ON (P,Q)-BINOMIAL EXTENSION OF COX-ROSS-RUBINSTEIN MODEL FOR OPTIMIZATION OF PORTFOLIO WITH NOISY OBSERVATIONS IN LIFE INSURANCE

BY

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Declaration

This thesis is my own work and has not been presented for a degree award in any other institution.

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Dedication

Grateful to the Almighty for life, energy, and health, with love and remembrance for my late daughter Amber. In loving memory of my late father, Patrick, whose wisdom and guidance continue to inspire me.

Abstract

In the realm of dynamic life insurance portfolio management, enhancing the Cox-Ross-Rubinstein (CRR) model becomes imperative amid the challenges presented by noisy observations. This quest arises from a nuanced understanding of the intricacies involved in modeling life insurance portfolios, particularly in the presence of uncertainties and fluctuations represented by noisy observations. An augmentation of the CRR model is undertaken through a(p, q)-binomial framework introducing a novel parameter p. The objectives of this study are to; Develop a *(p,q)*-extension of CRR model with noisy observations; Establish optimization conditions for the extended model with noisy observations and Simulate the outcomes of the model in life insurance. The (p, q)-binomial distribution facilitates a (p, q)-random walk within the CRR model, aligning it with the Black-Scholes model. Our approach includes developing a utility function for investor preference analysis, examining noise sensitivity, and establishing constraints for structured optimization. Practical simulations of the extended model in life insurance contexts demonstrates the model's real-world applicability. The study's innovative approach to integrating noisy market data into the model is intended to facilitate the strategic modeling of optimal portfolios, while judiciously balancing risk considerations. Furthermore the study delves into a more mathematical exploration of the model's sensitivity and responsiveness to various market conditions. Future research could focus on enhancing the precision of the model by incorporating advanced mathematical techniques, such as stochastic calculus or machine learning algorithms, to better predict and optimize portfolio performance amidst market uncertainties.

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INTRODUCTION

1.1 Background to the study

An approach for approximating the value of options, which are financial contracts that allow holders to exercise their rights until they expire, is the Cox, Ross, and Rubinstein (CRR) market model in Cox [46]. The model supposes that the principal asset's price goes up and down by a fixed factor, as in a binomial tree. Further, it divides periods into smaller units, simplifying the calculation of complex options. The model extends the lattice binomial technique, which was created by Cox, Ross, and Rubinstein, thus can price options when there are two fundamental variables. It can be applied to the pricing of any contingent claim in this way. The approach has previously been used to compare the outcomes for situations when an exact answer has been found, such as in European markets, and to price the maximum and minimum value of options involving numerous assets. The lattice approach of the model is useful since it takes into consideration the early exercise of the American options. In order to handle situations that other models cannot, the method has also been used to value a number of financial instruments in Boyle and Tian [27].

Extending the CRR model constructs an appropriate repeating portfolio, and prices are given for several parameters at each trading interval. In doing so, it assists in resolving a few of the difficulties that academics encounter when valuing options, including the Black-Scholes formula. In order to price European options contracts by calculating how their prices vary over time, Robert Merton, Myron Scholes, and Fischer Black created the latter method. To price European Options, the model takes into account the interest rate, time to expiration, asset and strike price, and volatility of the underlying financial instrument Black and Scholes [20].It assumes the absence of arbitrage; that is, investors can hedge option risks by buying and selling the instruments over time. The model also assumes the absence of trading transaction costs and dividends, and that asset prices follow a geometric random walk (Wiener process) with constant variance and volatility in Glantz and Kissell [59]. These assumptions enable investors to calculate the fair value of European options easily. The Black-Scholes-Merton model, sometimes known as Black-Scholes, was developed for option prices by Boyle [27], who also showed that it is effective at estimating parameter values. Over time, numerous iterations of the Black-Scholes model have been created.

However, the Black-Scholes formula does not conform to reality, because it assumes that volatility is constant. Empirical research shows that volatility is significant in valuing options, and follows a stochastic pattern Sheraz and Preda [135]. Given that this assumption is unsatisfactory, researchers have added several frictions and determining the emerging volatility, so as to fix the Black-Scholes technique. Several researchers have considered a distribution-based approach to model the behaviors of these instruments. Samuelson [127] re-introduced the idea that asset returns could model Einstein-Wiener processes in the early 1960s. First put forth by Bachelier [9], the notion that "past, present, and even discounted future events are reflected in market price, but often show no apparent relation to price changes" in his PhD thesis, but it was overlooked by most economists. After its reintroduction in the literature by Samuelson [127], Osborne [106] showed that stock prices behave like molecules. The discrete model of option is generally favored in practice. Considering that the Black-Scholes formula is a limit case of the discrete binomial Cox-Ross-Rubinstein model, this was further developed by Cox, Ross and Rubinstein [46].

Generalization of the CRR model has raised lengthy discussions on the modeling of financial instrument returns. It is believed that the CRR model can help insurance agencies to optimize portfolios and mitigate risks. First, insurance firms mitigate risks by pooling them. That is, they effect the law of large numbers to predict their risk exposure, and the value of future capital to cover losses with sufficient accuracy in Tinungki [143]. The insurance agency acts as the insurer and assumes the risk on behalf of the insured for a fee known as the premium. Insurance can be broken down into two major categories; general insurance and life insurance where general insurance secures the insured's property from various mishaps such as theft, storms, accidents, or other calamities in Satish [129]. Conversely, life insurance protects people from certainties and uncertainties like old age, illnesses, or death, which may disrupt or destroy their ability to earn income. Therefore, life insurance enables the insured or assured to earn income after encountering risk. There are three life insurance categories: individual life insurance, group life insurance, and pension plans. These are subdivided into whole life insurance, term life insurance, money back policy, savings, endowment policy, retirement plans, investment plans, and unit-linked insurance plans(ULIPs) as seen in Borch [24].

A group life insurance contract typically covers several individuals at once. In most cases, the owner of the policy is an employer who wants to cover employees against risk in Tinungki [143]. Conversely, individual life insurance policies are paid for, and cover only one individual. Pension benefit schemes are any plan, fund, or scheme which provides retirement income. It requires that an employer contributes to a funding pool, which is allocated for an employee's future benefit. The investment returns generate income for employees upon their retirement. Term Life insurance are policies that cover individuals or groups for a limited time. Whole Life insurance refers to a set of insurance policies that cover individuals throughout their lifetime. An endowment policy is a life insurance arrangement that intends to pay the total premiums upon the holder's death or policy maturity. Money back policy gives coverage during the time of the contract. The maturity benefits are paid in installments, based on the survival of the policyholder; however, beneficiaries get the sum assured and any accrued bonuses upon the insured's death. Savings and Investment Plans are financial instruments that allow creating wealth for the future through saving income to meet various goals. Unit Linked Insurance Plans(ULIPs) is an insurance policy that provides both insurance and investment at a go. The insured pays premium invested in the funds and enjoys coverage via insurance policy.

The life insurance industry provides the means through which policyholders offer financial protection to their beneficiaries in case of the policyholder's death or any other significant reason for that matter. Life insurance is a long-term liability stream reaching several years into the future by representing future claims on policies issued as seen in Borch [24]. Such streams are stochastic; that is to say, it is not known at the moment when they will necessarily occur, and this fact depends on the length of the policy and option termination rights. In this regard, many regulatory agencies do not allow life insurance firms to engage in risky investments. This is upheld in several ways, for example, they are expected to demonstrate the ability to possess assets over liabilities, pay any debt, and have sufficient premiums to meet their long-term obligations. For that matter, insurance agencies should determine portfolios with cash flows that are analogous to the stream of liabilities. One sure way of doing this is finding a replicating portfolio by determining their liabilities' fair market value. This translates to the constant look for opportunities to expand their area of operation, increase their revenues and improve profitability. Portfolio growth can lead to increased revenues and an accumulation of risk for different business lines and the firm in Oyatoye and Arogundande [107].

Actuaries can ascertain if adding additional risk will strengthen or weaken an existing portfolio by using the stochastic method to insurance portfolio optimization, which is achieved through the use of the CRR Model. Recently, many works have employed diffusion models to model controllable risk exposures for insurance companies. One study by argued that the financial reserve of insurance companies should be modeled as a process with a constant diffusion coefficient (risk exposure) and positive drift. Biagini [17] also notes that this drift can be used to determine the potential profit when the number of sold policies is sufficiently large. Other scholars have since supported this view in Broadie and Jerome [34]. Considering the general input-output theory of insurance, the significant inputs of an insurance company are earned premium,realized capital gain, and investment income. The earned premium comprises: expense loading, pure risk premium, and profit loading. Major outputs of an insurance company are dividends to shareholders, dividends to policyholders, expenses and incurred claims. According to Biagini and Zhang [17], the latter consists of claims from the current accident year as well as adjustments to claim amounts for claims from its predecessor. From the input-output theory, we can also deduce that the inputs and outputs of any insurance business originate from either one or a combination of any of the types of life insurance mentioned earlier. In an attempt to try and maximize input and minimize output to enhance sustainability in the industry, a life insurance company should spread their risks to reinsurance firms, and pool of insurance to guarantee an optimum performance.

In the finance world, derivatives are contracts whose valuations are dependent on underlying assets, which can be bank accounts, bonds, commodities, currencies, or stocks. There are different kinds of derivatives. Options are derivatives that are traded only on single securities, such as stocks, currencies, or indexes, because their underlying assets are fixed, thus, cannot be changed as see in Thomsett[141]. A call option allows the holder to purchase a particular asset at a predetermined price. Conversely, a put option allows the holder to sell a financial asset at a predetermined price in Hull[65].A strike price is the monetary value at which users can exercise rights to sell or purchase an option. It is different from the stock's price, which is the option's underlying stock's price in Thomsett [142]. There are two generally accepted techniques in which options are exercised with regard to their expiration dates: American options can be exercised during the length of the contract, and including on their expiration dates. Conversely, European options are exercised only on their expiration.

Options also have intrinsic and extrinsic value. The former measures the strike and stock price difference, while the latter measures the premium and intrinsic value difference. The former accounts for internal price movements of the option from early exercise, while the latter accounts for both internal and external price movements. There are three main features that govern an American option's status (intrinsic value); description of asset, expiration date, and exercise price. Call options have intrinsic value if they are below the strike price; calls that enables buyers to purchase assets at values lower than market rates are more valuable than the inverse as it will be profitable upon being exercised.

Conversely, puts have intrinsic value if they are above the strike price; puts that enable their holders to sell assets at much higher rates than the market value is more valuable as it will be profitable upon being exercised.The exercise price is also important as they determine whether an option is profitable (in-the-money) or loss-making (out-of-the-money). Finally, the expiration date shows the option's internal value as they determine their flexibility. As a result of the high levels of flexibility offered by American options, they usually have higher valuations than European options .

The following are the main factors that influence an option's extrinsic; volatility, interest rates, and rate of stock growth. Volatility refers to the degree of dispersion of returns over a trading period. It is caused by buyer and seller speculation about the option's price due to uncertainty and changes in the external or internal business environment as seen in Guillermo, Lazo and Aurelio [60]. Volatility is positively correlated with the value of an option. High volatility increases the likelihood of an option being profitable,thus increasing its extrinsic value in Garven and Hiliard [55]. Prevailing interest rates also affects the value of options by affecting rho which is a variable used in many option-pricing models which measures the rate at which the price of an option contract rises or falls if the risk-free interest rate changes by 1%.

Generally, calls have a positive rho, which implies that rising interest rates increases their extrinsic value. On the other hand, puts have a negative rho; hence, rising interest rates decreases their extrinsic value. There are two parties involved in every option trade. Parties that offer options are called option writers while parties that obtains the option are called purchasers in Kwok [76]. Typically, the former faces more significant risk than the latter, who only risk losing their original premium . This is because they must sell or buy the options when they are exercised, at the terms specified in the options contract. As a result, they are required to post security deposits called margins that guarantee performance.

Based on modern portfolio theory (MPT), which states that investors want the highest possible returns for the lowest risks [84]. To achieve this, assets selected should have a low correlation with each other such that if one asset class under performs, the entire portfolio does not crash. Portfolio optimization is a 2-stage process that involves the selection of asset classes depending on relative weight, and the selection of particular assets and the quantity they want to include in the portfolio for optimum returns. The optimum return is, in most cases uncertain. Return uncertainty can be treated with three different mathematical methods: First, mean-variance analysis, secondly utility function analysis, and lastly through arbitrage or comparison analysis.

