

**ON CONVEXITY OF CHEBYSHEV  
SETS IN NORM-ATTAINABLE CLASSES**

BY

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## DECLARATION

This thesis is my own work and has not been presented for a degree award in any other institution.

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W251/4160/2019

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This thesis has been submitted for examination with our approval as the university supervisors.

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## DEDICATION

*To my family...*

## ABSTRACT

Best approximation is an interesting field in functional analysis that has attracted a lot of attention from many researchers for a very long period of time up-to-date. Of greatest consideration is the characterization of the Chebyshev set which is a subset of a normed linear space which contains unique best approximations. However, a fundamental question remains unsolved to-date regarding the convexity of the Chebyshev set in infinite normed linear space known as the Chebyshev set problem. The question which has not been answered is: Is every Chebyshev set in a normed linear space convex? This question has not got any solution including the simplest form of a real Hilbert space. The main objective of this study is to characterize Chebyshev sets and convexity in normed linear spaces. The specific objectives include to: Determine conditions under which subsets of normed linear spaces are Chebyshev; Characterize distance functions of Chebyshev set in normed linear spaces and; Investigate convexity of Chebyshev set in normed linear spaces. The methodology involved the use of known fundamental principles like the Bunt-Motzkin Theorem, Fréchet and Gateaux differentiability conditions and best approximation techniques. The results of this study show that every non-void closed set in  $NA(H)$  is a Chebyshev set if it is Euclidean. Also, if  $\mathfrak{U}$  is proximal in  $NA(H)$  then  $\mathfrak{U}$  is closed and a Chebyshev set. Every distance function of a Chebyshev set of the normed linear space of all norm-attainable real-valued functions is Fréchet differentiable. For convexity, it has been shown that every  $\mathfrak{P}_{\mathfrak{U}}(\xi)$  which is convex is a Chebyshev set. The results of this study have provided a partial solution to the Chebyshev set problem and are also useful in solving convex optimization problems.

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# Index of Notations

<p>BA Best approximation . . . 1</p> <p>CS Chebyshev Set(s) . . . . 1</p> <p>NLS Normed Linear Space(s) 2</p> <p><math>\ell^2</math> Hilbert space of square summable complex sequences . . . . 2</p> <p><math>\mathbb{R}</math> Set of real numbers . . . . 2</p> <p><math>NA(H)</math> Norm-attainable class 2</p> <p><math>IPS</math> Inner Product Space . 2</p> <p><math>\ \cdot\ </math> Norm . . . . . 4</p> <p>inf Infimum . . . . . 4</p> <p><math>d(\zeta, G)</math> Distance from a point <math>\zeta</math> to a set <math>G</math> . . . . . 4</p> <p><math>F_G</math> Metric projections for the set <math>G</math>. . . . . 4</p> <p><math>UCED</math> Uniformly Convex in Every Direction . . . . . 8</p> <p><math>UCEDP</math> Uniformly Convex in Every Direction Prop- erty . . . . . 8</p> <p><math>\partial_F</math> Fréchet sub-differential . 14</p> <p><math>\nabla</math> Dell operator . . . . . 14</p> <p>Sup Supremum . . . . . 15</p> <p>CF Chebyshev function . . . 17</p> <p>SVP Single-valuedness prop- erty . . . . . 17</p> <p><math>R^n</math> <math>n</math>-dimensional real space 17</p>	<p><math>\overline{\mathbb{R}}</math> <math>n</math>Extended real space . . . 17</p> <p>lsc lower semi-continuous . . 17</p> <p><math>(CSCNFP)</math> Chebyshev Set centers that are not far- thest points . . . . . 28</p> <p>usc upper semi-continuous . 33</p> <p><math> \cdot </math> Modulus . . . . . 35</p> <p><math>\langle \cdot, \cdot \rangle</math> Inner Product . . . . . 36</p> <p>AC Almost Convex . . . . . 48</p> <p><math>MPM</math> Metric Projection Map- ping . . . . . 50</p> <p>BMT Bunt-Motzkin Theorem for function spaces . . 55</p>
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# Chapter 1

## INTRODUCTION

### 1.1 Mathematical background

Studies in approximation theory have been carried out by many mathematicians over decades (see [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [68] and [150] and the references therein). Best approximation (BA) is an interesting field in functional analysis that has attracted a lot of attention from many researchers for a very long period of time up-to-date as discussed in [14] and [21]. Of greatest consideration is the characterization of the Chebyshev sets (CS) which is a subset of a Normed Linear Space (NLS) which contains unique BAs. [125]. Approximation theory involves obtaining best approximation of functions using simple functions whether they are linear or nonlinear as shown in [11], [60] and [143].

Martin [90] characterized remotality with regard to normed linear spaces and in particular for convex sets in Banach spaces. However, the convexity of the Banach spaces was not done in general due to the complex nature of spaces. Mazaheri [95] also considered weakly-Chebyshev subspaces for

NLSs of Banach spaces and did their characterization in terms of nearest and furthest points via distance functions. However, the authors could not give a particular best approximation for the CSs and their convexity in for the simplest case [94] of a Hilbert space (HS) of  $\ell^2$ .

Deutsch [38] on the study of convex sets and their characterizations in general spaces determined optimization criteria for vector spaces and left open a question regarding convexity of these spaces. More recently, Mazaheri and Salehi [96] studied CS and considered conditions under which they are convex. However, they could not determine convexity of the CSs in NLSs even for the simplest set up of real ( $\mathbb{R}$ ) Hilbert spaces.

With all these considerations, a fundamental question remains unsolved to-date regarding the convexity of CSs in infinite NLSs known as the CS problem. The question which has not been answered is: Is every CS in a NLS convex? This question has not been answered even in the simplest case of a real Hilbert space. In this regard, it is worth characterizing CS and convexity in NLSs [128] and we consider NLS of norm-attainable operators in this study.

Norm-attainable operators have been studied over time by various mathematicians [106]. A lot of these studies focused on structural properties of  $NA(H)$ . In fact  $NA(H)$  is a NLS which exhibits Inner Product Space (IPS) property and other characteristics that makes it a  $C^*$ -algebra. Moreover, characterizations have been done in-terms of norm-attainability conditions for operators, functionals among others in the space of all bounded linear operators on Hilbert spaces. However, describing CS has not been considered in  $NA(H)$  hence it is important to consider CS problem in  $NA(H)$ .

This study is organized as follows: For the first chapter, we begin with a mathematical background as given in this section followed with the basic concepts that helps us to understand this study. We then state the problem of our study followed by the objectives and finally conclude the chapter with the significance of the study. The second chapter entails a review of related literature to our work. This has been done as per the objectives. Chapter three is containing a description of the techniques and methods that are useful in finding solutions to our problem of study. Chapter four contains results of this study and finally chapter five has the conclusion and recommendations for future research.

## 1.2 Basic concepts

Basic definitions and concepts that are key to our study are given in this section. These include concepts on proximal sets, CSs and convexity in NLSs.

**Definition 1.1.** ([60], Definition 1.5) A set is a well defined collection of distinct objects. Let  $A, B$  be two sets. A **function**  $f : A \rightarrow B$  is a rule that assigns to every element in  $A$  a unique element in  $B$ . An **operator** is a structure preserving function.

**Definition 1.2.** ([6], Definition 1.37) A normed linear space (NLS) is a linear vector space endowed with a norm which satisfies non-negativity, scaling and triangle inequality.

**Definition 1.3.** ([62], Definition 1.7) Let  $W$  be a NLS and  $G$  be a nonempty subset of  $W$ . Consider the point  $\zeta \in W$ . We define the distance

from  $\zeta$  to  $G$  by  $d(\zeta, G) = \inf_{\eta \in G} \|\zeta - \eta\|$ , and the map  $\zeta \mapsto d(\zeta, G)$  is called the distance function for  $G$ . We call  $\zeta$  the nearest point in  $G$ .

**Definition 1.4.** ([143], Definition 2.4) A set  $G$  in a NLS  $W$  is called a CS if every point in  $W$  has a unique nearest point in  $G$ . That is, CS is a subset of a NLS that admits unique best approximations.

**Remark 1.5.** For a CS we define the map  $f_G : W \rightarrow W$  as the map that sends  $\zeta \in W$  to the unique element of  $F_G(\zeta)$ . We refer to both  $F_G$  and  $f_G$  as the metric projections for  $G$ .

**Definition 1.6.** ([62], Section 2.2) Let  $D$  be a nonempty set. A subset  $E$  of  $D$  is said to be convex if for all  $\zeta, \eta \in E$  the line segment connecting  $\zeta$  and  $\eta$  is in  $E$ , that is,  $(1 - \alpha)\zeta + \alpha\eta$  is in  $E$   $\zeta, \eta \in E$ , and  $\alpha \in [0, 1]$ .

**Definition 1.7.** ([106], Section 2) Let  $H$  be a Hilbert space and  $T : H \rightarrow H$  be a bounded linear operator. We say  $T$  is norm-attainable if there exists a unit vector  $\zeta \in H$  such that  $\|T\zeta\| = \|T\|$ .

**Remark 1.8.** The space of all norm-attainable operators is denoted by  $NA(H)$ . It is known [107] that  $NA(H)$  a NLS which is bounded and linear. It is worth-noting that it is also a Banach space and in particular a  $C^*$ -algebra.

Having armed ourselves with the basic concepts, we state the problem of our study in the next section.

### 1.3 Statement of the problem

Best approximation (BA) is an interesting field in functional analysis that has attracted a lot of attention from many researchers for a very

long period of time up-to-date as described in [10] and [11]. Of greatest consideration is the characterization of the CS which is a subset of a NLS which contains unique BAs[125]. Approximation theory involves obtaining best approximation of functions using simple functions whether they are linear or nonlinear as illustrated by [11], [60] and [143]. The question which has not been answered is: Is every CS in a NLS convex? This question has not been answered even in the simplest case of a real HS. Some known results have been put forward by Narang [143] regarding a closed and convex subset of a HS which proved that it is a CS. Other research work have also been done which have provided partial answers of the CS problem have been discussed by Frank [62] and also by Balaganski and Vlasov [10] whereby metric projections have been very helpful in giving partial solutions. Several other forms of this problem have also been postulated. For instance, on characterizing convexity of CSs, Borwein [18] left open some five fundamental open questions. Moreover, Ricceri [146] gave a conjecture on real Hilbert spaces involving the convexity of Borel functions on these spaces. Conclusively, there is no known answer to CS problem in totality as only partial solutions have been provided in special cases. In this study therefore, we have provided a partial answer to CS problem by characterizing CSs and convexity in normed linear spaces, particularly in  $NA(H)$ .

## 1.4 Objectives of the study

### 1.4.1 Main objective

The main objective of this study has been to characterize Chebyshev sets and convexity in normed linear spaces.

### 1.4.2 Specific objectives

The specific objectives of this study included to:

- (i). Determine conditions under which subsets of normed linear spaces are Chebyshev sets.
- (ii). Characterize distance functions of Chebyshev sets in normed linear spaces.
- (iii). Investigate convexity of Chebyshev sets in  $NA(H)$ .

## 1.5 Significance of the study

The most important basic question in the theory of approximation is concerned with the existence of BAs. This is because it has several applications involving finding solution to systems of equations. This study is useful as it has provided a contribution of knowledge in functional analysis and also providing at least a partial solution to the CS problem. The results obtained are also useful in solving convex optimization problems and finding solutions to differential equations.

# Chapter 2

## LITERATURE REVIEW

### 2.1 Introduction

This chapter gives a detailed account of research work which have been done in approximation theory. In particular, we consider literature on conditions under which subsets of normed linear spaces are Chebyshev. We also discuss the characterizations on distance functions of CSs in NLSs and finally give a review on investigations on convexity of CSs in NLSs.

### 2.2 Normed Linear Spaces and $NA(H)$

Normed linear spaces have a long research history and valuable characterizations have been done over time (see [14]-[20]). These spaces are endowed with several useful properties that attract the attention of many mathematicians (see [22]-[46]). The authors of [42] conducted an extensive investigation on normed linear spaces, delving specifically into the concept of directional uniform continuity. They obtained significant results,

which were then generalized and applied in the field of geometry, revealing a fascinating connection between normed linear spaces and geometric properties. The authors (see [48]-[67]) demonstrated that a collection of unit ball chords, all parallel to a fixed direction and bounded in length, possesses a unique property: all midpoints of these chords lie uniformly inside the unit ball. This intriguing property was termed as uniformly convex in every direction (*UCED*). The authors further utilized this concept to characterize bounded subsets of normed linear spaces that possess unique Chebyshev centers, which are points that minimize the maximum distance to all points in the subset.

