, 0)
(4, 6,7,8,1, ∞)	(3, 5, 9, 10, 0, 2)
(6, 8, 9,10,3, ∞)	(5, 7, 0, 1, 2, 4)
(8, 10,0,1,5, ∞)	(7, 9, 2, 3, 4, 6)
(10, 1,2,3,7, ∞)	(9, 0, 4, 5, 6, 8)
(1, 3,4,5,9, ∞)	(0, 2, 6, 7, 8, 10)

Exploring other possible values like;

Case 2: λ_2 = 4

The following are the resulting automorphic symmetric designs

(1, 3, 4, 5, 9,∞)	(0, 2, 6, 7, 8, 10)
(5, 7, 8, 9, 2,∞)	(4, 6, 10, 0, 1, 3)
(9, 0, 1, 2, 6,∞)	(8, 10, 3, 4, 5, 7)
(2, 4, 5, 6, 10,∞)	(1, 3, 7, 8, 9, 0)

(6, 8, 9, 10, 3,∞) (5, 7, 0, 1, 2, 4)
(10, 1, 2, 3, 7, ∞) (9, 0, 4, 5, 6, 8)
(3, 5, 6, 7, 0,∞) (2, 4, 8, 9, 10, 1)
(7, 9, 10, 0, 4, ∞) (6, 8, 1, 2, 3, 5)

Case 3: where λ_2=5
The following are the results

(1, 3, 4, 5, 9,∞) (0, 2, 6, 7, 8, 10)
(6, 8, 9, 10, 3,∞) (5, 7, 0, 1, 2, 4)
(0, 2, 3, 4, 8,∞) (10, 1, 5, 6, 7, 9)
(5, 7, 8, 9, 2,∞) (4, 6, 10, 0, 1, 3)
(10, 1, 2, 3, 7,∞) (9, 0, 4, 5, 6, 8)
(4, 6, 7, 8, 1,∞) (3, 5, 9, 10, 0, 2)
(9, 0, 1, 2, 6,∞) (8, 10, 3, 4, 5, 7)
(3, 5, 6, 7, 0,∞) (2, 4, 8, 9, 10, 1)

The existence of the parent design was tested using Bruck-Ryser-Chowla Theorem proposed by [2]. The parameters of the design were replaced in to the relationship describing the above theorem for existence test. A relevant example has been presented with the values of the parameters.
In our second illustrative results we provide an outcome of sum construction by considering design which is symmetric in nature, that is a BIBD whose parameters are v=b=7, r= k= 3 and λ=1 whose initial blocks are as below;
[[1 2 3]
[3 4 5]
[5 6 7]
[7 1 2]
[2 3 4]
[4 5 6]
[6 7 1]]

by letting $v = (1,2,\ldots,7)$ and choosing on any block say block containing treatments 1,2,3
We fix our λ's for the second design and use the sum construction technique to generate up new designs, that is;

Case 1: λ_2=2

The following are the resulting blocks of the generated new BIBD, which are automorphic in nature
These are the blocks of the generated design 1 corresponding to the above case (λ_2=2):
[[3 4 5]
[5 6 7]
[7 1 2]
[2 3 4]
[4 5 6]
[6 7 1]
[1 2 3]]

These are the blocks of the generated design 2 corresponding to the above case (λ_2=2):
[[5 6 7]
[7 1 2]
[2 3 4]
[4 5 6]
[6 7 1]
[1 2 3]
[3 4 5]]

These are the blocks of the generated design 3 corresponding to the case $\boldsymbol{\lambda_2}$ **=2**:
[[7 1 2]
[2 3 4]
[4 5 6]
[6 7 1]
[1 2 3]
[3 4 5]
[5 6 7]]

These are the blocks of the generated design 4 corresponding to the case $\boldsymbol{\lambda_2}$ **=2**:
[[2 3 4]
[4 5 6]
[6 7 1]
[1 2 3]
[3 4 5]
[5 6 7]
[7 1 2]]

These are the blocks of the generated design 5 corresponding to the case $\boldsymbol{\lambda_2}$ **=2:**
[[4 5 6]
[6 7 1]
[1 2 3]
[3 4 5]
[5 6 7]
[7 1 2]
[2 3 4]]

These are the blocks of the generated design 6 corresponding to the case $\boldsymbol{\lambda_2}$ **=2:**
[[6 7 1]
[1 2 3]
[3 4 5]
[5 6 7]
[7 1 2]
[2 3 4]
[4 5 6]]

These are the blocks of the generated design 7 corresponding to the above case $\boldsymbol{\lambda_2}$ **=2:**
[[1 2 3]
[3 4 5]
[5 6 7]
[7 1 2]
[2 3 4]
[4 5 6]
[6 7 1]]

It is worth noting that design 7 of the case $\boldsymbol{\lambda_2}$ **=2**, is similar to our initial blocks. The blocks of design 1 through to design 7 are observed to be having same the same treatments with an exception of the blocks appearing in different position. This is confirming the idea of a one to one mapping concept. This method of constructing new BIBD from existing blocks of a given already existing design is best described as the sum construction method. In this study an algorithm in python program has been utilized to aid the process.

Case 2: $\boldsymbol{\lambda_2 = 3}$

The following are the blocks of the generated design corresponding to the case $\boldsymbol{\lambda_2 = 3}$**:**

[[4 5 6]
[7 1 2]
[3 4 5]
[6 7 1]
[2 3 4]
[5 6 7]
[1 2 3]]

Our main observation in this case is that the resulting blocks of the generated new BIBD are automorphic in nature

It is worth noting that the blocks of the designs in all the cases when $\boldsymbol{\lambda_2 = 3}$, are all similar to our initial blocks.

B. 4.2 Efficiency of the designs

The efficiency of the designs used in the sum construction method has been computed using the relationship that was earlier proposed by [10] as below,

Efficiency (E) $= \frac{v}{v-1} \times \frac{k-1}{k}$

Taking the test design with the values of the parameters specified above as v=12, b=22, k= 6 and λ=2

The efficiency computed

Efficiency (E) $= \frac{12}{12-1} \times \frac{6-1}{6} = \frac{60}{66}$ = 90.9%.

For the design whose parameters are v=7, k= 3, b=7, r=3 and λ=1, the value of the computed efficiency is; Efficiency (E) $= \frac{7}{7-1} \times \frac{3-1}{3} = \frac{14}{18}$ = 77.78%.

V. CONCLUSION

We conclude that various Automorphic Symmetric Balanced Incomplete Block Design (AUSBIBD) have been constructed using the sum construction method developed from the theorem 3.1. It is gainfully worth noting that the blocks of the designs in all the cases when $\boldsymbol{\lambda_2 = 3}$**,** are all similar to our initial blocks. Thus a method of sum construction of BIBD which are automorphic has been successfully developed and proven viable. Generally, for the generation of any AUSBIBD using sum construction method, the following are key points:

(i). List your given design parameters in some very fixed order.

(ii). Choose on any block of the given symmetric design at random.

(iii). Identify the value that leads to a one to one mapping onto the design whose parameters are fixed as in (i) above

(iv). Perform the sum construction of the AUSBIBD using theorems described in this study.

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AUTHORS PROFILE

C. W. Otulo is a PhD in Applied statistics candidate at JOOUST-Kenya. He is currently working as a lecturer at Technical University of Mombasa University in Department of mathematics.

Z. M. Muga is a senior lecturer at JOOUST-Kenya.

N. B. Okelo has a Phd in Pure mathematics from Maseno University Kenya and was the AU-TWAS young scientist award winner in 2013.

J. O. Nyakinda is a senior lecturer at JOOUST-Kenya.