Mean-variance analysis is a technique used in modern portfolio theory (MPT) to calculate the variance of assets against expected returns. It was advanced by Markowitz [85]. It enables analyst to pick investments that have the biggest returns at particular levels of risk. On the other hand, utility function analysis enables investment managers to calculate the optimal portfolio that meets an investor's objectives and preference. This model was advanced by Neumann and Morgensten [100] who argued that rational investors should only select portfolios that maximize their wealth's expected utility from a set of investment alternatives (see also Scott and Horvath [130], and Rose [117]). Finally, arbitrage analysis is a mathematical technique used to determine underpriced assets and their expected rates of returns relative to systematic risks as in Chamberlain [38]. Insurance companies, Life Insurance Companies in particular, need a deeper understanding of their portfolio and its management to help achieve portfolio efficiency, risk mitigation, capital appreciation, investment goals, asset allocation, diversification, among other reasons.

This study presented a model that can enable actuaries to determine an optimum portfolio with noisy observations that a life insurance company, considering the inherent risk in the insurance industry, can carry at an agreeable risk level, bearing in mind the volatile nature of the industry's environment in developing countries.

1.2 Basic Concepts

Definition 1.1 ([138], Definition 1.5)**.** Derivative securities known as options confer rights, but not responsibilities, on the holder to acquire or surrender financial assets within a specified timeframe and subject to particular conditions.

Remark 1.2. Let us assume that someone holds a call option on a stock with strike price K and price at maturity S. The option's value is $S-K$ (see Baaquie [7]) if $S < K$, which indicates that the option's value is zero. If, on the other hand, $S > K$, the option's value is reported. As follows, both cases can be shown: A call option's value at maturity can be found explicitly in $C(S,T) = \max(0, S-K)$. Although the outcome is different in this instance as follows, the same logic applies to put options as well; The option is put at maturity as $P(S,T) = \max(0, K-S)$. When considering an option pricing curve with different dates to expiration, in Luenberger [80] states that the value of a call option increases with the amount of time till expiration.

Definition 1.3 ([76], Definition 2.1.4)**.** Let $C(S,T)$ and $P(S,T)$ be the prices of a European put and call, whose underlying stock price is S and with a strike price K. Then, the put-call parity (the association between put and call prices), will be determined by the equation $C(S,T) - P(S,T) + dK = S$, where d is the discount factor of the underlying asset as the options approach their expiration date.

Remark 1.4. This combination is implemented to enhance speculative strategies. According to (see luenberger) [80], combining options with stock makes it possible to determine the investment's returns.

Definition 1.5 ([65], Definition 1.2.4)**.** A portfolio is a collection of assets or financial investments that a person or an organization owns. Cash equivalents, art, real estate, mutual funds, equities, bonds, and commodities could all be included in this mix. Through the use of portfolios, customers can diversify their investments, lowering risks and optimizing profits.

Hull [66] notes that stochastic processes enable investors to efficiently model the dynamics of stocks' prices. That is, given a stock, $S(t) = S_i(t)$, where $S_i(t)$ is the price of the i^{th} stock at time t. Consequently, the investors holding positions, $H_t = (H_1(t), ..., H_n(t))$, react according to the stochastic processes that are under their control. Using this knowledge, we can determine that a portfolio value process V_ϱ will be determined by,

$$V_\varrho(t) = \sum_{i=1}^{n} H_i(t) S_i(t). \tag{1.2.1}$$

Supposing from Equation 1.2.1 there is no consumption from the portfolio, no additional capital invested, and no cash dividend payout, investors can only earn money from the price fluctuations of their underlying assets, which can lead to a self-financing portfolio. The latter can be defined as outlined in,

Definition 1.6 ([80], Definition 2.1.1)**.** A portfolio ϱ is called self-financing if the value process V_ϱ satisfies

$$dV_\varrho(t) = \sum_{i=1}^{n} H_i(t) dS_i(t).$$

From above, we can also determine the nature of a relative portfolio as outlined in Definition (1.2.8).

Definition 1.7 ([80], Definition 1.1.2)**.** Given portfolio ϱ, the relative portfolio $y(t) = [y_1(t), ..., y_n(t)] = y_i(t)$ will be such that,

$$y_i(t) = \frac{H_i(t)S_i(t)}{V_\varrho(t)},$$

where $i = 1, ..., n$ and $\sum_{i=1}^{n} y_i(t) = 1$.

This shows that with n different stocks, free control is only on $n - 1$ of them. Given this constraint, we set $y_n = 1 - \sum_{i=1}^{n-1} y_i(t)$, and from this, the firm can only calculate the value of $(y_1, ..., y_{n-1})$. The following proposition shows the price movement in self-financing portfolio as expressed through y

Proposition 1.8 ([105], Proposition 4.2.1)**.** *A relative portfolio y is self-financing only if the value process's $V(t)$ dynamics is defined as,*

$$dV_y(t) = V_y(t) \sum_{i=1}^{n} y_i(t) \frac{dS_i(t)}{S_i(t)},$$

where, $dV_y(t)$ is the control process of $y(t)$. The primary goal of any investor is to maximize the returns on their portfolio by controlling its holdings. However, this paves way to the optimal stochastic control problem, a challenge that investors faced when trying to control the value process's movement, $V(t)$ through y_t.

Definition 1.9 ([80], Definition 3.2.1)**.** Portfolio management combines financial instruments with the right tools to generate optimum return downsizing risk within a given time horizon. One aspect of portfolio management is portfolio optimization.

Definition 1.10 ([80], Definition 1.1.7)**.** Portfolio Optimization is a process in which investors factor the minimization of risk and maximization of expected returns during the selection of one out of a possible set of probable portfolios.

Definition 1.11 ([80], Definition 3.1.2)**.** Diversification of a portfolio is the process of investing in different assets to minimize the portfolio's overall risk.

Remark 1.12. Investors can reduce the variance of their portfolio's returns by including many assets in the portfolio. Often, portfolios with fewer assets attract high-levels of risk, because they have more considerable variance.

Definition 1.13 ([80], Definition 4.1.1)**.** Hedging is a strategy in which investors reduce the risk of adverse price movements in an asset.

Remark 1.14. Hedging enables investors to protect themselves against risk, thus is crucial to the core functioning of financial markets. Insurance can be considered a form of hedging, since people pay premiums so that they can protect themselves against certain risks. Upon the occurrence of such risks, an individual is liable for compensation [80].

Definition 1.15 ([80], Definition 2.1.12)**.** Strategy is a general direction set to achieve a desired state in the future.

Definition 1.16 ([80], Definition 1.2.1)**.** Insurance is a contract between two parties. The first is the policyholder, the person, firm, or company confronted by risk. The other is the insurer, a person, firm, or company specializing in assuming the risk the exposure units face and making their losses collectively predictable.

Definition 1.17 ([80], Definition 1.1.12)**.** A Stochastic process refers to processes and events that produce random outcomes and variables.

Remark 1.18. For non-stochastic events, the observations and outcomes at certain times can be random; however, for stochastic events, the observed value at each time is a random variable.

Definition 1.19 ([80], Definition 3.1.1)**.** Risk refers to future uncertainty in the deviation of expected earnings. As such, it is a measure of the uncertainty that will be experienced before an investor profits from a particular investment.

Definition 1.20 ([80], Definition 2.2.5)**.** Noisy observations are the error between true and observed values due to a lack of accuracy in measurement. In most cases, noise represents the risk encountered in an investment.

Definition 1.21 ([80], Definition 4.1.7)**.** Life insurance is a contract between the insured and the insurer. The insurer agrees to pay a sum of money in exchange for a premium after agreed time or upon maturity or upon death of the insured.

Definition 1.22 ([21], Definition 5.1.1)**.** (The Black-Scholes Formula) Assume that r is the interest rate and that Equation 1.2.8 controls a security's price. The price of this security's derivative, $f(S,t)$, satisfies the partial differential equation.

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} rS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = rf. \tag{1.2.2}$$

1.2.1 Portfolio Return

Portfolio return is the amount of money generated by a particular investment (see Markowitz [84]). To visualize it, let's assume that we bought an asset at t_0 and sold it one year later. If X_0 and X_1 are amounts invested and amounts received respectively, then R_ϱ is the total return. Usually, scholars use the term return to denote the total return of a portfolio. The rate of return denoted by r_ϱ can be summarized as follows,

$$r_\varrho = \left(\frac{X_1 - X_0}{X_0}\right) X\ 100\%.$$

The two notions, the rate of return and total return are related by $R_\varrho = r_\varrho + 1$ (see Markowitz [84]). In essence, this equation describes that the rate of return behaves like the interest rate.

Taking n assets, a portfolio can be created by dividing X_0 among them. Next, for each $i = 1, 2, 3...n$, we choose amounts X_{0i}, so that $\sum\limits_{i=1}^{n} X_{0i} = X_0$, where X_{0i} denotes the amount invested in the i-th asset. Fractions of the total amount of money invested can be used to describe the amounts, as in $X_{0i} = w_i X_0, \imath = 1, 2, 3..., n$, where denotes the weight of the i-th asset. The return is $R_{\varrho i} X_{0i} = R_{\varrho i} w_i X_0$, and it is evident that $\sum\limits_{i=1}^{n} w_i = 1$. The amount of money created by the i-th asset is $R_{\varrho i} X_{0i} = R_{\varrho i} w_i X_0$, assuming that $R_{\varrho i}$ denotes the total return of asset i. This portfolio's overall receipt is consequently $\sum\limits_{i=1}^{n} R_{\varrho i} w_i X_0$. As a result, we can conclude that the portfolio's overall total return is,

$$R_{\varrho} = \frac{\sum\limits_{i=1}^{n} R_{\varrho i} w_i X_0}{X_0} = \sum_{i=1}^{n} w_i R_{\varrho i}.$$

Summarily, a portfolio's return rate and total ROI rate are equal to the sum of weighted corresponding individual asset returns. Here, an asset's weight is described as its relative cost in the portfolio, such that: $R_{\varrho} = \sum\limits_{i=1}^{n} w_i R_{\varrho i}$, $r_{\varrho} = \sum\limits_{i=1}^{n} w_i r_{\varrho i} = w_1 r_{\varrho 1} + w_2 r_{\varrho 2} ... + w_n r_{\varrho n}$.

1.2.2 Portfolio Variance

One cannot speak of portfolio returns without mentioning the variance of a portfolio; this, in a nutshell, is the measure of a portfolio's average returns' variance from its mean. It sheds some light on the portfolio's total risk. It is deduced by measuring the variances and mutual correlation between a portfolio's assets. Suppose we have a portfolio with n assets, these assets will have the random return rates $r_{\varrho 1}, r_{\varrho 2}, ..., r_{\varrho n}$ such that their expected values are $E(r_{\varrho 1}) = \bar{r}_{\varrho 1}$, $E(r_{\varrho 2}) = \bar{r}_{\varrho 2}$,...,$E(r_{\varrho n}) = \bar{r}_{\varrho n}$. If we create a portfolio ϱ using the weights $w_i, i = 1, 2..., n$ then the rate of return on individual assets can be described by $r_\varrho = w_1 r_{\varrho 1} + w_2 r_{\varrho 2} ... + w_n r_{\varrho n}$. Using the principle of linearity, we can take both sides' expected values to obtain $E(r_\varrho) = w_1 E(r_{\varrho 1}) + w_2 E(r_{\varrho 2}) ... + w_n E(r_{\varrho n})$. From this, the rate of return of the portfolio's variance, denoted by σ_ϱ^2, can be determined by considering asset i with asset j's covariance of returns, which is denoted by σ_{ij} and can be illustrated as follows:

$$\begin{aligned}
\sigma_\varrho^2 &= E\Big[(r_\varrho - \bar{r_\varrho})^2\Big] \\
&= E\Bigg[\Bigg(\sum_{i=1}^{n} w_i r_{\varrho i} - \sum_{i=1}^{n} w_i \bar{r_{\varrho i}}\Bigg)^2\Bigg] \\
&= E\Bigg[\Bigg(\sum_{i=1}^{n} w_i (r_{\varrho i} - \bar{r_{\varrho i}})\Bigg)\Bigg(\sum_{j=1}^{n} w_j (r_{\varrho j} - \bar{r_{\varrho i}})\Bigg)\Bigg] \\
&= E\Bigg[\sum_{i.j=1}^{n} w_i w_j (r_{\varrho i} - \bar{r_{\varrho i}})(r_{\varrho j} - \bar{r_{\varrho j}})\Bigg] \\
&= \sum_{i.j=1}^{n} w_i w_j \sigma_{ij}.
\end{aligned}$$

From this outcome, we can see that the return variance of a portfolio is suppose that there are n mutually uncorrelated assets and each individual asset's rate of return has a mean m and variance $\sigma_{\varrho i}^2$, then we can construct a portfolio from equal portions of these assets; $w_i = \frac{1}{n}$ for each i. Then the overall rate of return of this portfolio is $r_{\varrho i} = \frac{1}{n} \sum_{i=1}^{n} r_{\varrho i}$. The mean value m, is independent of n and the corresponding variance is $var(r) = \frac{1}{n^2} \sum_{i=1}^{n} \sigma^2 = \frac{\sigma^2}{n}$.