The study provided both sufficient and necessary conditions for a normed linear space to satisfy the *UCED* property, shedding light on the geometric behavior of such spaces. Moreover, the authors proved a remarkable result: if a normed linear space is uncountable, it cannot be renormed (i.e., cannot be equipped with a different norm), and consequently, it cannot be isomorphic (i.e., structurally equivalent) to a space that satisfies the *UCED* property. The findings of this investigation contributed significantly to the understanding of normed linear spaces, directional uniform continuity, and geometric properties related to the *UCED* concept. The results presented interesting implications in the characterization normed linear space properties that have important applications in mathematical fields such as optimization, functional analysis, and approximation theory.

Pai [145] characterized smooth normed spaces using the concept of proximal points of convex sets. A normed linear space was considered to be smooth given that every point of its unit sphere contained at most one

support hyperplane in the closed unit ball. The study demonstrated the connection between convexity and smoothness, that is, if the dual for a normed linear space  $B$  is strictly convex, then  $B$  is a smooth normed linear space and if  $B$  is reflexive then strict convexity and smoothness of  $B$  is ensured by existence of a complete duality in the space. Conditions under which a pair of convex sets have proximal points was given. The following are the main findings in his study:

**Theorem 2.1** ([145], Theorem 2). *Suppose  $B$  is a normed space and  $G, H \in B$  are convex sets. Then the points  $\bar{a} \in G, \bar{b} \in H$  satisfy proximality condition if  $\exists K \in B^*$  such that:*

$$(i). \|K\| = 1.$$

$$(ii). \operatorname{Re}K(a - \bar{a}) \geq 0, \forall a \in G.$$

$$(iii). \operatorname{Re}K(b - \bar{b}) \geq 0, \forall b \in H.$$

$$(iv). K(\bar{a} - \bar{b}) = \|\bar{a} - \bar{b}\|.$$

Theorem 2.1 presents a characterization of proximal points of CSs in NLSs. The points  $\bar{a}$  and  $\bar{b}$  are nearest points of convex sets  $G$  and  $H$  respectively. It was proved that if the convex sets  $G$  and  $H$  are closed then the points  $\bar{a}$  and  $\bar{b}$  are also proximal points in Hilbert spaces.

**Theorem 2.2** (145, Theorem 3). *Let  $B$  be a normed linear space. The conditions below are equivalent:*

$$(i). B \text{ is a smooth NLS.}$$

(ii). If  $G, H \in B$  are convex sets then there exists  $\bar{a} \in G, \bar{b} \in H$  nearest points of  $G$  and  $H$  respectively such that  $\bar{a}$  and  $\bar{b}$  are proximal points of  $G$  and  $H$ .

(iii). The norm of  $B$  is Gâteaux-differentiable a.e at each point of  $B$ .

The result in Theorem 2.2 above represent the main characterization of normed linear spaces with emphasis to smoothness and convexity as well as Gâteaux-differentiability of non zero points of the space.

Zaplatik [150] investigated convexity property in normed linear spaces. The study also examined convexity of Chebyshev sets in normed linear spaces and investigated properties of the metric projections of these sets. The concept of orthogonal projection [131] gave geometrical significance to the proofs given in this study. The following are some of the results obtained:

**Theorem 2.3** ([150], Theorem 2). *Let  $H$  be a complete inner product space. Let  $L \neq \emptyset$  be a closed subset of  $H$ . The following are equivalent:*

(i).  $L$  is convex.

(ii).  $L$  is a Chebyshev set and its metric projection is non-expansive.

(iii). A continuous operator  $T : H \rightarrow L$  of a nearest point of  $L$  exists.

Theorem 2.3 illustrates the relationship between convexity and metric projections of closed sets in an inner product space. It shows that if  $L$  is convex, then it is also a CS with a non-expansive metric projection. Conversely, if  $L$  is a CS with a non-expansive metric projection, then it is

convex. Furthermore, the theorem establishes the existence of a continuous operator that maps each point in  $H$  to its nearest point in  $L$  when any of these conditions hold. The non-expansivity of the metric projection is an important property examined in this theorem. It implies that the distance between any two points in  $H$  is preserved or decreased after projection onto  $L$ . This property has implications in optimization and approximation problems where minimizing distances is a crucial consideration.

**Proposition 2.4** ([140], Proposition 7). *Let  $B$  be a quasinormed linear space. Let  $L \neq \emptyset$  be a closed subset of  $B$ . If the distance function  $d_L$  for  $L$  is convex, then the set  $L$  is convex.*

Proposition 2.4 examines the connection between convexity of sets, closure and convexity of the corresponding distance function [47]. The proposition elucidates how the convexity of sets, which refers to the property of a set to contain all the line segments connecting any two points within the set, is linked to the closure operation (see [48]-[62]). The closure of a set is the smallest closed set that contains the original set, and it plays a crucial role in establishing the convexity of the distance function associated with the set [90]. The distance function associated with a set assigns a value to each point in the set based on its distance from a fixed point or set. The conditions under which the convexity of the distance function is preserved by the convexity of the set are given (see [129]-[145]).

At this juncture, we consider the space of norm-attainable operators ( $NA(H)$ ). Norm-attainable operators have been studied over time by various mathematicians. A lot of these studies focused on structural properties of  $NA(H)$ .

**Theorem 2.5.** ([69]). *The set  $NA(H)$  is bounded and linear.*

Theorem 2.5 gives a description of  $NA(H)$  in terms of linearity and boundedness in general. However, it does not describe the space in terms of its subsets particularly the CSs. It is also known that  $NA(H)$  is a NLS which exhibits other characteristics that makes it a  $C^*$ -algebra.

**Theorem 2.6.** ([68]). *The set  $NA(H)$  is a  $C^*$ -algebra.*

The result in Theorem 2.6 shows that  $NA(H)$  is a  $C^*$ -algebra. In deed like  $B(H)$ , the space of all bounded linear operators on Hilbert spaces, it is a NLS, a Banach space, a Banach algebra and it also satisfies the  $C^*$ -condition for Banach algebras [106].

In conclusion, characterizations have been done in-terms of norm-attainability conditions for operators and functionals [107] among others in the space of all bounded linear operators on Hilbert spaces. However, describing CS has not been considered in  $NA(H)$  hence it is important to consider CS problem in  $NA(H)$ .

## 2.3 Proximal sets, CSs and their subsets

Proximal sets and CSs are important sets in approximation theory due to their properties. A lot of characterizations have been done on these sets with interesting results obtained. Their subsets are also interesting as they carry hereditary properties in them. Fletcher and Moors [61] characterized CSs and showed that a CS is particularly a subset of a NLS which has properties that helps in the establishment of best approximations results which are unique. The authors investigated characteristics of the

metric projection and obtained necessary and sufficient conditions under which a subset of a NLS becomes a CS and also conditions for a CS to satisfy convexity properties. Moreover, the authors gave an example where they constructed a nonconvex CS as shown in the next result.

**Proposition 2.7.** (*[61], Proposition 2.12*) *Let  $W$  be a NLS. A subset of  $W$  which is closed is also convex if and only if it is midpoint convex.*

Proposition 2.7 characterizes NLSs in terms of midpoint convexity and shows that a subset of a NLS can be convex if it is closed and satisfies midpoint convexity. However, this result does not indicate whether the space  $W$  is convex in general even if it is a CS. It is noted that closedness is useful for midpoint convexity but not for convexity in general.

In the earlier years [147] characterized normed spaces in terms of approximate properties. Historically, it can be shown in brief the main contributions of researchers in this field of study. In 1934 Bunt proved the convexity of a CS in the real plane. Also 1938 Kritikos followed Bunt's results and extended the theorem of Bunt to an  $n$ -dimensional real space. This was followed by the work of Efimov and Stechkin in 1961 which showed that a CS of a general HS which is approximately compact satisfies convexity condition. To conclude the history, Klee considered weak closedness and proved that CS which is weakly closed satisfies the convexity condition. This history and more details can be found in Mantegazza [89].

The author in [125] considered proximal sets which are related to Chebyshev sets and gave the result below.

**Theorem 2.8.** *Let  $W$  be a NLS. Consider  $J$  as a proximal set of  $W$ . Then  $J$  is nonempty and closed.*

Theorem 2.8 considered proximal sets and characterized closedness and the content of the sets. However, it becomes very difficult to come up with a structure of the metric projection function in terms of its geometry. Nonetheless, an exception on this assertion can be considered when  $J$  is a subspace. Several mathematicians have delved into the realm of Chebyshev sets and proximal sets over time. A study by [86] focused on Chebyshev sets and their relationship with proximality. The research established that a set can be considered convex if and only if it satisfies the condition of being finite dimensional. Moreover, the study demonstrated that if a set possesses the closure property, then it can be classified as a Chebyshev set. Furthermore, in the case of convex sets in rotund reflexive spaces, the study revealed that such sets are also Chebyshev. In [114] the author also provided a characterization of  $CS$  (Chebyshev) sets in Euclidean spaces, as depicted in the following theorem:

**Proposition 2.9** ([48], Proposition 6). *Let  $L$  be a nonempty subset of a Euclidean space  $E$ , then the following statements are equivalent:*

- (i).  $L$  is Chebyshev.
- (ii).  $P_L$  is continuous and has at most one value.
- (iii).  $d_L^2$  is Fréchet differentiable a.e and  $\nabla_F \frac{d_L^2}{2} = 1 - P_L$ .
- (iv). The Fréchet sub-differential  $\partial_F (-d_L)^2(w) \neq \emptyset, \forall w \in E$ .

The next result by Borwein [16] follows from the fact that norms of closed sets and weak convergence relate on Hilbert space spheres.

**Theorem 2.10** ([18], Theorem 15). *Let  $H$  be a Hilbert space. Suppose the  $CS$  set  $L$  is weakly closed. If  $L \in H$  then  $L$  is convex.*

Theorem 2.10 was extended to general Banach spaces where it was proved that if a *CS* set  $L$  has a continuous projection in a Banach space  $B$  endowed with a rotund dual norm then  $L$  is a convex set [147].

Ricceri's contribution in [146] consisted of providing necessary conditions for the existence of non-convex CSs in infinite dimensional spaces, which extends the well-known result that closed and convex sets are Chebyshev for all spaces, but only convexity holds for weakly sequentially closed Chebyshev sets in finite dimensional spaces. Ricceri's work focused on the scenario of infinite dimensional spaces and showed that there exist special Chebyshev sets in such spaces that do not satisfy the convexity property as seen in [67].

The conjecture for this problem was formulated as follows:

**Theorem 2.11** ([146], Conjecture 1). *Let  $B$  be infinite dimensional NLS. Then a Borel function  $g : B \rightarrow \mathbb{R}$  that is not convex exists such that given  $t \in \inf_B g$  and  $\eta \in [0, \infty]$  the conditions below are satisfied:*

- (i).  $\sup_{w \in B} \frac{|g(w)|}{1 + \|w\|^2} < \infty$ ;
- (ii).  $\forall x \in B, \forall \beta \in [0, \eta]$  the function  $w \mapsto \|w - x\|^2 + \beta g(w)$  must attain at most one global minimum in  $B$ ,  $w_{x,\beta}^*$  such that the given a Borel map  $x \rightarrow w_{x,\beta}^*$  the following inequality holds  $\|w_{x,\beta}^*\| \leq L_\beta(1 + \|x\|)$  given that  $L_\beta$  does not depend on  $x$ ;
- (iii). Let  $\eta < \infty, \forall x \in g^{-1}[t, \infty]$  then  $w \mapsto \|w - x\|^2 + \beta g(w)$  attains no global minimum in  $B$ ;
- (iv).  $\forall q \in L^2([0, 1], B)$  and  $\int_0^1 g(q(x))dx > t$ , then  $\{p \in L^2([0, 1], B) : \int_0^1 \|p(x) - q(x)\|^2 dx + \beta \int_0^1 g(p(x))dx \leq r\} : \beta \in [0, \eta], r \in \mathbb{R}$  satisfies compactness property.

From this Conjecture the following findings regarding nonconvex CSs in infinite dimensional spaces were drawn for the global minimum (gm):

**Theorem 2.12** ([146], Theorem 1). *Suppose that the Conjecture 2.11 holds true. Let the function  $g$  be such that all the axioms of 2.11 are satisfied. Then,  $\{p \in L^2([0, 1], B) : \int g(p(x))dx \leq r\}$  in a non convex CS set.*

The result in his Theorem 2.12 characterizes in terms of nonconvexity in normed spaces.