This indicates that the variance decreases if individual returns are uncorrelated and if n increases. Conversely, it is more challenging to reduce the portfolio's variance if the returns are positively correlated. In this regard, when investors use the mean-variance approach, which was developed by Harry [83], they must make a trade-off between the variance explicit and the mean.

1.2.3 Random Walks and Wiener Process

A continuous-time model will emerge if the period length is shortened and the limit considered as this exact length goes to zero. By doing so, a special function of time is produced called the random walks and Wiener processes as shown by (see Luenberger [80]). Suppose that there are n periods of length Δt, we can define an additive process W such that $W(t_{k+1}) = W(t_k) + \epsilon(t_k)\sqrt{\Delta t}$, where $t_{k+1} = t_k + \Delta t$ for $k = 0, 1, 2, ..., n$ and $\epsilon(t_k)$ is a normal random variable. Elements in $\epsilon(t_k)$ are mutually uncorrelated, i.e., $E[\epsilon(t_j)\epsilon(t_k)] = 0$ for $j \neq k$. This process is called random walk which is started by setting $W(t_0) = 0$ from which a path is formed according to the behavior of the random variable $\epsilon(t_k)$ (see Durrett [52]).

Definition 1.23 ([103], [139], Definition 1.1.12)**.** A Wiener process $W(t)$ or Brownian motion has the following properties:

(i). For any $s < t$, the quantity $W(t) - W(s)$ is a normal random variable with mean zero and variance $t - s$;

(ii). $W(t_0) = 0$, with probability 1;

(iii). For any $0 \leq t_1 < t_2 \leq t_3 < t_4$, the random variables $W(t_2) - W(t_1)$ and $W(t_4) - W(t_3)$ are independent;

(iv). The process $W(t)$ has continuous trajectories.

1.2.4 Generalized Wiener Processes and Itô Process

Many natural processes behave like Wiener process or Brownian motion. We can generalize these processes by adding noisy observations in an ordinary differential equation, which can be extended such that,

$$dX(t) = \mu(t)dt + \sigma(t)dW(t), \tag{1.2.3}$$

where each t has a stochastic variable $X(t)$ with μ and σ being constants and $W(t)$ is a Wiener process.
It is worth noting that we can derive white noise from Brownian motion, but that derivative is ordinarily non-existent (see Hull [66]). Additionally, we can determine an analytic solution by integrating both sides. More precisely,

$$\int_0^t dX(s) = \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW(s), \tag{1.2.4}$$

$$X(t) = X(0) + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW(s), \tag{1.2.5}$$

$$X(t) = X(0) + \mu t + \sigma W(t). \tag{1.2.6}$$

An Itô process is somewhat more general and its equation takes the form

$$dX(t) = \mu(X,t)dt + \sigma(X,t)dW(t), \tag{1.2.7}$$

where $W(t)$ denotes a Wiener process, $\mu(X,t)$ and $\sigma(X,t)$ may depend on X and t.

1.2.5 Itô's Lemma

Fundamental Lemma: Itô's Lemma plays a crucial role in the stochastic calculus, particularly in the modeling of random processes. The lemma provides a method to differentiate and integrate functions of stochastic processes. The following lemma, as stated in Luenberger [80], is pivotal in understanding this concept.

Lemma 1.24 ([80], Lemma 11.2)**. *Definition of Itô Process:*** *Consider the Itô process defining the random process X:*

$$dX(t) = \mu(X,t)dt + \sigma(X,t)dW(t),$$

where $W(t)$ represents a standard Wiener process.

Itô's Equation: *Suppose there exists a process $y(t)$, defined by the function $y(t) = F(x,t)$. The process $y(t)$ adheres to Itô's equation if and only if the following condition is met:*

$$dy(t) = \left(\frac{\partial F}{\partial x}\mu + \frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2} \right) dt + \frac{\partial F}{\partial x}\sigma dW(t). \tag{1.2.8}$$

This lemma is instrumental in the analysis of stochastic differential equations, providing a framework for understanding the dynamics of processes influenced by random fluctuations.

1.2.6 Martingale Analysis

Martingales are a pivotal concept in the realm of discrete stochastic variables. Characterized by their unique property where the expected future value at any given time equals the present value, irrespective of past values, they form the backbone of understanding stochastic processes and the notion of fair games. The foundational aspects of martingales encompass a sequence of random discrete variables within a measurable mapping and a defined probability space. For in-depth theoretical underpinnings, the works of Ethier and Kurtz [51], along with Rolski et al. [119], are seminal.

Information Flow and Conditional Expectation

Consider the flow of information, denoted as $\{F_t\}_{t\geq 0}$. In this context, for any stochastic variable X, the expected value based on the information at time t becomes crucial in Bjork [22],

$$E[X|F_t],$$

which signifies the expected value of X considering the information available at time t (see Bjork [22]).

Core Propositions in Martingale Theory

Propositions: Central to the understanding of martingales are the following propositions (see Bjork [22]):

Proposition 1.25 ([22], Proposition 4.5). *For stochastic variables X and Y, where Y is F_t-Measurable, the equation*

$$E[Y \cdot X|F_t] = Y \cdot E[X|F_t].$$

is upheld.

Proposition 1.26 ([22], Proposition 4.5). *Given X as a stochastic variable and for times s and t with $s < t$, the following relationship is maintained:*

$$E[E[X|F_t]|F_s] = E[X|F_s].$$

The first proposition deals with the conditioning of $E[X \cdot Y|F_t]$ on all available information at time t. If X is part of F_t, it is treated as a constant in the conditional expectation. The second proposition, known as the "law of iterated expectations," parallels the law of total probability.

Defining Martingale

Definition: A stochastic process X is termed an F_t-martingale if it adheres to the following conditions as in Bjork [22]:

(i) X is in alignment with the filtration $\{F_t\}_{t \geq 0}$.

(ii) For every t, the condition $E[|X(t)|] < \infty$ holds true.

(iii) For any s and t with $s \leq t$, the equation $E[X(t)|F_s] = X(s)$ is valid.

Beyond Martingale: Supermartingales and Submartingales

Beyond martingales, the concepts of supermartingales and submartingales are significant. A supermartingale, defined as a uniformly integrable stochastic process, satisfies the inequality:

$$E[X(t)|F_s] \leq X(s),$$

for all s and t with $s \leq t$ (see Najim, Ikonen and Daoud [99]).

In contrast, a submartingale, which is uniformly differentiable, adheres to the inequality:

$$E[X(t)|F_s] \geq X(s),$$

for all s and t with $s \leq t$.

1.2.7 Bernoulli Distribution

This distribution describes data that emerges from Bernoulli trials, experiments which can result in either success $x = 1$, or failure $x = 0$. Its short-form is $X \sim \text{Bernoulli}(p)$ to describe that a stochastic variable X has a parameter p, where $0 < p < 1$, and is Bernoulli distributed. The probability density function of such a variable with success probability p has the probability mass function

$$f(x) = p^x(1-p)^{1-x}, \quad x = 0, 1$$

for $0 < p < 1$. The cumulative distribution function of $X \sim \text{Bernoulli}(p)$ is

$$F(x) = P(X \leq x) = \begin{cases} 0 & x < 0 \\ 1-p & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

The median of X is 0 if $p < 0.5$ and 1 if $p > 0.5$. If $p = 0.5$, both 0 and 1 can be considered medians. The moment generating function of X is

$$M(t) = E[e^{tX}] = (1-p) + pe^t, \quad -\infty < t < +\infty.$$

The population mean of X is

$$\text{Mean} = E[X] = p, \quad \text{Variance} = V[X] = p(1-p),$$

$$\text{Skewness} = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] = \frac{1-2p}{\sqrt{p(1-p)}},$$

$$\text{Kurtosis} = E\left[\left(\frac{X-\mu}{\sigma}\right)^4\right] = \frac{3p^2-3p+1}{p(1-p)}.$$

(see Kwok [75])

1.2.8 Binomial Distribution

This is a probability distribution that describes the successes of n mutually independent Bernoulli trials, each with a probability of success p. Its short-form is $X \sim \text{binomial}(n, p)$, indicating that X, a stochastic variable, satisfies $0 < p < 1$, and has the binomial distribution for positive integer parameter n and real p satisfying $0 < p < 1$. The probability mass function of X is

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \ldots, n.$$

The moment generating function of X is

$$M(t) = E[e^{tX}] = ((1-p) + pe^t)^n, \quad -\infty < t < \infty.$$

The mean, variance, skewness, and kurtosis are given by

$$\text{Mean} = E[X] = np, \quad \text{Variance} = V[X] = np(1-p),$$

$$\text{Skewness} = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] = \frac{1-2p}{\sqrt{np(1-p)}},$$

$$\text{Kurtosis} = E\left[\left(\frac{X-\mu}{\sigma}\right)^4\right] = 3 + \frac{1-6p(1-p)}{np(1-p)}.$$

(see Johnson, Kemp and Kotz [68])

1.2.9 Normal Distribution

The Normal distribution, often referred to as the Gaussian distribution, is a fundamental distribution in statistics and probability theory. It is characterized by its symmetric, bell-shaped curve centered around the mean, and is widely used in various fields due to its natural occurrence in many phenomena. The key features of the Normal distribution are as follows:

(i) The distribution curve is symmetric and bell-shaped, reflecting the equal distribution of data around the mean.

(ii) The distribution is continuous across all real numbers, with every interval having a non-zero probability, denoted as $-\infty \leq X \leq +\infty$.

(iii) The shape of the distribution is determined by the mean (μ) and standard deviation (σ), making it a family of distributions.

(iv) The probability density function of the Normal distribution is given by

$$f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

(v) Approximately 68.26% of the data falls within one standard deviation from the mean, i.e., $P(\mu - \sigma \leq X \leq \mu + \sigma) = 0.6826$.

(vi) About 95.44% of the data lies within two standard deviations from the mean, expressed as $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) = 0.9544$.

(see Kwok [75])

1.2.10 Log-normal Distribution

The Log-normal distribution is a probability distribution of a random variable whose logarithm is normally distributed. It is particularly useful in modeling phenomena where the variable is constrained to be positive, such as stock prices or lifetimes of components. The Log-normal distribution is denoted as $X \sim$ Log-normal(μ, σ^2), where μ and σ are the mean and standard deviation of the variable's natural logarithm. The probability density function of the Log-normal distribution is given by:

$$f(x;\mu,\sigma) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}}, \quad x > 0$$

$$\text{Mean} = E[X] = e^{\mu+\frac{\sigma^2}{2}}, \quad \text{Variance} = V[X] = (e^{\sigma^2}-1)e^{2\mu+\sigma^2}$$

$$\text{Skewness} = (e^{\sigma^2}+2)\sqrt{e^{\sigma^2}-1}, \quad \text{Kurtosis} = e^{4\sigma^2} + 2e^{3\sigma^2} + 3e^{2\sigma^2} - 6$$

(see Bowman[26])

1.2.11 Poisson Distribution

The Poisson distribution is a discrete probability distribution that expresses the probability of a given number of events occurring in a fixed interval of time or space if these events occur with a known constant mean rate and independently of the time since the last event. The Poisson distribution is commonly denoted as $X \sim \text{Poisson}(\lambda)$, where λ is the event rate. The probability mass function of this distribution is given by:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, \ldots$$

$$\text{Mean} = E[X] = \lambda, \quad \text{Variance} = V[X] = \lambda$$

$$\text{Skewness} = \frac{1}{\sqrt{\lambda}}, \quad \text{Kurtosis} = \frac{1}{\lambda} + 3$$

(see Ross[118])

1.2.12 Exponential Distribution

The Exponential distribution is commonly used to model the time between events in a Poisson process. It is defined by the equation:

$$f(x;\lambda) = \lambda e^{-\lambda x}, \quad x \geq 0$$

$$\text{Mean} = E[X] = \frac{1}{\lambda}, \quad \text{Variance} = V[X] = \frac{1}{\lambda^2}$$

$$\text{Skewness} = 2, \quad \text{Kurtosis} = 9$$

(see Ross[118])

1.2.13 Weibull Distribution

The Weibull distribution is a flexible life distribution model used in reliability and life data analysis. It is defined by the equation:

$$f(x;\lambda,k) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k}, \quad x \geq 0$$

$$\text{Mean} = E[X] = \lambda\Gamma(1+\frac{1}{k}), \quad \text{Variance} = V[X] = \lambda^2[\Gamma(1+\frac{2}{k})-(\Gamma(1+\frac{1}{k}))^2]$$

(see Rinne [114])

1.2.14 Geometric Distribution

The Geometric distribution models the number of Bernoulli trials needed to get one success. It is defined by the equation:

$$f(k;p) = (1-p)^{k-1}p, \quad k \in \{1,2,3,...\}$$

$$\text{Mean} = E[X] = \frac{1}{p}, \quad \text{Variance} = V[X] = \frac{1-p}{p^2}$$

(Johnson, Kemp and Kotz [68])

1.2.15 Pareto Distribution

The Pareto distribution, named after the economist Vilfredo Pareto, is used to describe the distribution of wealth. It is defined by the equation:

$$f(x;x_m,\alpha) = \alpha x_m^{\alpha} x^{-\alpha-1}, \quad x \geq x_m$$

$$\text{Mean} = E[X] = \frac{\alpha x_m}{\alpha - 1}, \quad \text{Variance} = V[X] = \frac{x_m^2 \alpha}{(\alpha-1)^2(\alpha-2)}$$

(see Arnold [4])

1.2.16 Gompertz Distribution

The Gompertz Distribution is a significant statistical tool, widely recognized for its effectiveness in modeling and analyzing time-dependent phenomena across various research domains. It is defined by the equation:

$$f(x;\eta,b) = b\eta e^{bx} e^{-\eta(e^{bx}-1)}, \quad x \geq 0$$

$$\text{Mean} = E[X] = \int_0^\infty x f(x) dx, \quad \text{Variance} = V[X] = \int_0^\infty x^2 f(x) dx - (\text{Mean})^2$$

(see Johnson, Kotz and Balakrishnan [67])

1.2.17 Volatility

Volatility denoted by σ is substantial degree of the scattering of returns stock. In most cases, high volatility implies the stock is less secure (see Kwok [76]). It is measured using the standard deviation or variance. In the stock markets, volatility is attributed to huge fluctuations in the price movement of an asset, commodity, class, or derivative. For example, if we consider a stock X whose asset price rises and falls by more than one percent over a sustained period of time, we can define this as a volatile market and analysts use an asset's volatility to choose contracts. The two frequently techniques with regard to analyzing the volatility of an asset and choosing contracts is to use either the historical or implied volatilities (see Bjork[22]).

Historic volatility

This technique uses the historical data of a particular asset's period to estimate its volatility for a period with a similar length. The rationale for using this technique is that the volatility of most assets fluctuates over long periods of time. Mathematically, it implies observing the discrete equidistant points $t_0, t_1, ..., t_n$ so as to obtain σ of a stock process at $n+1$. Further, we can deduce that the difference between two equidistant points denoted by Δt is the sampling interval, i.e., $\Delta t = t_i - t_{i-1}$. Using this technique, analysts also observe $S(t_0), ..., S(t_n)$ so as to estimate σ, since S has a log-normal distribution. Given $\xi_1, ..., \xi_n$ such that.

$$\xi_i = \ln\left(\frac{S(t_i)}{S(t_{i-1})}\right).$$

ξ_i are independent, normally distributed random variables with

$$E[\xi_i] = (\alpha - \frac{1}{2}\sigma^2)\Delta t,$$

$$Var[\xi_i] = \sigma^2 \Delta t.$$

Thence, we can approximate σ with the equation:

$$\sigma = \frac{S_\xi}{\sqrt{\Delta t}},$$

and determine the sample variance S_ξ^2 by

$$S_\xi^2 = \frac{1}{n-1}\sum_{i=1}^{n}(\xi_i - \xi)^2, \tag{1.2.9}$$

$$\bar{\xi} = \frac{1}{n}\sum_{i=1}^{n}\xi_i. \tag{1.2.10}$$

Implied Volatility

If the objective is to price stock concerning other assets already priced by the market, then the future market expectation of the volatility is very effective. We can determine the market expectation by observing and approximating future market price data using "benchmark" options.

The latter must be written on similar underlying stocks as those under valuation. In this sense, the implied volatility is the scattering of returns implied by the benchmark option (see Bjork [22]).

The (p, q)-coefficient

The (p, q)-coefficient, a dynamic mathematical concept, denoted by the parameter pair (p, q), enhances adaptability in mathematical models, contributing to precision and flexibility. It introduces adaptability to diverse models, enhancing their robustness in addressing real-world complexities. Formally expressed as a parameter pair (p, q), this coefficient significantly influences the flexibility and precision of mathematical models. Its role in refining accuracy and adaptability is highlighted across various mathematical frameworks.

1.2.18 Cox-Ross-Rubinstein Model

Generally, this model states that the value of American options in discrete markets can be understood by dividing them into small-time lengths of length Δt using a binomial lattice, which can either go up or down by a fixed factor. This behavior is then repeated n times, and a total of 2^n times, and the possible terminal values of the underlying securities are determined at each node Wrigglesworth [148]. The discounted rate i is then obtained by moving backward through the tree and determining the value of each node $V_{j,n}$, which is repeated until the last node is reached. This behavior can be represented by a binomial tree.

Consider a stock whose initial price is S. After the interest rate period;

its price can either move up uS with probability q or down dS with probability $1-q$. Hypothetically, if $0 < d < u$, and a common risk-free interest rate r applies for every borrowing or lending opportunity in the markets, then $R = 1 + r$. This also implies that one must have $d < R < u$ to avoid arbitrage. Hence, the risk-free value is a deterministic degenerate derivative of the stock as seen in Luenberger [80].

If $R \geq u > d$ and that $0 < q < 1$, then this implies that the stock has much poor returns than the risk-free asset. Hence, one could short the stock and use the returns to loan others. This way, their profits will be $R - u$ or $R - d$. In this case, the profit is positive, but the setup cost is zero. Investors cannot achieve such feats without arbitrage opportunities. Also, suppose a call option on the stock has a strike price K; we can use an argument without arbitrage to find its value at expiration. The risk-free value is deterministic. If the value of S is known, then it is also possible to determine other values in the one-step lattice (see Luenberger [80]).

This determination of all values is true, but for the call $C(S,t)$, which can only be deduced from the other values. We can achieve this by constructing a pattern of outcomes, and combining the proportion of the lattices. Such that

$$C_u(S,t) = \max(uS - K, 0), \tag{1.2.11}$$

$$C_d(S,t) = \max(dS - K, 0). \tag{1.2.12}$$

To duplicate the above outcome, we can purchase stock worth x amounts, and a risk-free asset worth b, so that the portfolio will either be $ux+Rb$ or $dx+Rb$. For simplicity, using C_d and C_u instead of $C_d(S,t)$ and $C_u(S,t)$, we have,

$$ux + Rb = C_u, \tag{1.2.13}$$

$$dx + Rb = C_d. \tag{1.2.14}$$

Solving for x and b from Equation 1.2.13 and Equation 1.2.14 we obtain

$$x = \frac{C_u - C_d}{u - d}, b = \frac{uC_u - dC_d}{R(u-d)}. \tag{1.2.15}$$

Combining these two, we find that the value of the portfolio is

$$\begin{aligned}
x + b &= \frac{C_u - C_d}{u-d} + \frac{uC_u - dC_d}{R(u-d)} \\
&= \frac{1}{R}\left(\frac{RC_u - RC_d + uC_d - dC_u}{u-d}\right) \\
&= \frac{1}{R}\left(\frac{(RC_u - dC_u) + (uC_d - RC_d)}{u-d}\right) \\
&= \frac{1}{R}\left(\frac{C_u(R-d) + C_d(u-R)}{u-d}\right) \\
&= \frac{1}{R}\left(\frac{R-d}{u-d}C_u + \frac{u-R}{u-d}C_d\right).
\end{aligned}$$

If we consider q to be the risk-neutral probability by taking the option's expected value, such that $q = \frac{R-d}{u-d}$ and discounting this to find C, then it will take the form of the risk-free rate shown below:

$$C = \frac{1}{R}\left[qC_u + (1-q)C_d\right]. \tag{1.2.16}$$

This procedure can also be used to value other securities. For instance, the formula for the underlying stock, which is the stock of interest, is given by,

$$C = \frac{1}{R}[quS + (1-q)dS]. \tag{1.2.17}$$

The above derivation shows the underlying concepts behind risk pricing, since the underlying stock's price is determined by combining a portfolio with stock and risk-free assets with the option's outcomes.

1.3 Statement of the problem

CRR model has been considered with a lot of extensions binomially for portfolio optimization. One major hindrance to this kind of optimization is the presence of noisy observations, which increases risks that leads to losses, particularly to insurance companies. In this regard, Mathematical models that solve for noisy observations are necessary. Breton, El-Khatib, Fan and Privault gave a q-binomial extension of CRR model. Furthermore, Breton posed an open question of whether a further extension of the CRR model is required to reduce noise in the observation. Therefore, this study carries out a (p,q)- extension of CRR model with noisy observations p as a new parameter.

1.4 Objectives of the study

1.4.1 Main objective

Main objective of this study is to establish (p, q)-binomial extension of Cox-Ross-Rubinstein model for optimization of portfolio with noisy observations in life insurance.

1.4.2 Specific objectives

The specific objectives of this study are to;

(i). Develop a *(p,q)*- extension of CRR model with noisy observations;

(ii). Establish optimization conditions for the extended model with noisy observations;

(iii). Simulate the outcomes of the model in life insurance.

1.5 Significance of the Study

This research, grounded in the Cox-Ross-Rubinstein (CRR) model a cornerstone in the domain of financial derivatives pricing has successfully extended the model's application, specifically catering to the portfolio optimization needs of insurance companies. The study's innovative approach to integrating noisy market data into the model has proven instrumental in facilitating the strategic modeling of optimal portfolios, while judiciously balancing risk considerations.

The implications of this research for the insurance industry are substantial. By leveraging the refined CRR model, insurance companies can now more accurately assess risk levels, strategize portfolio retention under diverse risk conditions, and deepen their understanding of reinsurance dynamics during high-claim periods. The study also sheds light on the critical thresholds of risk tolerance, guiding firms in decisions regarding co-insurance when faced with excessive risk exposures.

Beyond its immediate practical applications, this study marks a significant contribution to the field of financial mathematics. It not only enhances the theoretical framework of the CRR model but also sets the stage for future research endeavors. The findings offer valuable insights into risk management and portfolio optimization strategies, particularly relevant in today's volatile and unpredictable business landscape.

In conclusion, this research has not only achieved its objective of adapting the CRR model for a specific sector but has also opened new pathways for academic inquiry and practical application in financial risk management and investment strategy optimization.

Chapter 2

LITERATURE REVIEW

2.1 Introduction

In this chapter, we review existing literature related to the study. The author consideres a q-extension of the binomial model whose primary setting comes from the CRR model.

2.2 Option Pricing

Options are not recent inventions. They have existed for a long time, and have been priced differently over the years. They date back to the days of the Phoenicians and the Romans, who used contracts that were similar to options in shipping. However, they were only formalized in Japan during the 1650's, where buyers were able to trade on Yodoya rice futures in Osaka [15]. Option contracts were also used as speculative tools during the famous Dutch Tulip Bubble [140]. Nonetheless, option-pricing techniques became available only after Fischer Black, Robert Merton, and Myron

Scholes developed the famous Black-Scholes technique in 1973.

Before they developed their famous formula, option-pricing was considered very elusive. In 1900, Bacchelier (1900) invented Brownian motion to model options based on French bonds, by deriving a normal distribution for stock-price movements using the central limit theorem [9] (see also Davis and Etheridge, [48]). Einstein did not have the idea of Brownian. His work introduced the concept of modeling successive changes in stock prices by assuming that they are independent random variables, that is to say, stock prices have no memory as their present or future price movements are independent of previous valuations as in Boyle [30]. Bacchelier's work assumed a normal distribution in stock prices, such that the variance was proportional to the time period. Nonetheless, despite its importance it was largely overlooked by most economists, until it was circulated by Samuelson [127] and published in English by Cootner [43] (see also Kushwah et al., [74]).

Bacchelier's work had several limitations but it paved the way for option-pricing research. In the 1960's, Samuelson [127] used geometric Brownian motion to model long-term equity bonds at exercise. His work assumed the option's expected rate of return α, and discounted value at exercise β were dependent upon the underlying stocks unique characteristics. This was later disproven as different observers could propose different valuations depending on their risk aversion levels.

In 1972, Black-Scholes was formulated by Black, Merton, and Scholes to solve the challenge of pricing European option contracts based on a different approach. Their work assumed that the underlying stock's price had fixed volatility, and followed the Wiener process. Furthermore, it assumed that the underlying stock did not pay dividend or make other distributions. Their model predicted option price movements significantly, hence, is often considered the first option-pricing formula.