**Theorem 2.13** ([75], Theorem 3.2). *Let  $L \neq \emptyset$  be set. Let  $\varrho$  and  $\vartheta$  be two functions mapping  $L$  into  $\mathbb{R}$  such that given  $\beta \in [0, \infty]$ ,  $\varrho + \eta\vartheta$  is a function attaining at most one global minimum if  $\eta \in [0, \beta]$  and fails to have a global minimum if  $\forall \beta < \infty, \eta = \beta$ . Furthermore, Suppose  $w_0$  is the unique global minimum for  $\varrho$  and let  $\inf_L \vartheta < \vartheta(w_0)$ . Then, the family  $\{\{w \in L : \varrho(w) + \eta\vartheta(w) \leq r\} : \eta \in [0, \beta], r \in \mathbb{R}\}$  satisfies compactness property. Moreover,  $\forall r \in [\inf_L \vartheta < \vartheta(w_0)]$  the restriction of  $\varrho$  to  $\vartheta^{-1}(r)$  attains at most one global minimum.*

In the case of function attaining norms have global minima as shown in Theorem 2.13.

**Theorem 2.14** ([51], Theorem 4.1). *Let the Borel function  $g : L \rightarrow \mathbb{R}$  be such that for the global minimum,  $\sup_{w \in L} \frac{|g(w)|}{1+\|w\|^2} < \infty$ . If there exists  $r \in [\inf_L g, \sup_L g]$ , then the set  $\{p \in L^2([0, 1], B) : \int g(p(x))dx \leq r\}$  is weakly closed. Hence,  $g$  is convex.*

In [146] the author has given the conditions necessary for CS sets to exist in infinite dimensional spaces. These CS sets were proved to be

nonconvex as illustrated in the three theorems above.

Wang [148] conducted a comprehensive investigation into the properties of Chebyshev functions (CF) and Klee functions. In particular, the lower semi-continuity property of functions was utilized to define proximal mapping and farthest point mapping. The research demonstrated that if a farthest point mapping is single-valued almost everywhere, then it can be classified as a Chebyshev function. Furthermore, the study established that any function which is bounded below and single-valued must be convex. Conversely, it was shown that a function is strongly convex if and only if it is single-valued. These results were further supported with a concrete example, which highlighted that a continuous farthest point mapping does not necessarily have to be locally Lipschitz. As a result, the investigation of the single-valuedness property of farthest point mapping and proximal mapping was approached from a global perspective, rather than a local one.

**Lemma 2.15.** ([94]) *Let  $g : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$  be a proper lower semi-continuous (lsc) function which is essentially strictly convex. Then argming is a singleton set or empty.*

Lemma 2.15 characterize single-valuedness of proximal functions by use of lower semi-continuity.

**Lemma 2.16.** ([92]) *If a proximal function  $g : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$  is not restricted to be convex, then  $e_\eta g$  and  $h_\eta g$  satisfy  $\eta^{-1}j(q) - e_\eta g(q) = (g + \eta^{-1}j)\eta^{-1}q$  and  $h_\eta g(q) = (g + \eta^{-1}j)^{**}(q) - \eta^{-1}j(q)$ . Given that  $g$  is proper, prox-bounded with  $\eta_g$  threshold and lsc. Then  $\forall \eta \in (0, \eta_g)$  and  $\forall q \in \mathbb{R}^n$ ,  $\partial(g + \eta^{-1}j)^*(q) = \text{conv}P_\eta g(\eta q)$   $\partial h(q) = \text{conv}P_\eta h(q)$ , where  $h = \eta(g + \eta^{-1}j)^*(\eta^{-1})$ .*

The main contribution of this study is the identification of the necessary conditions for a function  $g$  such that  $P_\eta g$  and  $Q_\beta g$  are single-valued almost everywhere on  $\mathbb{R}^n$ . This finding represents the cornerstone of the research, providing valuable characterizations and properties of proximal and farthest point mappings in relation to their single-valuedness properties.

In their research paper [144], Narang and Sangeeta conducted a study on the properties of Chebyshev sets in metric spaces with a focus on convex metric spaces. The study resulted in several interesting findings. Firstly, they proved that a Chebyshev set  $L$  in a metric space  $(B, d)$  with a lower semi-continuous metric projection is a convex  $\delta$ -sun. This means that if the metric projection of  $L$  onto a closed convex set is lower semi-continuous, then  $L$  can be characterized as a convex  $\delta$ -sun. A convex  $\delta$ -sun is a set that has a special geometric structure, where every point in the set can be connected to a "sun" point outside the set by a path that stays within a certain distance  $\delta$  from the set. This result provides geometric properties of Chebyshev sets in metric spaces with lower semi-continuous metric projections.

Furthermore, Narang and Sangeeta's [125] study also showed that if the metric projection of  $L$  is continuous, then  $L$  is a convex  $\gamma$ -sun. This implies that if the metric projection of  $L$  onto a closed convex set is continuous, then  $L$  can be characterized as a convex  $\gamma$ -sun. A convex  $\gamma$ -sun is a more refined version of a convex  $\delta$ -sun, where the distance from the set to the "sun" points is controlled by a continuous function  $\gamma$ . This finding provides additional properties of Chebyshev sets in metric spaces with continuous metric projections.

In a nutshell, Narang and Sangeeta's study [144] sheds characterized Chebyshev sets in convex metric spaces, particularly in relation to the continuity of the metric projection. These findings contribute to the understanding of the geometric structure of Chebyshev sets and their relationship with the metric properties of the underlying metric space.

In their work [61], Fletcher and Moors conducted an exhaustive investigation on Chebyshev sets, coining key conditions that determine when a normed linear space can be considered a Chebyshev set. One of the significant findings of Fletcher and Moors was the proof that any nonempty Chebyshev set in a normed linear space must be closed. This result sheds light on the topological properties of Chebyshev sets, establishing that closure is one of the requirements for these sets to exist. It was shown that a closed set in a normed linear space can only be convex if the set is midpoint convex. This highlighted the connection between convexity and Chebyshev sets, elucidating that convexity can only be preserved if the set exhibits midpoint convexity. Hence the study provided insights into the geometric structure of Chebyshev sets and the conditions necessary for convexity to hold. The authors also investigated special classes of maps acting as metric projections on Chebyshev sets. Additionally conditions necessary for convexity to hold in Chebyshev sets were given. The authors further constructed examples of Chebyshev sets that are concave in inner product spaces. The results below represent Fletcher and Moors's work on Chebyshev sets:

**Lemma 2.17** ([61], Lemma 2.22). *Let  $(W, \|\cdot\|)$  be a normed linear space. If  $L \subset W$  such that  $L \neq \emptyset$ , then the metric projection  $w \rightarrow P_L(w)$  is a locally bounded map on  $W$ .*

This Lemma 2.17 asserting that all metric projections on  $CS$  sets in  $NLS$  are locally bounded is used together with the property of boundedly compactness of sets to show that to that if a set is boundedly compact then it is proximal and therefore closed.

**Theorem 2.18** ([61], Theorem 2.26). *et  $(W, \|\cdot\|)$  be a normed linear space. Let  $L \subset W$  be boundedly compact. If there is  $w \in W$  for which  $P_L(w)$  is singleton, then the metric projection  $u \rightarrow P_L(u)$  is continuous at  $w$ .*

Theorem 2.18 is an illustration of the conditions necessary for a metric projection mapping to be continuous.

**Corollary 2.19** ([61], Corollary 2.27). *Let  $(W, \|\cdot\|)$  be a finite dimensional normed linear space. If  $L \subset W$  is a  $CS$  set then the metric projection  $w \rightarrow P_L(w), \forall w \in W$  is continuous.*

In their research [61] Fletcher and Moors made significant contributions to the study of Chebyshev sets, providing insights into their topological properties, convexity conditions, and examples of their structures in  $NLS$  and inner product spaces. To prove the converse of the assertion that all proximal or Chebyshev sets are closed, the study showed that all closet sets which are nonempty in finite-dimensional Banach space is proximal and so Chebyshev. A counter example was given showing that not all closed convex sets are proximal.

The research paper published in reference [135] provides significant insights into the characteristics of Chebyshev sets and proximal sets in complete normed linear spaces. The authors presented key characterizations of these concepts and established conditions for their convexity.

Moreover, the study investigated the conditions under which the continuity of metric projections is ensured for both Chebyshev sets and proximal sets.

The researchers' findings shed light on important results that emerged from their study. Firstly, they identified crucial characterizations of Chebyshev sets and proximal sets in the context of complete normed linear spaces. These characterizations provide a deeper understanding of the behavior of these sets in such spaces and contribute to the existing literature on the topic.

Furthermore, the researchers established conditions for the convexity of Chebyshev sets and proximal sets. This is a significant contribution as convexity is an important property in many mathematical and optimization problems. The identified conditions provide insights into the geometric structure of these sets in complete normed linear spaces, and they can be used in various practical applications. Additionally, the study examined the continuity of metric projections for Chebyshev sets and proximal sets. Metric projections are commonly used in optimization and approximation problems, and the continuity of these projections is crucial for their effective use. The researchers' findings on the conditions under which the continuity of metric projections is guaranteed for Chebyshev sets and proximal sets provide valuable insights into the behavior of these sets in complete normed linear spaces. The following are some of the main results obtained in this study:

**Theorem 2.20.** *Let  $B$  a strictly convex Banach space. Let  $A$  be a finite dimensional subspace of the Banach space  $B$ . Then  $A$  is Chebyshev.*

Theorem 2.20 demonstrates a significant result [150]: finite dimensional

subspaces of strictly convex Banach spaces are not only closed sets, but also Chebyshev sets. This observation is crucial because the convexity property of these Banach spaces ensures the closure of sets within them. Moreover, it has been previously established that closed sets in strictly convex or reflexive Banach spaces are Chebyshev sets. Therefore, Theorem 2.20 provides valuable insights into the conditions under which a closed set can be a Chebyshev set. The concept of Chebyshev sets, which are sets that admit nearest point solutions, plays a fundamental role in optimization and approximation theory. The closure of sets in strictly convex Banach spaces is a crucial property that ensures the existence of nearest point solutions within these sets. The convexity of the Banach spaces guarantees that closed sets are convex, and the strictly convex property further guarantees their strict convexity. These properties are essential for establishing the existence of unique nearest point solutions in Chebyshev sets.

**Theorem 2.21** ([142], Theorem 2.2). *Let  $B$  be a Banach space and let  $L$  be a proximal set in  $B$ . If  $\widehat{L}$  is given to be convex, then  $L$  is a Chebyshev set.*

Theorem 2.21 above provides a crucial characterization of Chebyshev sets by leveraging the concept of metric complement. In this particular case, the metric complement of the set  $L$ , denoted as  $\widehat{L}$ , can be explicitly defined as follows:  $\widehat{L} = \{(p, q) : (p \geq 0 \text{ and } q \leq 0) \text{ or } (p \leq 0 \text{ and } q \geq 0)\}$ , where  $L \cap \widehat{L} = \{0\}$ , meaning that the only point of intersection between  $L$  and  $\widehat{L}$  is the origin  $(0, 0)$ , and  $\widehat{L}$  is a closed set. Furthermore, the theorem establishes that  $L$  and  $\widehat{L}$  do not share any points other than the origin, which implies that they are disjoint sets. Additionally,  $\widehat{L}$  is closed, meaning that it

contains all of its boundary points, and hence its closure is equal to itself.

**Theorem 2.22** ([57], Theorem 2.3). *Let  $B$  be a strongly convex Banach space and the convex closed set  $L$  be in  $B$ . The conditions below are equivalent:*

- (i).  *$L$  is a proximal set.*
- (ii).  *$L$  is weakly approximately compact.*
- (iii).  *$L$  is approximately compact.*

Theorem 2.22 examines the concept of approximative compactness and proximality of sets. A Banach space is shown to be approximatively compact if it satisfies reflexivity. This theorem is generalized to show that if a set satisfies the approximative compactness property in Banach spaces then it is proximal. It was evident that in as much as approximative compactness implies proximality as shown by Theorem 2.22 the consequent cannot imply the hypothesis.

**Theorem 2.23.** ([105] Theore4.3) *Let  $B$  be a Banach space and  $L$  be closed in  $B$  such that  $\|B\|$  is uniformly Gateaux differentiable and also Frechet differentiable. Let the dual of  $B$  ( $B^*$ ) be Frechet differentiable. The conditions below are equivalent for  $q \in B \setminus L$ :*

- (i).  *$d_L$  is Frechet differentiable at  $q$ .*
- (ii).  *$P_L$  is continuous at  $q$ .*
- (iii). *Any minimizing sequence in  $q \in L$  converges.*

Narang [105] conducted research on the relationship between Chebyshev sets and proximal sets in metric spaces, as documented in [143]. The research showed that sets in metric spaces that are strongly proximal are also Chebyshev sets, and furthermore, their metric projections are continuous. It was also proven that if sets in Hilbert spaces are strongly proximal, then they must be convex. The next Lemma presents a relationship between  $CS$  sets and proximal sets.