Despite its successes, the Black-Scholes model had several limitations. For one, its proponents assumed a fixed risk-free interest rate and volatility, which was very unrealistic. To this end, several scholars developed continuous time models. It wasn't until 1978 that Sharpe [133] outlined the benefits of discrete-time approaches to option-pricing. Cox, Ross and Rubenstein developed the CRR model (also known as the binomial options pricing model), which was a discrete-time pricing option technique for pricing American options that used a binomial system to model price movements. The CRR model enhanced the simplicity of pricing of complex derivatives, which had only previously been achieved using complex methods. Several studies found the CRR model converged with the Black-Scholes formula, and was able to predict price movements for several complete market structures in the case of continuous sample paths. In 1977, Merton developed a formula that was based on CRR to model discontinuous cases by replacing jump probabilities with Arrow-Debrau prices (see also Madan et al. [81]). Since then, several scholars have derived several models from the CRR technique to model price movements in a variety of conditions. Currently, option pricing is a very important component of finance, as many things can be represented as options.

Beyond the CRR model, two important option pricing techniques have been developed to model the behavior of options by Harrison and Kreps [61] and [62]. The former authors were largely inspired by the concept of risk neutral probabilities in the CRR model, which led them to develop a discounted price technique that focused on equivalent martingale measures. Consequently, this increased the understanding of the risk-neutral approach. On the other hand, Harrison and Pliska [62]extended the former model to include the use of square-integrable martingales in discrete time and finite sample spaces. However, these models are very complex, and cannot be implemented easily. Therefore, this study found the CRR model appropriate.

2.3 Discrete-Time Multi-Period Markets

CRR model was built to model conditions in discrete-time multi-period markets, hence, it is important to understand their characteristics and underlying assumptions. Discrete-time multi-period markets are characterized by trades that are aggregated at discrete time points, and the market price is set at equilibrium (see Musiela nad Rutkowski [97]). That is, the number of dates in which an option is traded is finite. In such markets, the investor is allowed to invest in multiple risky and risk-less assets. Furthermore, an underlying security's price-movements are apparent; that is, one sees what is happening at time period 1 until $t = T$ which can be represented as $T = 0, 1, ...T^n$.
With every price movement, investors know more about the option's terminal value of the option than in subsequent time periods. That is, investors know more about the option's closing price at $t = 1$ than at $t = 2$. This implies that the stock can be visualized as a tree with a root node and a depth T. Given an arbitrary set of points $\Omega = (w_1...w_d)$, and an arbitrary set $F = F_{T^n}$ as the σ -field of all subjects of Ω, we can con-

struct a filtered probability space (Ω, F, P) for multi-period markets such that $\mathbb{F} = [(F)]_{t=0}^{Tn}$ where P is the filtered probability measure such that $P(\omega j) > 0$ for every $j = 1, ...d$ in Musiela and Rutowski [97]. These variables can be used to develop pure and mixed-method trading strategies. Discrete models are mostly used to model call options. For European call options with a stock S, and a terminal date T, the stock's value can be modeled as S_t for any time moment and S_T for the stock's terminal date. According to Khan [71], the option's value can be modeled as: $Y = (S_T - K) = \max(S_T - K, 0)$, where K is the exercise price. Several elements in this equation can be substituted to enable the estimation of call options at the time of expiry C_T, which can be denoted as $C_T = (S_T - K)$.

2.4 CRR model

The CRR model is very popular as it provides simple option pricing methods, unlike most complex continuous-time models used in finance. Its simplicity arises from the assumptions it makes about market movements and returns. First, it assumes the chance that a stock will go up or down is independent of previous price movements. These price movements are then used to construct binomial trees, from which users can calculate the benefits of each node and path. Secondly, it assumes that the probability of an option rising or falling is constant at every time step. This assumption implies that only four parameters are essential in predicting a stock's price, that is, $r > -1, u > 0,\ d > 0$, and $S(0)$.

Finally, it assumes there are no arbitrage opportunities. Simply put, arbitrage implies that a portfolio has both long and short stocks, hence a trader can benefit from risk-free profits by purchasing and selling the same asset at two different prices. The absence of arbitrage restricts the portfolio to equilibrium, such that the asset's price is the same as in the replicating portfolio; hence, no trader can make risk-free profits by purchasing and selling the same asset at two varying valuations. The no-arbitrage assumption made by the CRR model makes it easy for options traders to construct portfolios. Several variations of the CRR model have been introduced to model the behavior of securities in multi-period discrete-time markets.This section analyzes the classical discrete CRR model, the generalized discrete CRR model, and the conditional generalized CRR model.

2.5 Classical Discrete CRR model

The classical discrete CRR model was introduced by Wright [149] and established in Cox, Ross and Rubinstein [46] to model American options for stocks that do not pay any dividends in discrete-time markets. It assumes lattice, which enables investors to potentially earn two varying returns from their movements, i.e., up uS and down dS at every time step as shown in Figure 2.5. The classical CRR model paves the way to option pricing formulas, as it is convergent to the Black-Scholes pricing formula. It constructs the equivalent binomial lattice in a manner similar to the moment matching properties laid forth by Donsker's theorem, which states that as the sequence of random walk goes towards 0, it tends to converge to a Wiener process. CRR models price movements as random geometric walks, which converge towards the Wiener process when time is assumed between 0 and 1 (see Malaeb, Tarhini and Alnouri [82]).

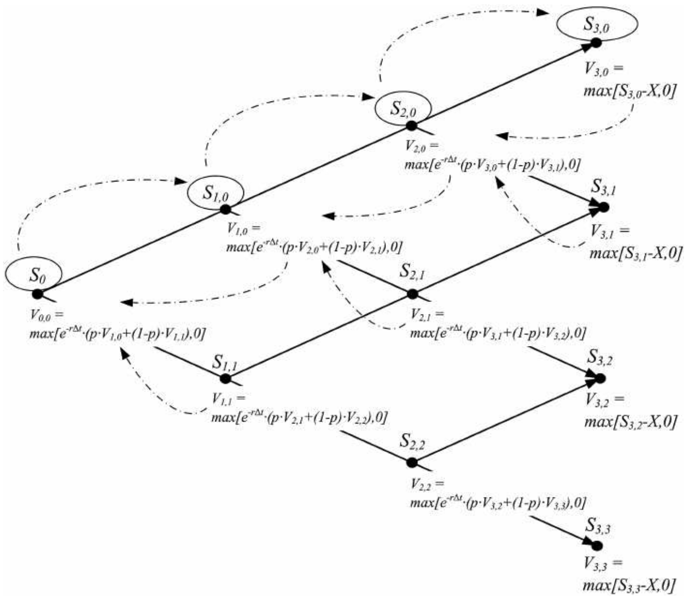

Figure 2.5.1: Binomial Lattice

The CRR model is multiplicative as suggested in Luenberger [80]; with $u > 0$, $d > 0$,because its construction does not allow the stock's price to turn negative. Accordingly, it incorporates a variable called the yearly expected growth rate, which is denoted by v. Consequently $v = E\big[\ln(\frac{S(T)}{S(0)})\big]$ where $S(0)$ refers to the starting price, and $S(T)$ is the price after 1 year. Likewise, we can also incorporate the stock's yearly standard deviation, σ, such that $\sigma^2 = var\big[\ln(\frac{S(T)}{S(0)})\big]$. Consequently, the binomial lattice's parameters can be denoted as:

$$\begin{aligned} p &= \frac{1}{2} + \frac{1}{2}(\frac{v}{\sigma})\sqrt{\Delta t}, \\ u &= e^{\sigma\sqrt{\Delta t}}, \\ d &= e^{-\sigma\sqrt{\Delta t}}. \end{aligned}$$

Where Δt is the chosen time period chosen, but is smaller than one. Given the choices above, we can deduce that the binomial model will approximate the values of v and σ.

In Equation 1.2.6.12, a quantity q which is a probability, was defined as $q = \frac{R-d}{u-d}$ to approximate a one-period call option's price whose underlying asset is modeled using the binomial lattice. In this sense q denotes the risk-neutral probability, which can be deduced by ensuring that the underlying stock's risk-neutral formula is such that Luenberger [80]:

$$S = \frac{1}{R}[quS + (1-q)dS].$$

2.6 Multiperiod options

Acoording to Fares [53] the underlying stock's initial value can fluctuate depending on its up or down movement through the lattice, and should be analogous to the end point in the binomial. That is to say, the final nodes prices the option, such that: $C_{uu} = \max(u^2S - K, 0)$, $C_{ud} = \max(udS - K, 0)$, $C_{dd} = \max(d^2S - K, 0)$ and q is defined again as $q = \frac{R-d}{u-d}$ where

R represent's the asset's single-period risk-free return. Here, C_u and C_d represent the option's price, C, and can be represented by applying them to the risk-neutral discounting formula, such that:

$$C = \frac{1}{R}[qC_u + (1-q)C_d]. \tag{2.6.1}$$

The same technique can be applied to model the behavior of lattice with more periods. By working backward from the final period, analysts can repeatedly compute the option's value through single-period risk-free discounting at all nodes in Hull [66].

The classical CRR model has also been found to converge to the Black-Scholes formula. According to Cox, Ross and Rubinstein [46], the former can be represented as:

$$C = S\Phi[a; n; p\prime] - Kr^{-\hat{n}}\Phi[a; n; p\prime],$$

where, C is the n-period option's value, S is the stock's price, $\Phi[a; n; p\prime]$ is the probability that the sum of n-random variables, each of which can take the value of 1 with probability p, and 0 with probability $1-p$, a is the tiniest non-negative integer greater than $\frac{\log(\frac{K}{dS})}{\log(\frac{u}{d})}$, $p = \frac{(r-d)}{(u-d)}$ and $p\prime = (\frac{u}{r})p$, K is the exercise price and r is the risk-free rate of return.

The authors noted that this formula converged to the Black-Scholes formula, when t was subdivided into intervals and d, u, $\hat{r}$, and q are chosen such that a lognormal multiplicative binomial probability distribution. That is; The Black-Scholes formula can be represented as:

$$C = SN(x) - Kr^{-1}N(x - \sigma\sqrt{t});$$

where,

$$x \equiv \frac{\log(\frac{S}{Kr^{-1}})}{\sigma\sqrt{t}} + \frac{1}{2}\sigma\sqrt{t}.$$

There are several similarities between the Black-Scholes and the Cox-Rubenstein. For example, Cox et al. state that r^{-1} can be substituted for $\hat{r}^{-n}$. The authors also note that the random value's of the complement of $\Phi[a;n;p\prime]$ has a mean np and standard deviation $\sqrt{np(1-p)}$, which can be represented as

$$1 - \Phi[a;n;p\prime] = Prob\left[\frac{j-np}{\sqrt{np(1-np)}} \leq \frac{a-1-np}{\sqrt{np(1-np)}}\right].$$

We can apply this formula to a stock S with upside probability p, and downside probability $1-p$, which changes the stock's price to uS or dS, such that the mean of the stock is

$$\hat{\mu} = p(\log\frac{u}{d}) + \log d,$$

while the variance of the rate of return is

$$\sigma_p^2 = p(1-p)[\log(\frac{u}{d})^2].$$

Applying these equations, they find that

$$\frac{j-np}{\sqrt{np(1-np)}} = \frac{log(\frac{s*}{s}) - \hat{\mu_p}n}{\hat{\sigma_p}\sqrt{n}}.$$

Substituting $a-1$ (from the binomial formula) into the formula, we can develop an equation with the form

$$\frac{a-1-np}{\sqrt{np(1-np}} = \frac{log(\frac{K}{S}) - \hat{\mu}_p n - \varepsilon log(\frac{u}{d})}{\hat{\sigma}_p \sqrt{n}}.$$

Hence,

$$1-\phi[a;n;p\prime] = prob\left[\frac{log(\frac{s*}{s}) - \hat{\mu}_p n}{\hat{\sigma}_p \sqrt{n}} \leq \frac{log(\frac{K}{S}) - \hat{\mu}_p n - \varepsilon log(\frac{u}{d})}{\hat{\sigma}_p \sqrt{n}}\right].$$

The central limit can then be applied to this equation to evaluate the parameters $\hat{\mu}_p n, \sigma_p^2$,and $log(\frac{u}{d})$, thus showing that $log(\frac{u}{d}) \to 0$ can be represented as $n \to \infty$, $\hat{\mu}_p\ n \to (logr\frac{1}{2}\sigma^2)t$, and $\hat{\sigma}_p\sqrt{n} \to \sigma\sqrt{t}$, which are properties of the Black-Scholes formula.
Cox et al. found that,

$$\frac{log(\frac{K}{S}) - \hat{\mu}_p n - \varepsilon log(\frac{u}{d})}{\hat{\sigma}_p \sqrt{n}} \to z = \frac{log(\frac{K}{S}) - (logr - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}},$$

such that $1-\phi[a;n;p\prime] \to N(z)$ could be represented as:

$$N\left[\frac{log(\frac{Kr^{-t}}{S})}{\sigma\sqrt{t}} + \frac{1}{2}\sigma\sqrt{t}\right].$$

And its complement,
$\phi[a;n;p\prime] \to N(-z) = N\left[\frac{log(\frac{Kr^{-t}}{S})}{\sigma\sqrt{t}} - \frac{1}{2}\sigma\sqrt{t}\right] = N(x-\sqrt{t})$, which is a function within the Black-Scholes formula. This way, the authors proved that the Black-Scholes formula acted as a limiting case in the classical CRR model.