**Lemma 2.24** ([143], Lemma 1.1). *Let  $(Y, d)$  be a metric space. If a set  $L$  is strongly proximal in  $Y$ , then  $L$  is Chebyshev in  $Y$ .*

This relationship between Chebyshev sets and proximal sets holds true for all metric spaces and metrizable topological spaces, as stated in 2.24.

**Theorem 2.25.** *Let  $(Y, d)$  be a metric space. If a set  $L \neq \emptyset$  is strongly proximal in  $Y$ , then the function  $P_L$  mapping the nearest point of  $L$  is continuous.*

The theorem immediately above below illustrates continuity property of  $CS$  sets projection maps in metric spaces. However, it doesn't show the work in other Banach spaces.

The following theorem characterizes proximal sets in Hilbert spaces.

**Theorem 2.26** ([143], Theorem 1.3). *Let  $H$  denote a Hilbert space. Is  $L$  is strongly proximal in the space  $H$ , then  $L$  is a convex set.*

Narang [143] showed that in metric spaces Chebyshev sets coincide with strongly proximal sets and their metric projection maps are continuous.

the study also proved that for Hilbert spaces all strongly proximal sets are convex.

## 2.4 Distance Functions of Chebyshev Sets

Distance functions are also important in characterizing CSs. By the result of Asplund [4], the CS problem was given a different dimension to consider metric projections as shown in the next proposition.

**Proposition 2.27.** (*[61], Proposition 2.2*) *Let  $W$  be a NLS. Consider  $J$  as a CS of  $W$ . Then the distance function for  $J$  satisfies nonexpansivity and continuity conditions.*

Proposition 2.27 describes CSs and their distance functions in terms of continuity and nonexpansivity in a general set up. The studies referenced in [28], [147], and [149] investigated distance functions of Chebyshev sets in various spaces, with a particular focus on normed linear spaces.

One key result in [139] is that a Chebyshev set  $L$  in a Banach space with a rotund dual is convex if and only if its distance function  $d_L$  is regular on  $L^c$  and strictly Gateaux differentiable on  $L^c$ . This implies that the differentiability of the distance function is closely related to the convexity of the Chebyshev set. Another result from [46] is that in a reflexive Banach space  $B$  with a smooth Kadec norm, a set  $L$  is weakly closed if the projection operator  $P_L$  is continuous. This result provides a characterization of weakly closed Chebyshev sets in reflexive Banach spaces with smooth Kadec norms.

Furthermore, [45] proposed that if the norm and dual norm of the Banach

space  $B$  are Frechet differentiable, then a set  $L$  is convex if the distance function  $d_L$  is Frechet differentiable on  $L^c$ , the complement of  $L$ . This provides another condition for the convexity of Chebyshev sets in Banach spaces with Frechet differentiable norms and dual norms. The following represent the findings obtained:

**Theorem 2.28** ([58], Theorem 1.1). *Let  $B$  be a Banach space endowed with a rotund dual  $B^*$ , then  $L \neq \emptyset$  is convex if the distance function  $d_L$  satisfies the equality below:  $D^+d_L(q) = \limsup_{\|p\| \rightarrow 0} \frac{d_L(q+p) - d_L(q)}{\|p\|} = 1, \forall q \in B \setminus L$ .*

Theorem 2.28 presents sufficient conditions under which a  $CS$  is convex. The theorem is applicable in reflexive Banach spaces, which are mathematical spaces that possess certain properties, including the property of reflexivity [126]. Furthermore, the theorem requires that the Banach spaces admit a smooth Kadec norm [132], which is a specific type of norm with particular smoothness properties. The conditions for weakly closed sets are also implied in Theorem 2.28. The theorem outlines the necessary conditions that such sets must satisfy in order for the convexity of  $CS$  to hold [135].

Overall, the studies in [35], [94], [99] and [129] contribute to the understanding of the properties of Chebyshev sets in normed linear spaces, and provide conditions under which these sets are convex based on the Gateaux differentiability and Frechet differentiability of the associated distance functions in various normed linear spaces.

In the study conducted by the authors in reference [107], the focus was on investigating the concept of Chebyshev sets in relation to Bregman distances. Bregman distances were found [122] to be a useful tool in es-

establishing the convexity of sets [88], with a key condition being that each point in the Euclidean space is closest to only one point in a closed set (see [82]-[87]). Furthermore, the authors delved into characterizing sub-differentiable functions and their properties in relation to Bregman nearest distance. This involved utilizing nonsmooth analytic methods and employing maximal operators to form monotone sequences (see [93]-[96]), which were key elements of the methodology used in the study. Bregman distances were found to be effective in proving the convexity of sets, provided that each point in the Euclidean space is nearest to only one point in a closed set. The study also provided insights into the properties of sub-differentiable functions in relation to Bregman nearest distance, shedding light on the behavior of these functions in the context of Bregman distances and Chebyshev sets. The following findings are some of the study's main results:

**Proposition 2.29** ([113], Proposition 5.1). *Let a function  $g$  be twice continuously differentiable on a Euclidean space  $W$ . then the left Bregman distance function  $\overleftarrow{d}_L$  is locally Lipschitz on  $W$  and  $\overleftarrow{d}_L = g^* \circ \nabla g - (g + l_L)^* \circ \nabla g = [g^* - (g + l_L)^*] \circ \nabla g$ .*

Proposition 2.29 above shows that if a function is twice differentiable in the Euclidean space then it is locally Lipschitz in that space [89].

**Lemma 2.30** ([69], Lemma 6.2). *Let  $w \in \mathbb{R}^k$ . Then we have,  $\partial(g + l_L)(w) = \{r \in \mathbb{R}^k : w \in \overleftarrow{P}_L(\nabla g^*(r))\} = (\overleftarrow{P}_L \circ \nabla g^*)^{-1}(w)$ , and it follows that  $\partial(g + l_L) = (\overleftarrow{P}_L \circ \nabla g^*)^{-1}$ .*

Lemma 2.30 gives a characterization of *CS* sets in relation to Bregman

distances. Overall, this study contributed to the understanding of Chebyshev sets, Bregman distances, and sub-differentiable functions, and provided important insights into their relationships and properties [26]. In their research paper [147], the authors conducted an extensive investigation on the concept of Chebyshev centers that are not necessarily farthest points in Banach spaces. They made a significant discovery that the existence of such centers, referred to as (*CSCNFP*), depends on the convexity properties of the Banach space, for a nonempty bounded subset. The authors provided a comprehensive characterization of *CS* (Chebyshev center) centers and farthest points in two-dimensional real convex spaces. Also authors in [143]-[144] explored properties such as centerability and M-compactness in uniformly convex Banach spaces, shedding light on their significance in the context of Chebyshev centers. Concrete examples were presented to substantiate the optimality of the characterizations proposed in the study [108].

Furthermore, the authors in [64] -[67] proved that in a finite-dimensional Banach space  $\chi$ , convexity can only be achieved when all its nontrivial subsets are *CSCNFPs*. The property of M-compactness played a crucial role in establishing that in a strictly convex Banach space  $\chi$ , if every nonempty subset of  $\chi$  is M-compact [74], bounded [77], and centerable [78], then it is a *CSCNFP*. The research also delved into the case of  $L^p$ -spaces that are non-Hilbert, where  $p \neq 2$ . The authors demonstrated that every subset of a non-Hilbert  $L^p$ -space admits a Chebyshev center that is a farthest point (see [109]-[112]), denoted as *CSCFP*. This result further expanded the understanding of Chebyshev centers in various types of Banach spaces (see [115]-[123] and the references therein). The

following findings represent the main results:

**Lemma 2.31** ([125], Proposition 1.1). *Let  $L \neq \emptyset$  be a bounded subset of a Banach space  $\chi$  with a CS center  $CS_L$ . Assume that  $CS_L$  is also a farthest point of  $L$  from some point  $w \in \chi$ . Denote CS radius of  $L$  by  $r$  and let  $D = \|CS_L - w\|$ . Then  $r \leq D$  and a subset  $V = B[CS_L, r] \cap B[w, D]$  satisfies the following properties:*

- (i).  $L \subseteq V$
- (ii). The  $CS_L$  of  $V$  is equal to  $r$
- (iii).  $CS_L$  is a center of  $V$
- (iv).  $CS_L$  is a farthest point of  $V$  from  $w$

The Lemma 2.31 provides a relationship between the CS center  $CS_L$  of a bounded subset  $L$  in a Banach space and the distance between  $CS_L$  and a given point  $w \in \chi$ . It also introduces the concept of CS radius and a subset  $V$  that satisfies various properties [130]. This lemma can be useful in analyzing the properties of bounded sets in Banach spaces and their relationship with CS centers and farthest points [134].

**Theorem 2.32** ([137], Proposition 4.4). *Let  $\xi$  be a Banach space. Let  $L_{t,w} = B_\chi \cap B[t, w]$  be a set with radius  $r_{t,z}$ . Then the following conditions are equivalent in Banach spaces:*

- (i).  $\chi$  is a CSCNFP space.
- (ii). For each  $w$  in the open sphere of  $\chi$  and for each  $0 < t \leq 1$  the inequality  $r_{t,w} < t$  holds.

(iii).  $CS_L$  is a center of  $V$

(iv). For each  $\xi \in (0, 1]$ ,  $\exists t_0 \in (0, \xi)$  satisfying  $r_{t,w} < t, \forall w \in S_\xi, \forall t \in (0, t_0)$ .

This Theorem establishes a connection between the properties of a Banach space  $\chi$  and the behavior of the radius of certain sets  $L_{t,w}$  with respect to  $t$  and  $w$ . It characterizes the conditions under which a Banach space is a *CSCNFP* space, which has implications in the study of stability and farthest point properties in functional analysis[133].

**Theorem 2.33** ([14], Proposition 2.4). *Let  $\chi$  be a real Banach space. If  $L \neq \emptyset$  is a bounded CS subset of  $\chi$  which admits its CS center  $CS_L$ . Then  $L$  is CSCNFP.*

Theorem provides a result about the properties of bounded sets in real Banach spaces that have CS centers. It asserts that such sets satisfy the *CSCNFP* property, which can be relevant in the study of stability and farthest point properties in real Banach space as seen also in [136]-[141].

**Theorem 2.34** ([34], Proposition 5.4). *Suppose  $(\chi, \sigma, \mu)$  is a finite measure space with a disjoint triple  $\{\Lambda_i\}_{i=1}^3 \subset \sigma$  of finite positive measure subsets. Then  $\forall p \in [1, 2) \cup (2, \infty)$ ,  $L^p = L^p(\chi, \sigma, \mu)$  is a CSCFP space.*

Theorem establishes the property of *CS* with respect to farthest points for the  $L^p$  spaces associated with a finite measure space with certain conditions on the measure subsets. It provides insights into the properties of  $L^p$  spaces with respect to stability and farthest point properties, which can be relevant in functional analysis and measure theory [47].

Findings presented in [147] shed new light on the intricate properties and characteristics of Chebyshev centers in Banach spaces. These findings have significant implications for various aspects of convexity properties, M-compactness, and different types of Banach spaces, including finite-dimensional and  $L^p$ -spaces. Valuable characterization of convexity properties of Chebyshev centers in Banach spaces is provided. The research states the conditions under which Chebyshev centers exist in convex sets, and provides insights into the geometric and topological properties of these centers, shedding light on their relationship with convexity. Furthermore, the connection between Chebyshev centers and M-compactness in Banach spaces is illustrated.

## 2.5 Chebyshev sets and convexity

This section reviews literature on the key question of this study, that is, Is every CS in HS convex? We begin with the following result Borwein [16] on closedness, reflexivity and rotundity of Hilbert spaces.

**Lemma 2.35.** *([16], Fact 3) All Chebyshev sets are closed and all closed sets satisfying convexity condition are Chebyshev in a rotund reflexive space. Particularly, all nonempty closed sets satisfying convexity condition in Hilbert space is Chebyshev.*

Lemma 2.35 gives an elaborate characterization of Chebyshev sets and convexity in terms of reflexivity and rotundity that requires uniqueness property however this does not answer question 1 in general. Numerous mathematicians have conducted extensive research on the convexity of

Chebyshev sets in various spaces, yielding interesting findings. It has been observed that  $CS$  sets are convex in smooth finite-dimensional  $NLS$ , as well as in finite-dimensional Hilbert spaces. Furthermore, in uniformly Banach spaces, compact  $CS$  sets are convex. In the case of strictly convex reflexive Banach spaces, it is well-known that  $CS$  sets are convex if a closure condition for the sets is satisfied, and this result extends to Hilbert spaces as well. Additionally, in strongly smooth spaces, all  $CS$  sets that have a continuous metric projection are convex.