2.7 Generalized CRR model in Discrete-Time Markets

The classical CRR model revolutionized the option pricing process. However, some of its assumptions did not model reality. For example, the model assumed that volatility and risk-free interest rates were constant. As a result, the model failed in cases where uncertainties influenced stock price movements. To solve these issues, scholars developed the Generalized CRR model. This model mostly arose out of the work of [71], such that the underlying asset's price changes by multiplication of the values u and d at different values and periods, allowing stock price changes to be modeled by the values $\mu[X(t)]_{(t\leq T)}$, and $\sigma[X(t)]_{t\leq T}$. Generalized CRR models the stock's value at time t with the equation,

$$S(t) = \xi_{(t-1)}S_{(t-1)}, \tag{2.7.1}$$

for all values of $t \leq T$, where, $S(0) = \xi(0)S(0)$ and $S(0)$ is a positive constant, and $\xi(t) = X(t)v(t)$, for all values of $t \leq T$, $[X_t]_{(t\leq T)}$ and $[(v(t))]_{(t\leq T)}$ are mutually independent variables, with $[v(t)]_{t\leq T}$ comprise of Bernoulli random values u and d with probability p and $1-p$ respectively. The two variables share the same probability space (Ω, F, P).
This gives us the probability $\mathbb{P}\{X(t)v(t) = \frac{xu}{X(t)=x}\} = p = 1-\mathbb{P}\{X(t)v(t) = \frac{xd}{X(t)} = x\} \quad \forall t \leq T^*$. the model introduces a string of stochastic variables $\Gamma m = (1..., m)$, such that $m \in \mathbb{N}$ to give the random set

$$I_{j,m}(x) = \left\{ J \subset \Gamma_m, |J| = j, x \prod_{k\in J} \xi^u_{T-k} \prod_{k\in J} \xi^d_{T-k} > K \right\},$$

where $|J|$ stands for the cardinality of the set. Using this random set we can derive

$$p_{T-k} = \frac{\hat{r} - \xi^d_{T-k}}{\xi^u_{T-k} - \xi^d_{T-k}}, \quad \bar{p}_{T-k} = \frac{\xi^u_{T-k}}{\hat{r}} p_{T-k},$$

$$q_{T-k} = 1 - p_{T-k}, \quad \bar{q}_{T-k} = 1 - \bar{p}_{T-k}.$$

Which can be shortened to

$$\bar{P}(J^{(j)}, T) = \prod_{k \in J} \bar{p}_{T-k} \prod_{k \notin J} \bar{q}_{T-k},$$

$$P(J^{(j)}, T) = \prod_{k \in J} p_{T-k} \prod_{k \notin J} \bar{q}_{T-k},$$

$$\Xi(J^{(j)}, T) = \prod_{k \in J} \xi^u_{T-k} \prod_{k \notin J} \xi^d_{T-k}.$$

Kan [71] showed that from this set of equation, we can derive the CRR of an option pricing model such that

$$C_{T-m} = S_{T-m} \sum_{j=0}^{m} \Bigg(\sum_{J \in I_{j,m}(S_{T-m}} \bar{P}(J^{(j)}, T) - \frac{K}{\hat{r}^m} \sum_{J \in I_{j,m}(S_{T-m}} P(J^{(j)}, T) \Bigg),$$

where $m = \{0...T\}$.

To prove this equation, Kan [71] used a method of Mathematical induction of the equation of an option $C_T = (S_T - K)^+$, such that $m = 0$, and option price C_T. The author then showed that the European call option's price C_{T-m}, at time $T-m$, and a portfolio $\phi_{T-m-1} = (\alpha_{T-m-1}, \beta_{T-m-1})$ at time $T-m-1$, where, α_{T-m-1} represents the number of shares held at time $T-m-1$ and β_{T-m-1} is the cash investment at time $T-m-1$ could satisfy the relation, $C_{T-m} = V_{T-m}(\phi)$ where $V_{T-m} = \alpha_{T-m-1}S_{T-m} + \beta_{T-m-1}\hat{r}$. Kan [71] then noted that one could derive the following system of equations for a stock price that could move either by a factor u and d,

$$\begin{cases} \alpha_{T-m-1}\xi^u_{T-m-1}S_{T-m-1} + \beta_{T-m-1}\hat{r} = C^{\xi^u_{T-m-1}}_{T-m}, \\ \\ \alpha_{T-m-1}\xi^d_{T-m-1}S_{T-m-1} + \beta_{T-m-1}\hat{r} = C^{\xi^d_{T-m-1}}_{T-m}, \end{cases}$$

such that

$$C^{\xi^u_{T-m-1}}_{T-m} = \frac{1}{\hat{r}^m}\sum_{j=0}^{m}\sum_{J\in I_{j,m}(\xi^u_{T-m-1}S_{T-m})} P(J^{(j)},T)(\xi^u_{T-m-1}S_{T-m-1}\Xi^{u,d}(J^{(j)},T) - K),$$

$$C^{\xi^d_{T-m-1}}_{T-m} = \frac{1}{\hat{r}^m}\sum_{j=0}^{m}\sum_{J\in I_{j,m}(\xi^d_{T-m-1}S_{T-m})} P(J^{(j)},T)(\xi^d_{T-m-1}S_{T-m-1}\Xi^{u,d}(J^{(j)},T) - K).$$

Since $t \leq T$ and $d < u$ and $\xi^u_t > \xi^d_t$, then the set $I_{j,m}(\xi^d_{T-m-1}S_{T-m-1}) \subseteq I_{j,m}(\xi^u_{T-m-1}S_{T-m-1})$.

Additionally, we can derive that the number of shares at time $T-m-$

$1(\alpha_{T-m-1})$ can be derived by equation,

$$= \frac{C_{T-m}^{\xi_{T-m-1}^u} - C_{T-m}^{\xi_{T-m-1}^d}}{S_{T-m-1}(\xi_{T-m-1}^u - \xi_{T-m-1}^d)},$$

while the cash investment at time $T-m-1$

$$= \beta_{T-m-1} = \frac{\xi_{T-m-1}^u C_{T-m}^{\xi_{T-m-1}^d} - \xi_{T-m-1}^d C_{T-m}^{\xi_{T-m-1}^u}}{\hat{r}(\xi_{T-m-1}^u - \xi_{T-m-1}^d)}.$$

Substituting the two gives,

$$\begin{aligned} C_{T-m-1} &= \alpha_{T-m-1} S_{T-m-1} + \beta_{T-m-1} = \frac{1}{\hat{r}}\big(C_{T-m-1}^{\xi_{T-m-1}^u} p_{T-m-1} + C_{T-m-1}^{\xi_{T-m-1}^d} q_{T-m-1}\big) \\ &= \frac{1}{\hat{r}^{m+1}} \sum_{j=0}^{m+1} \sum_{J \in I_{j,m+1}(S_{T-m-1})} P(J^{(j)}, T)(S_{T-m-1}\Xi^{u,d}(J^{(j)}, T) - K)^+, \end{aligned}$$

thus proving the proposition.

2.8 Conditional Generalized CRR model in Discrete-Time Markets

The conditional generalized CRR model is an approach to binomial pricing, which considers the multinomial distribution of random variables in its assessment of the underlying securities. The model is derived from the generalized CRR, and employs the use of multinomiality parameters. It was developed by Kan [71], who noted that stock price could be modeled not only by multiplying the underlying price with the parameters u and d, but also embedding the process with $\{X(t)\}_{t<T}$, thus, changing the

way the stock price can be modeled using the values $uX(t)$ and $dX(t)$. Kan [71] also considered a set of random variables $C_k = c_1..., c_k$ $k \in \mathbb{N}$, $c_i > 0$ and all values $i = 1..., k$, experiencing the events $\{X(t) = c_j\}$, $t = 1..., T$, $j = 1..., k$. The researcher noted that the probabilities $p_1 = ... = p_k = \frac{1}{k}, c_j$ were equally likely events in the set; hence, were multinomiality parameters.

From them, [71] developed a random set of events $X_{T-m}, ..., X_T$, and used it to calculate the mean of conditional expectation C_{T-m}, to find the corresponding mean's price, such that, $\tilde{C}_{T-m} = E\{E\{C_{T-m}|X_{T-m}, ..., X_T\}\} = E(C_{T-m})$ and $X_{T-m}(\omega), ..., X_T(\omega)$ is a sequence of random variables that can be denoted as $N_{c_1}, ..., N_{c_k}$, and satisfies the condition $N_{c_1} + ... + N_{c_k} = m$. Kan [71] also applied the local Richter limit theorem to determine the option's price limit. The local Richter limit theorem states that amultinomial random variable's probability distribution limit is a multivariate normal distribution. That is,

$$\lim_{n \to \infty} \sup_{(x_1, ..., x_k) \in G_n} \left| p_n(n_1, ..., n_k)(2\pi n)^{(k-1)/2}(p_1, ..., p_k)^{\frac{1}{2}} e^{\frac{1}{2}\sum_{j=1}^{k} \frac{nx_j^2}{p_j}} - 1 \right| = 0.$$

Where G_n represents a set of points $x = (x_1, ..., x_k) \in R$ for which $|x_j| \leq A_{n-\beta}$ for $j + 1, ..., k$, where $A > 0$ and $\frac{1}{3} < \beta < \frac{1}{2}$ are randomly chosen (but fixed afterwards).

$$P(Z_n = (n_1, ..., n_k)) = p_n(n_1, ..., n_k) \sim \frac{1}{\sqrt{2\pi} n^{k-1} \sqrt{p_1 ... p_k}} e^{1\frac{1}{2}\sum_{j=1}^{k} \frac{nx_j^2}{p_j}}$$

is likelihood of the multinomial distribution of X stochastic variables that can assume k different values, which can be modeled as $\beta_1, ..., \beta_k$

with probabilities $P(X = \beta_j) = pj,\ j = 1, ..., k$, and parameters n and $p = (p_1, ..., p_k)$, while $(x_1, ..., x_k) = \frac{1}{n}(n1 - np1), ..., nk - npk) \in G_n$ and $n \to \infty$. Using this theorem, Kan [71] found that,

$$\begin{aligned} \hat{C}_{T-n} &= S_{T-n} \sum_{\mathcal{M}_{2k}(n,p,x)} \frac{1}{(2\pi)^{k/2}} \frac{1}{\sqrt{(\sum^k i = 1)\bar{p}_{k+1 \ldots \bar{p}_{2k}}}} \exp\left\{ \frac{(\sum_{i=1}^k \bar{x}_i)^2}{\sum_{i=1}^k \bar{p}_i} + \sum_{i=k+1}^{2k} \frac{\bar{x}_i^2}{\bar{p}_i} \right\} \\ &- \frac{K}{\hat{r}^n} \sum_{\mathcal{M}_{2k}(n,\bar{p},x)} \frac{1}{(2\pi)^{k/2}} \frac{1}{\sqrt{(\sum^k i = 1)p_{k+1 \ldots p_{2k}}}} \exp\left\{ \frac{(\sum_{i=1}^k x_i)^2}{\sum_{i=1}^k p_i} + \sum_{i=k+1}^{2k} \frac{x_i^2}{p_i} \right\}, \end{aligned}$$

$$\mathcal{M}_{2k}(n,p,x) = \begin{cases} (n_1^k, m_{k+1}, ..., n_{2k}) : n_1^k + ... + n_{2k} = n \\ x \leq n_1^k \leq n(p_1 + ... + p_k) + An^\gamma, \\ np_i + An^\gamma \leq n_i \leq np_i + An^\gamma, \forall i = k+1, ..., 2k, \end{cases}$$

for $A > 0t, \frac{1}{2} < \gamma < \frac{2}{3},\ k \leq \frac{n}{2}$ and $x_i = \frac{n_i - np_i}{\sqrt{n}},\ \bar{x}_i = \frac{n_i - np\bar{p}_i}{\sqrt{n}}$.
Furthermore, the researcher found that the conditional generalized CRR approximates the Black-Scholes formula, given the rates u, d, the probabilities p, q and the limit $\Delta t = 0$, such that the conditional distribution of $\ln[S(j+1)]$ is an asymptotically normal distribution with the expectation parameter $\ln[S_j + [[b - \frac{1}{2\sigma}]^2]^{\Delta t}]$ and converges to Brownian motion. Over the years, several scholars have tested this model.