In a recent study by [77], the conditions for convexity of  $CS$  sets in Hilbert spaces were investigated. The open problem of whether  $CS$  sets in strictly convex reflexive Banach spaces, particularly Hilbert spaces, need to be convex was addressed, and necessary conditions for the convexity of Hilbert space  $CS$  sets were presented, as shown in the following results:

**Theorem 2.36** ([76], Theorem 3.1). *Let  $B$  be a reflexive space endowed with a Kadec norm. If the set  $L$  is weakly closed in the space  $B$ , then  $P_L$  is a continuous metric projection.*

**Corollary 2.37** ([77], Corollary 3.6). *If  $L$  is a weakly closed  $CS$  set in a Hilbert set, then  $L$  is convex.*

These results provide insights into the convexity of  $CS$  sets in various spaces and contribute to the understanding and extension of the convexity notion of  $CS$  sets.

In the work by [120], the authors extended the conditions for convexity of Chebyshev sets in Hilbert spaces. They demonstrated that an approximately compact Chebyshev set in a Hilbert space is convex, which is a significant result. Moreover, the study established the classical result that

every Chebyshev set in a smooth finite-dimensional normed linear space is convex. Additionally, the authors showed that boundedly compact sets that are Chebyshev in complete normed linear spaces are also convex. Furthermore, the study revealed that all Chebyshev sets in a strongly smooth Banach space are convex, which provides further insights into the convexity properties of Chebyshev sets in different functional spaces. Interestingly, the authors also proved that in strictly reflexive complete normed linear spaces, every closed set is convex, highlighting the connection between reflexivity and convexity. The study presented several key findings, including the following theorems:

**Theorem 2.38** ([52], Theorem 2.6). *Let  $H$  be a Hilbert space. Then  $L \neq \emptyset$  in  $H$  is uniformly convex.*

**Theorem 2.39.** *Let  $B$  be a Banach space. If  $B$  is uniformly convex, then  $B$  is reflexive. Let  $L \neq \emptyset$  be in  $B$ , then  $\|L\|$  is a Kadec norm.*

These two theorems contribute to the understanding of the convexity properties of Chebyshev sets in different functional spaces and provide valuable insights into the relationship between uniform convexity, reflexivity, and Kadec norms in Banach spaces. The findings of this study have important implications in functional analysis and related areas of mathematics. In their paper [8], the authors established conditions that determine the convexity of a Chebyshev set in a Banach space using the concept of Gauteaux differentiability of the distance function. The main result of their study can be stated as follows:

**Theorem 2.40.** *Let  $B$  be normed linear space and  $B^*$  be its strictly convex dual. Suppose that  $L \subseteq B$  is a CS set and  $\partial d_L(w)$  is singleton for*

each  $w \in B \setminus L$ . The following statements are equivalent:

(i). The set  $L$  is convex.

(ii). The distance function  $d_L$  is convex.

(iii).  $d_L$  is Gateaux differentiable at  $w$ .

(iv).  $\exists y \in B$  such that  $\lim_{r \rightarrow 0^+} \frac{d_L(w+ry) - d_L(w)}{r} = 1$ .

(v).  $\limsup_{\|z\|} \frac{d_L(w+z) - d_L(w)}{\|z\|} = 1$ .

The authors' result establishes the equivalence between the convexity of the set  $L$ , the convexity and Gateaux differentiability of the distance function  $d_L$ , and the existence of a certain limit in (iv) and (v). This result provides insight into the geometric and differentiable properties of Chebyshev sets in Banach spaces, and sheds light on the relationship between convexity and differentiability in this context. In [142], the author, Moors, conducted a comprehensive quantitative analysis on the convexity of Chebyshev sets in relation to remotal sets. The study aimed to establish a correlation between the magnitude of the nearest point sets and the convexity measure of the underlying set. The findings revealed that unique remotal sets are directly related to non-convex Chebyshev sets, as it was proven that non-convex Chebyshev sets can only exist in Hilbert spaces if a non-singleton set exists that is uniquely remotal.

Furthermore, the study presented necessary conditions for  $\varepsilon$ -Chebyshev sets to be  $\varepsilon$ -convex. This provided valuable insights into the geometric properties of  $\varepsilon$ -CS sets. Additionally, open problems involving  $\varepsilon$ -uniquely remotal sets and  $\varepsilon$ -Chebyshev sets were addressed and solved, with examples provided to illustrate the results. Some of the notable findings from

the study include novel results on the relationship between remotal sets and Chebyshev sets, shedding light on the convexity properties of these sets in different contexts. The study also contributed to the understanding of necessary conditions for  $\varepsilon$ -convexity of  $\varepsilon$ -CS sets, providing valuable information for further research in this area. The examples provided in the study helped to illustrate the theoretical findings and highlight their practical implications in various applications. Some of the results obtained include:

**Proposition 2.41** ([142], Proposition 3). *Let  $L$  be a non singleton remotal set in a NLS  $B$ . Let  $p \in B$  and  $q \in P_L(p)$ . If  $\forall \eta \in (0, 1), p_\eta = p + \eta(q - p)$ , then  $\frac{r(p_\eta, L) - r(p, L)}{\|p_\eta - p\|} \leq (1 - \frac{\|q - p_\eta\|}{r(p, L)})$  with  $q_\eta \in P_L(p_\eta), \forall \eta \in (0, 1)$ .*

Proposition 2.41 establishes a relationship between a radial function and the farthest point mapping. The proof of this theorem relies heavily on the concept of upper semi-continuity. The theorem provides a characterization of radial functions in relation to the farthest point mappings, which is a mapping that associates each point in a set with its farthest point in another set. The theorem highlights the crucial role of metric upper semi-continuity, which is a mathematical concept that deals with the continuity of functions from above, in establishing this relationship. The proof of this theorem, as presented in [142], demonstrates the significance of metric upper semi-continuity in establishing the desired relationship between the radial function and the farthest point mapping.

The study [113] presents an extensive investigation of convexity properties of CS sets in NLS. The focus of the study is on the farthest and nearest points in Hilbert spaces as well as general inner product spaces. Moreover, the authors also explore special classes of sets categorized as "suns" and

”sunrises” in normed linear spaces. One of the key results of the study is the proof that for a normed linear space  $B$  with  $L = \{q \in B : |q| = 1\}$  and  $p \in B$ , the set  $L$  is both a sun and a sunrise set [130]. This finding demonstrates a duality in the convexity properties of  $L$  in relation to the concepts of sun and sunrise sets. Furthermore, the study establishes that if the convex set  $L$  in a normed linear space  $B$  is remotal and unique, then  $L$  is a sunrise set. This result sheds light on the relationship between remotality, uniqueness, and convexity in normed linear spaces, providing valuable insights into the geometric properties of convex sets [124]. Additionally, the study proves that every  $CS$  set in a complete inner product space is convex, indicating a generalization of the convexity property for  $CS$  sets in this particular setting. Moreover, the continuity of the associated proximity maps is verified, highlighting the topological properties of  $CS$  sets in inner product spaces [83]-[85]. The results below are among the study’s main findings:

**Theorem 2.42.** ([93], Theorem 2.4) *Let the convex set  $L \neq \emptyset$  be remotal in a normed linear space  $(B, \|\cdot\|)$ . Given that  $p \in B$  and  $q \in L$ , if  $r = \eta p + (1 - \eta)q$  and  $q = F_L(p)$ . Then  $\|r - q\| \geq \|r - (\eta s + (1 - \eta)q)\|, \forall s \in L$ .*

**Theorem 2.43.** ([106], Theorem 2.5) *Let  $(B, \langle \cdot, \cdot \rangle)$  be an inner product space and  $L$  be a non-empty subset of  $B$ . Then for each  $p \in B$  and  $q \in L$ ,  $q \in P_L(p)$  if and only if  $\langle p - \frac{q+r}{2}, r - q \rangle \leq 0$  for any  $r \in L$ .*

The two theorems discussed here relate to the convexity of Chebyshev sets in normed linear spaces and inner product spaces, specifically in the context of best approximation points and worst approximation points [95]. These theorems highlight the properties of remotality, uniqueness, and

convexity that characterize Chebyshev subsets of these spaces. Furthermore, the properties of remotality, uniqueness, and convexity guarantee that a Chebyshev subset  $L$  of a normed linear space is a sunrise set. A sunrise set is a set that does not have any "holes" or "gaps" in it, and it extends towards the boundary of the normed linear space in a convex and unique manner. In other words,  $L$  fills the space between itself and the boundary of the normed linear space in a convex and remote manner, without any overlaps or repetitions [140].

In summary, the findings in [143] establish the relationship between convexity and Chebyshev sets with respect to best approximation points and worst approximation points in normed linear spaces and inner product spaces, and highlight the important properties of remotality [86], uniqueness [94], and convexity that characterize Chebyshev subsets in these spaces [104].

The authors in [115]-[120] investigated convexity of  $CS$  sets in finite dimensional Euclidean spaces. The Bunt-Motzkin theorem was proved in this study. Induction methods and hyperplanes were used to prove that each  $CS$  set in finite dimensional Euclidean spaces is convex. The concept of orthogonality was also applied by repeated use of pythagorean theorem. It was shown that the intersection between a Chebyshev set and a hyperplane of a finite dimensional affine subspace results in a Chebyshev set. Thus by applying mathematical induction on the dimension of this set convexity of the Chebyshev set is deductively proved.

The study attempted to solve the longstanding open problem that seeks a proof of the theorem that asserts that a Chebyshev set in an infinite dimensional Hilbert space is convex. One of the pivotal results obtained

in their study, denoted as Lemma 2.44 below, showed that for a hyperplane  $H$  intersecting a  $CS$  set  $L$ , any point  $w_0$  in  $H$  has a unique nearest point in the intersection of  $L$  and  $H$ . This implies that every point in  $H$  contains a unique nearest point in the set  $L \cap H$ , rendering  $L \cap H$  a Chebyshev set with respect to the affine subspace  $H$ .

**Lemma 2.44.** ([144], Lemma 4] *Let  $H = H(x, l)$  denote an hyperplane intersecting a  $CS$  set  $L$ . Let  $w_0 \in H$  and  $r \in \mathbb{R}$ , then  $P_{L \cap H}(w_0 + rl) = P_{L \cap H}(w_0)$ .*

Lemma 2.44 means that  $P_{L \cap H}(w_0) = P_{L \cap H}(w_0 + rl)$  implying that  $w_0$  has a unique nearest point in the intersection of a Chebyshev set  $L$  and a hyperplane of a finite dimensional affine subspace  $H$ . Therefore every point in  $H$  contains a unique nearest point in the set  $L \cap H$ . Hence the set  $L \cap H$  is a Chebyshev set with respect to the affine subspace  $H$  and by induction it follows that  $L \cap H$  is convex.

**Lemma 2.45** ([94], Lemma 5). *Let  $H = H(x, l)$  denote an hyperplane intersecting a  $CS$  set  $L$ . Let  $w_0 \in H$  and  $r \in \mathbb{R}$ , then the function  $g : \mathbb{R} \rightarrow \mathbb{R}$  given by  $g(r) = \langle P_L(w_0 + rl), l \rangle$  is continuous.*

Lemma 2.45 makes use of the intermediate mean value theorem to show that  $g$  is continuous. That is,  $\exists w_0 \in [m, M]$  such that  $g(w_0) = l$  or equivalently in this case  $P_L(w_0 + rl) \in H$ .

Finally, their seminal work culminated in Theorem 2.46, which unequivocally affirmed that any Chebyshev set  $L$  in  $n$ -dimensional Euclidean space  $\mathbb{R}^n$  is convex. This result not only resolved the long-standing open problem of convexity in Chebyshev sets, but also provided crucial insights into

the structure and properties of  $CS$  sets in finite dimensional Euclidean spaces. The theorem is formally stated as below:

**Theorem 2.46.** (*[118], Theorem 8*) *Suppose  $L \subseteq \mathbb{R}^n$  is a Chebyshev set. Then  $L$  is convex.*

In summary, the works of [89] significantly advanced the understanding of convexity in  $CS$  sets, providing rigorous proofs and novel insights into the convexity of Chebyshev sets in both finite and infinite dimensional spaces. Their findings have far-reaching implications in various fields, including optimization, approximation theory, and convex analysis, and have paved the way for further research in this area.

# Chapter 3

## RESEARCH METHODOLOGY

### 3.1 Introduction

We outline and discuss in details the fundamental principles like the Bunt-Motzkin Theorem, Frèchet and Gateaux differentiability conditions, best approximation techniques that are useful in obtaining the results.

### 3.2 Fundamental principles

Fundamental principles like Ekeland's principle, Hahn-Banach theorem and Chauder's fixed point theorem are very instrumental in construction of results. There are several principles which are useful to this study. We utilize them in Chapter 4.

**Theorem 3.1.** (*Ekeland's Principle [72]*) For any real number  $\epsilon > 0$  and any lower semicontinuous real-valued function  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  such that there exists  $x_0 \in \mathbb{R}^n$  satisfying  $f(x_0) < \inf_{x \in \mathbb{R}^n} f(x) + \epsilon$ , there exists  $x^* \in \mathbb{R}^n$  such that  $f(x^*) \leq f(x_0)$  and  $\|x^* - x_0\| < \frac{1}{\epsilon} (f(x_0) - \inf_{x \in \mathbb{R}^n} f(x))$ .

Next we state the Hahn-Banach Theorem as follows:

**Theorem 3.2.** (*Hahn-Banach Theorem [7]*) Let  $X$  be a normed vector space over the field  $\mathbb{R}$  or  $\mathbb{C}$ . Let  $Y$  be a subspace of  $X$ , and let  $f : Y \rightarrow \mathbb{R}$  (or  $f : Y \rightarrow \mathbb{C}$ ) be a linear functional. If  $f$  is bounded on  $Y$  and satisfies  $f(x) \leq p(x)$  for all  $x \in Y$ , where  $p : X \rightarrow \mathbb{R}$  (or  $p : X \rightarrow \mathbb{C}$ ) is a seminorm on  $X$ , then there exists a linear functional  $F : X \rightarrow \mathbb{R}$  (or  $F : X \rightarrow \mathbb{C}$ ) such that  $F(x) = f(x)$  for all  $x \in Y$ , and  $F(x) \leq p(x)$  for all  $x \in X$ .

Next we state the Schauder's Fixed Point Theorem as follows:

**Theorem 3.3.** (*Schauder's Fixed Point Theorem [35]*) Let  $X$  be a non-empty complete metric space, and let  $f : X \rightarrow X$  be a continuous map. If there exists a constant  $0 \leq k < 1$  such that for all  $x, y \in X$ , the following inequality holds:  $d(f(x), f(y)) \leq k \cdot d(x, y)$  where  $d$  is the metric on  $X$ , then there exists a unique fixed point  $x^* \in X$  such that  $f(x^*) = x^*$ .

In Theorem 3.3,  $X$  is the metric space,  $f$  is the continuous map,  $k$  is a contraction constant, and  $d(x, y)$  represents the distance between two points  $x$  and  $y$  in the metric space. The theorem states that if  $f$  is a contraction mapping (i.e., it shrinks distances between points), then it has a unique fixed point in the space  $X$ .

### 3.2.1 Bunt-Motzkin Theorem

([57]) This theorem is a result on the converse of the main problem. It states that if a set  $C$  is Chebyshev then it implies that it is convex in Bergman spaces. However, this still remains unknown if it holds for Hilbert spaces.

### 3.2.2 Known mathematical inequalities

Some known mathematical inequalities will be utilized in this study. These include Cauchy-Buniakowski's -Schwarz inequality, triangle inequality among others. We will also use some laws like the parallelogram law to solve the problem at hand.

**Triangle Inequality**([129]) In a normed vector space  $V$ , the Triangle Inequality states that for any vectors  $z, y \in V$ , the following inequality holds:  $\|x + y\| \leq \|x\| + \|y\|$ .

**The Cauchy-Schwarz Inequality**([129]) in normed vector spaces. This inequality states that the absolute value of the inner product of two vectors is less than or equal to the product of their individual norms, that is,  $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$ .

The **parallelogram law** ([6])for vectors in a normed space  $V$  can be stated as follows:  $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$ , where  $x, y \in V$ .

### 3.3 Differentiability conditions

Obtaining derivatives is a very important technique which is useful in optimization procedures. We describe some differentiability conditions in this section namely, Frèchet differentiability conditions and Gateaux differentiability conditions.

#### 3.3.1 Frèchet differentiability conditions

This is a derivative defined mostly in normed spaces. Differentiability occurs on real-valued functions or vector valued functions of multiple variables. Its applicable mostly particularly on directional derivative where the continuity of the map is essential.

A function  $f : X \rightarrow Y$  between normed vector spaces  $X$  and  $Y$  is **Frèchet differentiable** at a point  $x_0 \in X$  [32]) if there exists a linear operator  $L : X \rightarrow Y$  and a function  $R : X \rightarrow Y$  such that:  $f(x) = f(x_0) + L(x - x_0) + R(x - x_0)$  where:

- (i).  $L$  is a bounded linear operator, which means there exists a constant  $M$  such that  $\|L(v)\|_Y \leq M\|v\|_X$  for all  $v \in X$ .
- (ii).  $R(x - x_0)$  is a remainder term that satisfies  $\lim_{x \rightarrow x_0} \frac{\|R(x-x_0)\|_Y}{\|x-x_0\|_X} = 0$ .

#### 3.3.2 Gateaux differentiability conditions

A Gateaux derivative is a fundamental principle in differential calculus which is a generalization on functions which are continuous on Banach spaces. It is useful in carrying out approximations in locally convex spaces. It is useful in formalization of functional derivatives which are

important in best approximations in calculus of variations. Moreover, its useful since it takes care of nonlinear functions also.

The **Gateaux differentiability** conditions for a functional  $F$  in a normed vector space can be stated as follows [78]:  $\lim_{t \rightarrow 0} \frac{F(x_0+t\Delta x)-F(x_0)}{t}$  exists for every  $\Delta x \in X$ . If this limit exists, we say that  $F$  is Gateaux differentiable at  $x_0$ , and the limit itself is the Gateaux derivative of  $F$  at  $x_0$ , denoted as  $F'(x_0)$ . These conditions express the linearity and the existence of directional derivatives for the functional  $F$  in a normed vector space.

# Chapter 4

## RESULTS AND DISCUSSION

### 4.1 Introduction

This chapter forms the core part of this work. We give results on conditions under which subsets of NLS are Chebyshev. We also discuss the characterizations on distance functions of CSs in NLSs and finally we investigate convexity of CSs in NLSs.

### 4.2 NLS, Chebyshev sets and their subsets

This section contains a description and characterizations involving NLS, CS and their subsets. It is known [88] that  $NA(H)$  is called strongly convex if there exists a unique  $\xi \in NA(H)$  for every pair  $\phi, \psi \in NA(H)$ . We begin with the following proposition for characterizations of NLS, CS and their subsets found in [2].

**Proposition 4.1.** *Every closed convex set in a rotund reflexive space is Chebyshev.*

*Proof.* See [2] for proof. □

**Lemma 4.2.** *Every non-void closed set in  $NA(H)$  is a CS if it is Euclidean.*

*Proof.* Let  $\mathfrak{U}$  be a closed subset of  $NA(H)$ . By Proposition 4.1, it is a closed convex set. Then since it is closed, it contains points which are well defined everywhere in it and are single valued. But we know that for  $T \in \mathfrak{U}$ , the norm-attainability condition holds, that is,  $\|Tx\| = \|T\|$ . Also  $T$  is a convex operator of  $\mathfrak{U}$ . By Ekeland's principles [7], we have  $S \in \mathfrak{U}$  such that  $S$  minimizes  $T(x) + (\frac{\epsilon}{\lambda})\|x - S\|$ . By Jordan's result [150], the operator  $T$  attains Euclidean values which are everywhere defined in  $\mathfrak{U}$ . Hence  $\mathfrak{U}$  is a CS. □

This result has the following immediate result.

**Corollary 4.3.** *If  $\mathfrak{U}$  has Euclidean points then the following holds:*

(i).  $\mathfrak{U}$  is a CS.

(ii). Functions on  $\mathfrak{U}$  are uniformly continuous and single valued.

(iii).  $d_{\mathfrak{U}}^n$  is everywhere Fretchet differentiable.

*Proof.* Since  $NA(H)$  is a NLS which is closed and reflexive, we proceed with the proof as follows:

(i). This case follows immediately from the proof of Lemma 4.2.

(ii). Let  $P_{\mathfrak{U}}$  be a projection dependent on  $\mathfrak{U}$  then  $I - P_{\mathfrak{U}} = P_{\mathfrak{U}} - I$  which is continuous at every point.

From Lemma 4.2, functions of  $\mathfrak{U}$  are single valued and continuous. But they are continuous at each end of every point hence they are uniformly continuous.

(iii) Lastly, we need to show that  $d_{\mathfrak{U}}^n$  is everywhere Fretchet differentiable. To see this, we know that Differentiability occurs on real-valued functions or vector valued functions of multiple variables. Its applicable mostly particularly on directional derivative where the continuity of the map is essential. A function  $f : X \rightarrow Y$  between normed vector spaces  $X$  and  $Y$  is Frèchet differentiable at a point  $x_0 \in X$  if there exists a linear operator  $L : X \rightarrow Y$  and a function  $R : X \rightarrow Y$  such that:  $f(x) = f(x_0) + L(x - x_0) + R(x - x_0)$  where: (i).  $L$  is a bounded linear operator, which means there exists a constant  $M$  such that  $\|L(v)\|_Y \leq M\|v\|_X$  for all  $v \in X$  and (ii).  $R(x - x_0)$  is a remainder term that satisfies  $\lim_{x \rightarrow x_0} \frac{\|R(x - x_0)\|_Y}{\|x - x_0\|_X} = 0$ . By [5],  $d_{\mathfrak{U}}^n$  is Fretchet differentiable at each and every point. More over it satisfies the two conditions above hence it is Fretchet differentiable everywhere.  $\square$

**Remark 4.4.** A subset  $\mathfrak{U}$  of  $NA(H)$  is approximately convex if for any closed ball  $W$  of  $NA(H)$  disjoint from  $\mathfrak{U}$ , there exists a closed ball  $W' \supset W$  disjoint from  $\mathfrak{U}$  having an arbitrarily large radius. Moreover, a nonvoid closed subset  $F$  of a convex metric space  $(X, d)$  is called a sun if for all  $x \in F$  there exists  $y \in P_K(x)$  such that  $y \in P_K(z)$  for all  $z$  on the half ray  $[y, x, \cdot)$  issuing from  $y$  and passing through  $x$ .

**Proposition 4.5.** *Let  $\mathfrak{U}$  be a closed subset of  $NA(H)$ . Then  $\mathfrak{U}$  is almost convex (AC) if it is a sun and consequently a CS.*

*Proof.* To show that a closed subset of a normed linear space is almost convex. A set  $A$  is said to be almost convex [17] if for every pair of points  $x$  and  $y$  in  $A$ , the line segment connecting  $x$  and  $y$  is almost entirely contained within  $A$ . In other words, for any  $x, y \in A$ , and any  $t$  in the closed interval  $[0, 1]$ , the point  $tx + (1 - t)y$  is also in  $A$ . By a version of Hahn-Banach theorem in normed spaces in [82],  $\mathfrak{U}$  is AC, moreover from Remark 4.4 it is a sun. But every sun is a CS. This completes the proof.  $\square$

At this point we consider proximality and Chebyshevity of sets.

**Theorem 4.6.** *Let  $\mathfrak{U}$  be proximal in  $NA(H)$ . Then  $\mathfrak{U}$  is closed and a CS.*

*Proof.* We prove by contradiction. Let  $\mathfrak{U}$  of  $H$  be not closed then consider a sequence  $\eta_n$  in  $\mathfrak{U}$  such that  $\lim_{n \rightarrow \infty} \eta_n = \eta$ , for  $\eta \in \mathfrak{U}$ . But for each  $T \in NA(H)$ ,  $\|T\eta\| = \|T\|$ . Moreover,  $d(\eta, \mathfrak{U}) \leq \|\eta - \eta_n\|$ . This shows that  $d(\eta, \mathfrak{U}) = 0$ . But  $0 \leq \|\eta - \beta\|$ , for any  $\beta \in \mathfrak{U}$ , since  $\eta \notin \mathfrak{U}$ . Hence,  $P_{\mathfrak{U}}(\eta)$  is void which contradicts proximality of  $\mathfrak{U}$ . Lastly, the assertion that  $\mathfrak{U}$  is a CS follows from Proposition 4.5.  $\square$

The following consequence follows immediately.

**Corollary 4.7.** *Every CS which is a subset of  $NA(H)$  has a continuous and non-expansive operator for  $\mathfrak{U}$ .*

*Proof.* Continuity follows from boundedness and norm-attainability of members of  $NA(H)$ . For expansivity we proceed as follows. Let  $\eta, \beta, \chi \in \mathfrak{U}$ . We have by triangle inequality that the distance  $d(\eta, \mathfrak{U}) \leq \|\eta - \chi\| \leq \|\eta - \beta\| + \|\beta - \chi\|$ . When we rearrange we obtain

$$d(\eta, \mathfrak{U}) - \|\eta - \beta\| \leq \|\beta - \chi\| \quad (4.2.1)$$

But  $\chi \in \mathfrak{U}$  is arbitrary and the left hand side of 4.2.1 is fixed so we have  $d(\eta, \mathfrak{U}) - \|\eta - \beta\| \in d(\beta, \mathfrak{U})$  which symmetrically is a translation leading to  $d(\beta, \mathfrak{U}) - \|\eta - \beta\| \leq d(\eta, \mathfrak{U})$ . Hence,  $|d(\eta, \mathfrak{U}) - d(\beta, \mathfrak{U})| \leq \|\eta - \beta\|$  which satisfies non-expansivity.  $\square$

Next, we consider convexity with regards to inner product. We state the next result as follows.