2.9 Binomial versus Multinomial option pricing models

In the past few decades, scholars have applied the binomial and multinomial discrete-time market models in different ways and contexts with varying results. First, it is worth noting that multinomial models are the generalized binomial models. Secondly, despite the fact that most scholars note that the binomial option pricing formula has severe defects that make it hard to price options, they still have their applications. Benninga and Wiener [16] showed that the binomial methodology could be used to price exotic options such as Asian options, that is, options whose price trajectory depend on the underlying asset's path.
Nonetheless, its inherent assumptions of the classical binomial options pricing model make it unsuitable to use when performing option pricing in varying contexts (see Hull and White [64]) noted that the riskless rate assumption in many binomial models implied that the different stocks in a portfolio grew at the same rate, resulting in biased results Lee et al. [78] sort to solve the problem discovered by Hull and White [64] by treating investors as risk-neutral by introducing fuzzy volatility and fuzzy riskless interest rates into the generalized multinomial model. The authors developed three formulas to calculate the call price of an option under high, medium, and low volatility, such that:

$$C_u = e^{-R_t \Delta t}[P_u.C_{ur} + (1 - P_u).C_{dl}];$$

$$C_m = e^{-R_m \Delta t}[P_m.C_{ur} + (1 - P_m).C_{dl}];$$

$$C_d = e^{-R_k \Delta t}[P_d.C_{ul} + (1 - P_d).C_{dr}].$$

Where, C_u is call price of the option undergoing the greatest volatility, C_m is the call price of the option undergoing medium volatility, C_d is the call price of the option undergoing low volatility, $e^{R_t \Delta t}$ is the discounted riskless interest rate, P_u is the volatility of an up movement of an option undergoing great volatility, $1 - Pu$ is the probability of a down movement of an option undergoing great volatility, P_m is volatility of an up movement of an option undergoing medium volatility, $1 - P_m$ is the probability of a down movement of an option undergoing medium volatility, P_d is the volatility of an up movement of an option undergoing low volatility and $1 - P_d$ is the probability of a down movement the option undergoing low volatility.

Lee et al. [78] found that when the model was applied to study $S\&P$ 500 index options, the fuzzy number was closer to real market values than the generalized CRR model at different levels of sensitivities. However, they did not resolve the challenges posed by system bias, between the theoretical value and the real market value. Florescu and Viens [54] developed a multinomial option pricing technique, which relied on multinomial trees to estimate the value of options with stochastic volatility. Their model attempted to model real returns by solving the problem of constant volatility. According to the authors, most scholars stuck with models that assumed constant volatility after time 0, because modeling options that had random volatility was made almost impossible as it was not observable (See also, Bozdog et al., [31]). In this regard, their technique used a quadrinomial tree approach that recombined and converged in distribution to the price process. Their model sought to predict the result of the equation,

$$dX = \left(r - \frac{\sigma^2(Y)}{2}\right)dt + \sigma(Y)dW_t,$$

where, X_t was the logarithm of the price of the return S_t such that $X_t = \log S_t$, S_t was the derivative of the function of the price process $dS_t = \mu S_t dt + \sigma(Y_t)S_t dW_t$, Y_t was the derived from the volatility driving process's function $dY_t = \alpha(v - Y_t)dt + \psi(Y_t)dZ_t$. W_t and Z_t were independent Brownian motions and $\sigma(x)$ and $\psi(x)$ were different specifications of the stochastic volatility function. The authors applied their models to predict the price of European call options of blue chip stocks in the $S\&P$ 500. When they compared their model to the dynamic Monte Carlo method that worked by filtering stochastic volatility, they found that their model was better at pricing options than the dynamic generalized model since it approximated the bid-ask spread while the other did not. The scholars also noted that their results better approximated the Black-Scholes method.

Prigent et al. [111] used an autoregressive conditional binomial pricing method to model the behavior of IBM stock data. The model sought to solve the problem of continuous rebalancing of portfolios. According to the scholars, the Black-Scholes model assumed option traders did not hedge their assets depending on the underlying stock price's movement. However, they noted that this assumption was wrong because they would have to pay transaction costs, which made it impossible for them to continuously trade. They also noted that price variations typically did not follow binomial variables, hence the binomial discrete CRR methodology was inadequate when solving for empirical data. They proposed a conditional multinomial option pricing methodology in which the probability of price movements was fixed and occurred at random times. Furthermore, market participants in their model were able to rebalance their portfolio positions at random times by a fixed percentage such that,

$$C(t,S) = E\bigg[(S-\bar{K}) + \frac{\hat{\eta}}{\hat{\eta}} : \prod (1+r_j)^{-1} | \mathcal{F}\bigg],$$

where $C(t,S)$ was the price of a European call option's value that matured at price T, $\bar{K}$ was the strike price, $(S-\bar{K}) = max(0, S-\bar{K})$ was the final payoff and $\frac{\hat{\eta}}{\hat{\eta}}$ was the exponential of the Weibull distribution denoted by,

$$\frac{\hat{\eta}}{\hat{\eta}} = \prod \bigg(1 - \frac{\sigma(T,a)\pi + 1 + \sigma(T,-a)(1-\pi+1)}{\sigma^2(T,a)\pi + 1 + \sigma^2(T,-a)(1-\pi+1)}\sigma(T,Z)\bigg)$$
$$\exp\bigg(-\frac{(\sigma(T,a)\pi + 1 + \sigma(T,-a)(1-\pi+1))^2}{\sigma^2(T,a)\pi + 1 + \sigma^2(T,-a)(1-\pi+1)} \log G(min(T+1),T) - T)\bigg).$$

The authors used their technique to model the price of European call options of IBM stock using intraday trading data of its price movements which occurred over a period of 9 months (i.e. from January 2nd 1997 to September 30th 1997). They also accounted for the company's stock-split on $28th$ May 1997. Furthermore, the authors used different parameter estimates. Using $486,506$ trading points, the authors found that they could easily approximate the price of the options without the bid-ask price affecting their estimates. After, they tested their model using the

Monte-Carlo control variate S such that,

$$S = E\left[S : \prod(1 + r_j)^{-1}|\mathcal{F}\right],$$

and compared their results to the Black-Scholes. The results showed that their prices and annualized volatility were below the Black-Scholes market prices. Furthermore, their implied volatility was very close to the one derived from the Black-Scholes. Most recently, Cantarutti [37] used the multinomial technique to model the log-returns of options with variance gamma processes and constant drift. VG processes are pure Levy processes in which the jumps' magnitude tends toward infinitesimally-small values as their arrival rates tend towards infinity. Unlike Brownian motion, they have finite variations and lack continuous martingale components. Mathematically, Levy processes are stochastic processes occurring on a probability space $(\Omega, \mathcal{F}, F(t) \geq 0)$ that satisfies three characteristics:

(i). They are statistically continuous such that $\lim_{h \to 0} \mathbb{P}(|L_{t+h} - L_t| > \epsilon) = 0$;

(ii). L_t has stationary increments;

(iii). $L_0 = 0$.

They can be represented as the integral of a random Poisson measure such that:

$$L_t = b'l + \int_R xN(t, dx)$$

Conversely, variance gamma processes are represented by:

$$X_t = 0T_t + \sigma W_{T_t}.$$

T_t is a Gamma process that is a surbordinator i.e. it is a non-decreasing one-dimensional Levy process that is often used as a time variable. the statistical features of Levy processes closely resemble real financial data, as it enabled the computation of random and independent increments. Using the process, Cantarutti [37] found that the model converged with the Black-Scholes technique and enabled accurate pricing of different kinds of options in a much smaller time frame. According to Fares [53] after estimating the multinomial parameters $C_1, ..., C_k$ where $i = 1, ..., k$, both the stock price models were applied to several raw financial data using different levels of volatilities. The outcome was that incorporating multinomial parameters better approximated real data, particularly for data with high volatility. This showed that the multinomial parameters significantly affect the option's value, including a multinomial parameter; the multinomial option price approximated the option's actual payoff than the binomial. However, the option prices corresponding to both models have in the limit a Black-Scholes type formula that gives prices closer to the actual pay-off of the option. Given that the multinomial model gave a better approximation of the prices, the concept of improving it to conform to the reality of markets is worth considering.

Stock prices change daily, and what makes investors sell or buy a stock is influenced by many factors that can be political, social, or economic. Many researchers came up with many theories that tried to explain the price movements, but unfortunately, no theory can explain everything.

Depending on these factors that can affect price, it is logical to consider the fluctuations from one day to another are dependent,which is the main study behind Fares [53].

2.10 Black-Scholes model

Black-Scholes (BSM) is considered to be one of the best ways for pricing an option based on the no-arbitrage principle. The model is a simple differential equation that shows how a price option contract behaves relative to a given stock price, strike price of an option, risk-free rate, volatility and time to expiration. The formula derived here is the price of a call option based on the assumption that the prices will follow a random walk with no volatility and a lognormal distribution (see Duffie [50]) which results from application of log function to a data set to obtain a normal distribution.
Let S be the price of an underlying security that obeys a geometric Brownian motion over a time interval $[0, T]$ described by,

$$dS = \mu S dt + \sigma S dW(t), \tag{2.10.1}$$

where $W(t)$ is a standard Brownian motion or a Wiener process. Suppose also that there is a risk free asset e.g a bond, which attracts an interest rate r over $[0, T]$. The value of this bond denoted by B is described by,

$$dB = rBdt.$$

Lastly, consider a security that is derivative to S denoted by $f(S,t)$. To generate a nonrandom equation for the function $f(S,t)$ which will explicitly give the price of the derivative, Black-scholes equation's solution gives the function as defined below by Black [21]. At any given time that a portfolio is formed with combinations of stock and bond so that the newly formed portfolio equals the return of the derivative security, the value of this portfolio must equal the value of the derivative security. By It$\hat{o}$'s lemma equation which is an It$\hat{o}$'s process for the price of the derivative security which fluctuates randomly along with the stock price S and the Brownian motion $W(t)$. A portfolio of S and B that replicates the characteristics of the derivative security $f(S,t)$ is formed at each time t where an amount x_t of the stock and y_t of the bond is selected giving a total portfolio value of $G(t) = x_t S + y_t B(t)$. The investment gain is given by

$$dG = x_t dS + y_t dB. \tag{2.10.2}$$

The portfolio gain $G(t)$ behaves like the gain of f because of the matching of the coefficients of dt and $dW(t)$ by first matching the $dW(t)$ coefficients through setting

$$x_t = \frac{\partial f}{\partial S}. \tag{2.10.3}$$

Expanding Equation 2.10.2

$$dG = x_t dS + y_t dB, \quad (2.10.4)$$

$$= x_t(\mu S dt + \sigma S dW(t)) + y_t r B dt, \quad (2.10.5)$$

$$= (x_t \mu S + y_t r B)dt + x_t \sigma S dW(t). \quad (2.10.6)$$

$$(2.10.7)$$

Requiring $G = x_t S + y_t B$ and $G = f$, gives

$$y_t = \frac{1}{B}\left[f(S,t) - S\frac{\partial f}{\partial S}\right]. \quad (2.10.8)$$

Matching the coefficient of dt in Itô's lemma and substituting these expression in Equation 2.10.4 we obtain

$$\frac{\partial f}{\partial S}\mu S + \frac{1}{B}\left[f(S,t) - S\frac{\partial f}{\partial S}\right]rB = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2. \quad (2.10.9)$$

Simplification of Equation 2.10.9 gives the following results

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}rS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = rf,$$

which is the Black-Scholes equation.