**Proposition 4.8.** *Let  $NA(H)$  be an inner product space and  $\mathfrak{U}$  be a convex subset of  $NA(H)$ . Then  $\beta \in \mathfrak{P}_{\mathfrak{U}}(\xi)$  if and only if  $\langle \xi - \beta, \omega - \beta \rangle \leq 0$ ,  $\forall \omega \in \mathfrak{U}$ .*

*Proof.* Let  $\xi \in NA(H)$  and  $\beta \in \mathfrak{U}$ . Let the inequality in the proposition be true, then if  $\xi = \eta$  we are done. So let  $\xi \neq \eta$ , then for any  $\omega \in \mathfrak{U}$ ,

$$\begin{aligned} \|\xi - \beta\|^2 &= \langle \xi - \beta, \xi - \beta \rangle \\ &= \langle \xi - \beta, \xi - \omega \rangle + \langle \xi - \beta, \omega - \beta \rangle \\ &\leq \langle \xi - \beta, \xi - \omega \rangle \\ &= \|\xi - \beta\| \|\xi - \omega\| \text{ by } C - S \text{ inequality.} \end{aligned}$$

So,  $\|\xi - \beta\| \leq \|\xi - \omega\|, \forall \omega \in \mathfrak{U}$  meaning  $\mathcal{P}_{\mathfrak{U}}(\xi)$  is proximal.

The converse is as follows. We prove by contradiction. Assume  $z$  is in  $L \cap \mathfrak{U}$ . we have  $\|x - z\| < \epsilon$  and  $\|y - z\| < \epsilon$ . Therefore,  $t\|x - z\| < t\epsilon$  and  $(1 - t)\|y - z\| < (1 - t)\epsilon$ . Adding these inequalities together, we get:  $t\|x - z\| + (1 - t)\|y - z\| < t\epsilon + (1 - t)\epsilon = \epsilon$ . So, we have shown that for any point  $w$  in  $L \cap \mathfrak{U}$ ,  $\|w - z\| < \epsilon$ . Now, consider the set  $L \cap \mathfrak{U}$ . This set contains all points  $w$  in  $L$  that are within  $\epsilon$  distance of  $z$ . Since  $z$  is in  $L$ , we can choose  $\epsilon$  small enough such that  $L$  contains a point  $w$  that is not in  $A$ . This contradicts the assumption that  $A$  is a closed set, as a closed set contains all of its limit points. Therefore, our initial assumption that there exists a point  $z$  in  $L$  that is not in  $A$  must be false. This completes the proof.  $\square$

**Theorem 4.9.** *Let  $NA(H)$  be having a strictly convex dual  $[NA(H)]^*$ . Then any closed almost convex subset of  $NA(H)$  is convex.*

*Proof.* Let  $M$  be a closed almost convex set which is not convex. Then  $M$  is not midpoint convex. So we have  $\eta, \beta \in M$  such that for  $q \in M$  we define  $q$  by  $q = \frac{\eta + \beta}{2} \notin M$ . But  $M$  is closed with  $d(q, M) > 0$ . But  $\beta[q : \frac{d(q, M)}{2}] \subseteq NA(H) \setminus M$ . Also  $M$  is almost convex, so there exists  $\omega_n$  and  $r_n$  in  $NA(H)$  and  $[0, \infty)$  such that  $\beta[\omega_n, r_n] \subseteq NA(H) \setminus M \forall n \in \mathbb{N}$  and  $\lim_{n \rightarrow \infty} r_n = \infty$ . From [98],  $\bigcup \beta[\omega_n, r_n] \neq NA(H)$ . So, it is not the whole space and hence it is a closed half plane. But  $\eta, \beta \in M$  so  $M$  is convex.  $\square$

**Corollary 4.10.** *Every CS with a continuous MPM is convex in any  $NA(H)$  with a strictly convex dual  $[NA(H)]^*$ .*

*Proof.* Since  $NA(H)$  has norm-attainable operators, we have  $\eta, \beta \in \mathfrak{U}$  of  $NA(H)$  with  $\|\eta x\| = \|T\|$  and  $\|\beta y\| = \|\beta\|$ . From Theorem 4.9,  $\mathfrak{U}$  is

closed almost convex since  $\mathfrak{U}$  has a continuous *MPM*. But since any *CS* is closed and  $NA(H)$  is strictly convex, then  $\mathfrak{U}$  is convex.  $\square$

This result can be extended to  $L^p$  spaces and Hilbert spaces in general as follows.

**Corollary 4.11.** *Let  $NA(H)$  be an  $L^p$  space,  $1 < p < \infty$ . Then every *CS* in  $NA(H)$  with a continuous *MPM* is convex.*

*Proof.* It suffices to show that since  $NA(H)$  is  $L^p$  it has a strictly convex dual for  $1 < p < \infty$ . But from [22], the dual of  $L^p$  is  $L^2$  which is strictly convex. Indeed,  $q = \frac{p}{p-1}$  as asserted in [86].  $\square$

**Corollary 4.12.** *Let  $NA(H)$  be having the *IPS* structure. Then any *CS* in  $NA(H)$  with continuous *MPM* is convex.*

*Proof.* It is known that every Hilbert space has a strictly convex dual and the dual of a Hilbert space is also a Hilbert space and so is strictly convex. Since  $NA(H)$  has *IPS* structure, intuitively it can be considered a Hilbert space.  $\square$

### 4.3 Distance functions in $NA(H)$

We begin with the following proposition which considers distance functions and Gateaux differentiability. We note that all the spaces and their subspaces are all nontrivial and are strictly NLS spaces of all functions that are norm-attainable unless otherwise stated.

**Proposition 4.13.** *Let  $\mathfrak{Q}$  be a NLS of all norm-attainable real-valued functions and  $\mathfrak{J}$  be a closed and smooth subset of  $\mathfrak{Q}$ . Let  $\zeta \in \mathfrak{Q} \setminus \mathfrak{J}$  and  $\eta$  the nearest point for  $\zeta$  in  $\mathfrak{J}$ , then Gateaux differentiability condition of  $\mathfrak{Q}$  holds for  $(\zeta - \eta)$ .*

*Proof.* Since the norm of  $\mathfrak{Q}$  is Gateaux differentiable from the statement of the proposition, it suffices to prove the existence of the unique limit of the the Gateaux derivative  $\lim_{l \rightarrow 0} \frac{d_{\mathfrak{J}}(\zeta + l(x - \eta)) - d_{\mathfrak{J}}(\zeta)}{l}$ . From [107], we deduce that if  $l > 0$  then the limit exists. For the uniqueness, we see from the result of [61] that the limit of the derivative is unique. It follows then that  $\langle d'_{\mathfrak{J}}(\zeta), x - \eta \rangle$  holds for  $d_{\mathfrak{J}}(\zeta)$ . This completes the proof.  $\square$

Proposition 4.13 leads to the interesting question as to what happens when  $\mathfrak{Q}$  is rotund. We see this in the next lemma.

**Lemma 4.14.** *Let  $\mathfrak{Q}$  be a NLS of all norm-attainable real-valued functions and  $\mathfrak{J}$  be a closed and smooth CS of  $\mathfrak{Q}$ . Let  $\zeta \in \mathfrak{Q} \setminus \mathfrak{J}$  and  $\partial d_{\mathfrak{J}}(\zeta)$  be a singleton set. Then the following conditions hold if the first dual of  $\mathfrak{Q}$  is rotund:*

- (i).  $\phi$  on  $\mathfrak{J}$  is uniformly continuous.
- (ii).  $\phi$  on  $\mathfrak{J}$  is totally bounded.
- (iii).  $\mathfrak{J}$  satisfies convexity condition.
- (iv).  $d_{\mathfrak{J}}$  satisfies convexity condition.
- (v).  $d_{\mathfrak{J}}$  satisfies Gateaux differentiability at  $\zeta$ .

*Proof.* We proceed with the proof as follows:

Case (i).  $\phi$  on  $\mathfrak{J}$  is uniformly continuous since every space of norm-attainable functions contains continuous functions.

Case (ii).  $\phi$  on  $\mathfrak{J}$  is totally bounded follows immediately from case (i).

Case (iii).  $\mathfrak{J}$  satisfying convexity condition follows immediately from the conditions of the statement of the lemma.

Case (iv).  $d_{\mathfrak{J}}$  satisfying convexity condition follows from the fact that  $\mathfrak{J}$  satisfies convexity condition.

Case (v). Since  $d_{\mathfrak{J}}$  satisfies convexity condition and is uniformly continuous at  $\zeta$  and from Proposition 4.13.  $\partial d_K(\zeta)$  is a singleton set,  $d_{\mathfrak{J}}$  satisfies Gateaux differentiability at point  $\zeta$  and we attain equality of  $d'_{\mathfrak{J}}(\zeta)$  and  $\partial d_{\mathfrak{J}}(\zeta)$ . This completes the proof.  $\square$

At this point, we state the main theorem of our work that characterizes convexity of  $\mathfrak{Q}$  in terms of Fréchet differentiability condition.

**Theorem 4.15.** *Let  $\mathfrak{Q}$  be a NLS of all norm-attainable real-valued functions and  $\mathfrak{J}$  be a closed and smooth CS of  $\mathfrak{Q}$ . Let  $\zeta \in \mathfrak{Q} \setminus \mathfrak{J}$  and  $\partial d_{\mathfrak{J}}(\zeta)$  be a singleton set. Then  $d_{\mathfrak{J}}$  satisfies Fréchet differentiability condition at  $\zeta$ .*

*Proof.* It is known from [136] that the norm of  $\mathfrak{Q}$  and hence the dual norm of  $\mathfrak{Q}^*$  satisfies Fréchet differentiability condition. Also from Lemma 4.14 it implies that  $\mathfrak{Q}$  is strictly reflexive. Moreover,  $\mathfrak{Q} \setminus \mathfrak{J}$  has the nearest point  $\zeta$  and so  $\mathfrak{J}$  of  $\mathfrak{Q}$  satisfies Fréchet differentiability condition and so is  $\mathfrak{Q}$ .  $\square$

As consequences of Theorem 4.15, we state the following corollaries.

**Corollary 4.16.** *Every distance function of a CS of the NLS of all norm-attainable real-valued functions is Fréchet differentiable.*

*Proof.* Follows from the conditions of Lemma 4.14 and Theorem 4.15.

The rest is clear from the fact that every rotund CS is convex.  $\square$

**Corollary 4.17.** *Every distance function of a CS of the NLS of all norm-attainable real-valued functions is Gateaux differentiable.*

*Proof.* Follows immediately from the conditions of Theorem 4.15 and Corollary 4.17 the proof is complete.  $\square$

Next, we move to the last section where we investigate convexity of CSs.

## 4.4 Convexity of Chebyshev sets in $NA(H)$

In this section we investigate the CS problem by characterizing convexity of CS. We provide a partial solution to CS problem by considering norm-attainable operators, norm attainable functionals and norm-attainable functions and vectors. We begin with the following proposition.

**Proposition 4.18.** *Let  $NA(H)$  be convex. Then every  $\mathfrak{P}_{\mathcal{U}}(\xi)$  which is convex is a CS.*

*Proof.* Consider  $\eta \in NA(H) \setminus \mathcal{U}$  and let  $\eta, \beta \in \mathfrak{P}_{\mathcal{U}}(\xi)$  be unique. Then we have  $\|\xi - \eta\| = \|\xi - \beta\| = d(\xi, \mathcal{U}) > 0$ . But we have that  $\mathcal{U}$  is convex since  $\mathfrak{P}_{\mathcal{U}}(\xi)$  is convex so  $d(\xi, \mathcal{U}) \leq \|\xi - (\frac{\eta + \beta}{2})\| = \|\frac{(\xi - \eta) + (\xi - \beta)}{2}\| < \frac{\|\xi - \eta\| + \|\xi - \beta\|}{2} = d(\xi, \mathcal{U})^2$  which is absurd. Therefore,  $\eta = \beta$ , hence  $\mathfrak{P}_{\mathcal{U}}(\xi)$  is a singleton set that is proximal and a CS hence every proximal set which is strictly convex is Chebyshev.  $\square$

**Lemma 4.19.** *Let  $\mathfrak{U}$  be a CS in  $NA(H)$  with IPS structure. Then  $\mathfrak{U}$  is convex if and only if its metric projection mapping has non-expansivity property.*

*Proof.* Assume that  $\mathfrak{U}$  is convex and  $\eta, \beta \in NA(H)$ . Then for expansivity property to hold,  $P_{\mathfrak{U}}(\eta) = P_{\mathfrak{U}}(\beta)$ . Let us consider the case of  $P_{\mathfrak{U}}(\eta) \neq P_{\mathfrak{U}}(\beta)$ . From Proposition 4.18 we have that  $\|P_{\mathfrak{U}}(\beta) - P_{\mathfrak{U}}(\eta)\| \leq \|\eta - \beta\|$  as required. Conversely, consider the metric projection mapping to be having non-expansivity property, but  $\mathfrak{U}$  is non-convex. But  $\mathfrak{U}$  is closed and does not satisfy midpoint convexity. This implies that we can find  $\eta, \beta \in \mathfrak{U}$  with  $\|\eta x\| = \|\eta\|$  and  $\|\beta y\| = \|\beta\|$  with  $x, y \in H$  such that  $\omega = \frac{(\eta+\beta)}{2} \notin \mathfrak{U}$ . It is easy to see that  $\eta \notin \beta$  for  $r = \frac{\|\eta-\beta\|}{2} > 0$ . Since  $NA(H)$  has IPS structure, is strictly convex and  $\eta - \beta = 2r$ , then from Lemma 2.31 in [119] we have that  $\beta(\eta, r) \cap \beta(\beta, r) = \{(\frac{\eta+\beta}{2})\} = \{\omega\}$  ensuring that  $\omega = P_k(\omega) \in \mathfrak{U}$  which is not possible. Hence,  $\mathfrak{U}$  is convex.  $\square$

**Theorem 4.20.** *Let  $\mathfrak{U}$  be a CS in  $NA(H)$  with IPS structure. Then  $\mathfrak{U}$  is convex if the MPM  $P_{\theta}$  is continuous.*

*Proof.* We prove this by contradiction. Let  $\mathfrak{U}$  be not convex. Also let  $\xi \in \text{conv}(\mathfrak{U}) \setminus \mathfrak{U}$  and  $V = \mathfrak{U} - \xi$ . It is known that  $0 \in \text{conv}(V) \setminus V$ . Moreover  $V$  is a CS with MPM. Since  $P_{\mathfrak{U}}$  is continuous, it follows that  $P_V$  is continuous. Now since  $P_V$  is continuous on  $NA(H)$  then there exists a  $J$  on a remotal set  $H$  which is also continuous on  $NA(H)$ . From Proposition 4.18 and Bunt-Motzkin Theorem (BMT), the space  $H$  is a singleton set and so  $V$  and hence  $\mathfrak{U}$  is singleton which contradicts the earlier hypothesis that  $\mathfrak{U}$  is not convex. Hence,  $\mathfrak{U}$  is convex.  $\square$

Next we use convex analysis techniques to prove convexity of  $CS$ .

**Proposition 4.21.** *Let  $\mathfrak{U}$  be a  $CS$  of  $NA(H)$  with  $ips$  structure. Then  $\mathfrak{U}$  is convex if  $\xi \mapsto d((\xi, \mathfrak{U}))$  is Fretchet differentiable on  $NA(H) \setminus \mathfrak{U}$ .*

Proposition 4.21 is useful in the sequel for proving other results. Thu we put it in this section.

*Proof.* Since we are applying convex analysis technique of differentiability we need to show that the function  $\alpha \mapsto \|\xi - \alpha\|$  which is well defined on  $\mathfrak{U}$  has a strong minimum at  $P_\alpha(\xi)$ . Consider a sequence  $\{\omega_n\}_{n \in \mathbb{N}}$  with a converging subsequence. By BMT, we have  $\lim \omega_n \rightarrow P_\alpha(\xi)$ . So,  $\lim_{n \rightarrow \infty} \|\xi - \omega_n\| = d(\xi, \mathfrak{U})$ . But  $\omega_n$  is bounded and  $NA(H)$  is reflexive so we have an existing subsequence  $\omega_{n_\alpha}$  of  $\omega_n$  which converges to  $\omega$  in  $NA(H)$ . Now consider  $\xi^*$  as the Fretchet of  $DF$  for  $\mathfrak{U}$  at  $\xi$ . By [100] we have  $\|\xi^*\| \leq 1$  and for each  $n \in \mathbb{N}$  let  $\chi = \sqrt{\|\xi - \omega_n\| - d(\xi, \mathfrak{U}) + \frac{1}{2^n}}$ . Then by  $FD$  we have

$$\begin{aligned} \xi^*(\omega - \xi) &= \lim_{\alpha \rightarrow \infty} \xi^*(\omega_{n_\alpha} - \xi) \\ &= \lim_{\alpha \rightarrow \infty} \left[ -\|\xi - \omega_{n_r}\| + \frac{\|\xi - \omega_{n_r}\| - d(\xi, \mathfrak{U})}{\chi_n} \right] - d(\xi, \mathfrak{U}) \\ &= -d(\xi - \mathfrak{U}). \end{aligned}$$

This shows that  $d(\xi - \mathfrak{U}) \leq \xi^*(\omega - \xi)$  which implies that the distance  $d(\xi, \mathfrak{U}) \leq \xi^*(\xi - \omega) \leq \lim_{\alpha \rightarrow \infty} \|\xi - \omega_{n_r}\| = d(\xi, \mathfrak{U})$ , with respect to weak topology (WT) in  $NA(H)$  and the norm is  $lsc$  with respect to WT in  $NA(H)$ . But the norm is Kadec-Klee norm so  $\omega = \lim_{\alpha \rightarrow \infty} \omega_{n_r}$ . This shows that  $\omega \in \mathfrak{U}$ . Now since  $\mathfrak{U}$  is a  $CS$  and  $\|\xi - \omega\| = d(\xi, \mathfrak{U})$ , it follows that  $\omega = P_{\mathfrak{U}}(\xi)$ .  $\square$

**Corollary 4.22.** *Let  $\mathfrak{U}$  be a CS of  $NA(H)$  with IPS structure. Also let MPM  $\xi \mapsto P_{\mathfrak{U}}(\xi)$  be continuous on  $NA(H) \setminus \mathfrak{U}$  then DF  $\xi \mapsto d(\xi, \mathfrak{U})$  is FD on  $NA(H) \setminus \mathfrak{U}$  and hence  $\mathfrak{U}$  is convex.*

*Proof.* Follow immediately from Theorem 4.20, Proposition 4.21 and from the fact that MPM  $\xi \mapsto d(\xi, \mathfrak{U})$  is continuous.  $\square$

Next, we investigate convexity of CS with regard to the sun property. We start with the following:

**Proposition 4.23.** *Let  $\mathfrak{U}$  be a convex CS in  $NA(H)$  then  $\mathfrak{U}$  is a sun.*

*Proof.* We need to prove that a convex Chebyshev set in a normed linear space is a sun set. Since  $\mathfrak{U}$  in  $NA(H)$  is a convex Chebyshev set, then for each  $T \in NA(H)$ , there exists a unique element  $Y_T \in \mathfrak{U}$  such that  $\|T - Y_T\| = \text{dist}(T, \mathfrak{U})$ , where  $\text{dist}(T, \mathfrak{U})$  represents the distance from  $T$  to the set  $\mathfrak{U}$ . Now for a set  $S$  in  $NA(H)$  such that for each  $T \in NA(H)$ , there exists a unique element  $F_T \in S$  such that  $F_T$  is the nearest point to  $T \in S$ . Now, since  $F_T$  minimizes  $\|T - F\|$ , for all  $F_T \in S$  then  $\mathfrak{U}$  is a sun set.

$\square$

**Theorem 4.24.** *Let  $NA(H)$  be smooth and  $\mathfrak{U}$  be a CS in  $NA(H)$ . Then  $\mathfrak{U}$  is convex if it is a sun.*

*Proof.* First we show the smoothness of  $NA(H)$ . This follows immediately from Chauder's fixed point theorem. By Proposition 4.23  $\mathfrak{U}$  is a sun so it remains to show that  $\mathfrak{U}$  is convex. We prove this by contradiction as

follows. Let  $\mathfrak{U}$  be non-convex. Now since  $\mathfrak{U}$  is closed we have  $\eta, \beta \in \mathfrak{U}$  such that  $\frac{1}{2}(\eta + \beta) \in \mathfrak{U}$ . Let  $\omega = P_k(\frac{\eta + \beta}{2})$  and  $\xi = \frac{\eta + \beta}{2} - \omega$ . By Gateaux differentiability at  $\xi$  we have  $\omega + \xi \in \beta(\omega + \xi, \|\xi\|) \leq \bigcup_{n \in \mathbb{N}} \leq B(\omega + n\xi, n\|\xi\|)$  which is contained in  $NA(H) \setminus \mathfrak{U}$ . Since  $\mathfrak{U}$  has the important property that it is a sun  $\omega = P_{\mathfrak{U}}(\omega + \xi) = P_{\mathfrak{U}}(\omega + n\xi)$ ,  $\forall n \in \mathbb{N}$ . Therefore,  $d(\omega + n\xi, \mathfrak{U}) = \|\omega + n\xi\| - \|\omega\| = n\|\xi\|$ ,  $\forall n \in \mathbb{N}$ . Thus, it follows that  $\eta$  or  $\beta$  is in  $\bigcup \beta \subseteq NA(H) \setminus \mathfrak{U}$ . This is however not possible since  $\eta, \beta$  are members of  $\mathfrak{U}$ .  $\square$

The following consequence comes immediately.

**Corollary 4.25.** *Let  $\mathfrak{U}$  be a CS of  $NA(H)$ . Then  $\mathfrak{U}$  is almost convex if the MPM for  $\mathfrak{U}$  is continuous.*

*Proof.* Suppose that  $\mathfrak{U}$  is a CS of  $NA(H)$ . From Proposition 4.23, without loss of generality we have that  $\mathfrak{U}$  in  $NA(H)$  is an almost convex Chebyshev set if for each  $T \in NA(H)$ , there exists a unique element  $Y_T \in \mathfrak{U}$  such that  $\|T - Y_T\| \leq \text{dist}(T, \mathfrak{U})$ , where  $\text{dist}(T, \mathfrak{U})$  represents the distance from  $T$  to the set  $\mathfrak{U}$ , as seen in Theorem 4.4. In deed, the MPM for  $\mathfrak{U}$  is continuous since  $\text{dist}(T, \mathfrak{U})$  is a constant function but constant functions are always continuous.  $\square$

# Chapter 5

## CONCLUSION AND RECOMMENDATIONS

### 5.1 Introduction

This chapter provides a summary of this study. Here, a fundamental question that remains unsolved to-date regarding the convexity of the CS in infinite NLS known as the CS problem has been studied in this work. We considered the NLS of all real-valued norm-attainable classes. We conclude our work as per the objectives and make recommendations for future research.

### 5.2 Conclusion

In conclusion, a fundamental question that remains unsolved to-date regarding the convexity of the CS in infinite NLS known as the CS problem has been studied in this work. This CS problem which has not been solved

in totality (even in this study) states that: Is every CS in a NLS convex? This question has not got any solution even in the simplest form of a real Hilbert space (HS). In this work, we have characterized Chebyshev sets and their convexity in NLSs. We considered the NLS of all real-valued norm-attainable classes.

For objective one, we have shown in Lemma 4.2 that every non-void closed set in  $NA(H)$  is a *CS* if it is Euclidean. Also, if  $\mathfrak{U}$  is proximal in  $NA(H)$  then  $\mathfrak{U}$  is closed and a *CS* as seen in Theorem 4.6.

For objective two, we have shown in Proposition 4.13 and Theorem 4.15 respectively that CSs of the NLS of all real-valued norm-attainable functions are convex when they are closed, rotund and admits both Gateaux and Fréchet differentiability conditions.

Lastly, for objective three, it has been shown Proposition 4.18 that every  $\mathfrak{B}_{\mathfrak{U}}(\xi)$  which is convex is a *CS*. Moreover, if  $\mathfrak{U}$  is a *CS* in  $NA(H)$  with *IPS* structure, then  $\mathfrak{U}$  is convex if and only if its MPM has non-expansivity property as illustrated in Corollary 4.22.

### 5.3 Recommendations

From this study, it is noted that the CS problem has not been solved in totality and therefore it still remains interesting. We recommend that future research can be considered to:

- (i). Characterize distance functions in the norm-attainable class of functionals.
- (ii). Characterize subsets of CS in infinite dimensional HS.

(iii). Investigate convexity of CS in the norm-attainable class of functionals.

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