To find such a solution for a European call option (see Merton [87]). The formula uses the function $N(x)$ which is the standard cumulative normal probability distribution with a mean of 0 and variance of 1 described as follows

$$N(x) = \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{x} e^{-y^2/2}dy.$$

Consider a European call option with strike price K and expiration time T. If the underlying stock pays no dividends during the time $[0, T]$ and if interest is constant and continuously compounded at a rate r, the Black-Scholes solution is $f(S,t) = C(S,t)$ defined by

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2),$$

where,

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

Analogous formulas does not exist for other options, including an American put option (see Rubinstein and Leland) [120]. The following equation gives the Black- Scholes formula in integral form,

$$C = \frac{e^{-rT}}{\sqrt{2\Pi\sigma^2 T}} \int_{\ln k}^{\infty} (e^x - K) e^{-[x - \ln S(0) - rT + \sigma^2 T/2]^2/(2\sigma^2 T)} dx.$$

2.11 Replication, Synthetic Options and Portfolio insurance

When pricing options, it is important to factor in replication and volatility. According to Black and Scholes[21], a portfolio replicates the derivative security if a duplicate security can be formed by combining the underlying security with an appropriate risk-free asset whose proportions must be repeatedly adjusted over the length of the trading period, and no extra cash is required to self-finance the portfolio. For a call, replicating a portfolio requires investors to be bullish on the implicit asset's price movement. However, for a put, replicating the portfolio requires taking a short position on the underlying asset.That is, Suppose we have a stock A trading at $S = 100$, and the price can either go up or down by a factor of 10 in a single period such that $uS = 110$, and $dS = 90$, and it is possible to lend/ borrow from the markets at a 4% interest rate, we can set up a replicating portfolio with Δc units of the stock's call option and B units of borrowing, such that

$$\Delta c.(110) + B(1.04) = 10\Delta c.(90) + B(1.04) = 0$$

Solving this, will obtain $\Delta c = .50$ and $B = -43.26$ which implies that, we can replicate the call option by having a long position on .50 units of the stock and borrowing 43.26. The cost of the portfolio (C) will be given by: $(.50)(100) + (-43.26)(1) = 6.74$. Hence if $C > 6.74$, the call can be sold to buy a replicating portfolio. However, if $C < 6.74$, the call can be bought and the replicating portfolio sold. For the same underlying stock A, we can set up a replicating portfolio with Δp units of the stock's call option and B units of borrowing, such that,

$$\Delta.p(90) + B(1.04) = 10$$
$$\Delta.p(110) + B(1.04) = 0,$$

solving this will obtain $\Delta P = -.50$ and $+52.88$ which implies that, we can replicate the put option by having a short position on .50 units of the stock and borrowing 52.88. The cost of the portfolio (C) will be given by, $(-.50)(100)+(52.88)(1) = 2.88$ implying that if $C > 6.74$, the put option can be bought to sell a replicating portfolio. However, if $C < 6.74$, the put option can be sold and the replicating portfolio bought. The portfolio will be self-financing in both options.

Portfolio replication should be done to build synthetic securities, which are simply ways in which traders can create the risk profile and payoff of a particular asset by using combinations of the underlying asset and different instrument options. Synthetic options enable traders to hedge their positions against extreme volatility, while also minimizing the opportunity cost by enabling them to explore options with similar properties. The process of call option hedging at the starting point requires investors to determine C's theoretical value. This can be achieved by allocating the replicating portfolio with an amount C. This way, the delta and portfolio value will be analogous to the option's. After a given time interval, delta will change, and investors should repeatedly rebalance their positions. As the portfolio approaches the time to maturity approaches, it will mainly be constituted with stock if the price is above K, else, its value will decrease till it reaches zero (see Black and Scholes [21]). Research by Black et al [21] shows that several institutions owning significant stock portfolios seek to hedge their positions against significant market downturn risk. They can achieve this by simply buying a put that will be exercised at K. Leland and Rubinstein who introduced the idea of option-based portfolio insurance noted that insurance agencies and mutual funds could hedge their positions by simultaneously buying a stock and a put written on it. This way, the portfolio's value at maturity is always greater than the strike price even with excessive volatility. For example, a trader could buy a stock on the NASDAQ (National Association of Securities Dealers Automated Quotations Stock Market) and a put option on the same allowing him to sell the index at a particular price. If the put fell beneath the price, the trader can sell the put and use the profit to exercise any losses he faces from bearish market, while if the put rises the trader loses the amount paid for in the put, but continues to enjoy returns from stocks.

Creating synthetic portfolios is easy and uncomplicated. However, in the real world, listed options often do not have desired strike prices and maturities. Furthermore, synthetic options can be affected by jumps in transaction costs and stock prices. Therefore, there is a need to refine existing strategies, so as to improve the performance and efficiency of option-based portfolio insurance. The Cox Rubenstein model and Black Scholes formula provide insurance agencies with methods that they can use to guarantee minimum portfolio values at the end of their respective trading periods. However, the latter assumes a constant trading strategy, which makes it an impractical strategy in hedging constructed portfolios. Fares [53] showed that an extension of the classical generalized CRR model in discrete-time markets

2.12 Existing extensions

An earlier work from Breton, El Khatib, Fan and Privault [33] considered a scenario where $q = q_N$ correlates with N's total number of steps. They also studied its convergence with the q_N-binomial model on $\{1, ..., \text{N}\}$ as q_N approaches one, while N which had an additional stretch parameter $\theta > 0$. They found that in such scenarios, q_N can be described by $q_N = 1 + \eta(T/N)^{3/2} + o(N^{-3/2})$which would make the asset's returns $a_N = 1 - \sigma\sqrt{\theta T/N} + \zeta\sigma^2\theta T/(2N) + o(N^{-2})$ and $b_N = 1 + \sigma\sqrt{\theta T/N} + \zeta\sigma^2\theta T/(2N) + o(N^{-2})$. Hence, they concluded that these findings implied that the equation converges to the Black-Scholes formula through Musiela and Rutkowski [98],

$$dS_t = \sigma S_t dB_t + \zeta\frac{\sigma^2}{2}S_t dt + \frac{\sigma\eta\sqrt{\theta}}{1+\theta}tS_t dt.$$

$(B_t)_{t\in[0,T]}$ is a standard Wiener process with time-dependent drift at time $[0, T]$. The following proposition shows this convergence.

Proposition 2.1 (Breton [33], Proposition 4.1). *Let a_N, b_N be*

$$a_N = 1 - \sigma\sqrt{\theta T/N} + \zeta\sigma^2\theta T/(2N) + o(N^{-2}),$$

$$b_N = 1 + \sigma\sqrt{\theta T/N} + \zeta\sigma^2\theta T/(2N) + o(N^{-2}),$$

and assume that q_N depends on N as

$$q_N = 1 + \eta(\Delta t)^{3/2} + o(N^{-3/2}),$$

where $\Delta t = T/N$. We have the convergence of finite -dimensional distri-

butions (fdd)

$$(logS_{[Nt/T],N})_{t\in[0,T]} \Rightarrow (logS_0 + \frac{\sigma\eta\sqrt{\theta}}{2(1+\theta)}t^2 - (1-\zeta)\frac{\sigma^2}{2}t + \sigma B_t)_{t\in[0,T]},$$

where $(B_t)_{t\in[0,T]}$ *is a standard Brownian motion.*

2.13 q-binomial distribution

A parametrization has been used in Kemp [72] in developing the following parametrized binomial distribution.

Consider the distribution of a sequence of independent Bernoulli random variable, $(X_{k\geq 1})$,

$$\mathbb{P}_{\theta,q}(X_k = 0) = \frac{1}{1+\theta q^{k-1}}, \tag{2.13.1}$$

and

$$\mathbb{P}_{\theta,q}(X_k = 1) = \frac{\theta q^{k-1}}{1+\theta q^{k-1}}. \tag{2.13.2}$$

Proposition 2.2 (Breton [33], Proposition 2.1)**.** *Let* $(X_{k\geq 1})$ *be a sequence of independent Bernoulli random variables with distributions (Equation 2.1.0.1 and Equation 2.1.0.2). The sum* $Z_n = X_1 + ...X_n,\ n \geq 1$ *has the distribution*

$$\mathbb{P}_{\theta,q}(Z_n = k) = \frac{\theta^k q^{k(k-1)/2}}{(1+\theta)(1+\theta q)...(1+\theta q^{n-1}})\binom{n}{k}_q, \quad k = 0, 1, ..., n, \tag{2.13.3}$$

and the probability generating function

$$\mathbb{E}_{\theta,q}[t^{Z_n}] = \frac{(1+\theta tq)...(1+\theta tq^{n-1})}{(1+\theta q)...(1+\theta q^{n-1})}, \quad t\epsilon[0,1],$$

where

$$\binom{n}{k}_q = \frac{(1-q^n)...(1-q^{n-k+1})}{(1-q)...(1-q^k}, \quad k = 0, 1, ..., n,$$

is the q-binomial, or Gaussian binomial, coefficient.

Proof. By induction on $n \geq 0$, we have

$$\begin{aligned}
\mathbb{P}(Z_{n+1} = k) &= \frac{1}{1+\theta q^n}\mathbb{P}(Z_n = k) + \frac{\theta q^n}{1+\theta q^n}\mathbb{P}(Z_n = k-1) \\
&= \frac{1}{(1+\theta q^n)}\frac{\theta^k q^{\frac{k(k-1)}{2}}}{(1+\theta)(1+\theta q)...(1+\theta q^{n-1})}\binom{n}{k}_q \\
&\quad + \frac{\theta q^n}{(1+\theta q^n)}\frac{\theta^{k-1} q^{\frac{(k-1)(k-2)}{2}}}{(1+\theta q)...(1+\theta q^{n-1})}\binom{n}{k-1}_q \\
&= \frac{\theta^k q^{\frac{k(k-1)}{2}}}{(1+\theta)(1+\theta q)...(1+\theta q^n)}\left(\binom{n}{k}_q + q^{n-(k-1)}\binom{n}{k-1}_q\right) \\
&= \frac{\theta^k q^{\frac{k(k-1)}{2}}}{(1+\theta)(1+\theta q)...(1+\theta q^n)}\binom{n+1}{k}_q,
\end{aligned}$$

and the q-Pascal rule is applied thus the expression of the probability generating function emanates from the Gauss's binomial formula

$$\sum_{k=0}^{n} \theta^k q^{\frac{k(k-1)}{2}} \binom{n}{k}_q = \prod_{l=1}^{n} (1 + \theta q^{l-1}).$$

□

2.14 A q-binomial extension of the CRR

Researchers have considered a q-binomial extension of the CRR to enable traders to hedge their portfolios during sharp market movements such as bear markets. The classical CRR model is based on the idea of constant switching probabilities, thus does not account for such movements, meaning that investors can overreact to increasing probabilities over time (see Soroka [137]). Furthermore, the CRR model uses a narrow set of parameters, such that predictions involving the convergence rates of asset price movements are limited in their ability to model sharp market movements (see Ritchken [115] and Privault [112]).

The q-binomial extension solves these problems by introducing the q-binomial walk and time -dependent probabilities. The q-binomial walk is defined from q-binomial coefficients in q-binomial distributions. This way, the model's flexibility is extended while it still maintains its polynomial complexity since it can move up and down depending on a trend parameter, see Georgiadis [56]. Researchers in Breton [33] considered a risky asset price with initial value S_0. From Equation (2.8.0.1) and Equation (2.8.0.2) they noted that for $q > 1$ was a manifestation of an increase in the overall assets, while $q < 1$ showed a decline in the overall value of assets. On that note, they reasoned that as q approached 1, the constant likelihoods $1/(1+\theta)$ and $\theta/(1+\theta)$ were reflective of the standard CRR model. They suggested modeling the price of an asset, such that $A_0 = 1$ at time 0. $A_0 = 1$ was considered both risky and a riskless. Given a, b such that $-1 < a < b$, $n \geq 1$, they also considered,

$$A_n = \prod_{k=1}^{n}(1 + r_k),$$

which was a risky asset. With an initial price S_0 given in discrete time as,

$$S_n = S_0 b^{Zn} a^{n-Zn} = \begin{cases} bS_{n-1}, & X_n = 1, \\ \\ aS_{n-1}, & X_n = 0 \; n \geq 1 \end{cases},$$

where $(X_n)_{n\geq 1}$ is a distribution of independent Bernoulli functions that can be parametized as,

$$\mathbb{P}_{\theta,q}(S_n = S_0 b^k a^{n-k}) = \frac{\theta^k q^{k(k-1)/2}}{(1+\theta)(1+\theta q)...(1+\theta q^{n-1})} \binom{n}{k}_q, \; k = 0, 1, ..., n,$$

and

$$\mathbb{E}_{\theta,q}\left[\frac{S_n}{S_{n-1}}\right] = \frac{a + b\theta q^{n-1}}{1 + \theta q^{n-1}},$$

$$A_n = \Pi_{k=1}^{n}(1 + r_k),$$

at time $n \geq 1$, where r_k satisfies

$$1 + r_k = \mathbb{E}_{\theta,q}[\frac{S_K}{S_{k-1}}] = \frac{a + b\theta q^{k-1}}{1 + \theta q^{k-1}}, k \geq 1.$$

The researchers (see Breton, El khatib, Fan and Privault [33]) found that the discounted price process could be modeled as

$$\tilde{S}_n = \frac{S_n}{A_n} = S_n \Pi_{k=1}^{n} (1 + r_k)^{-1}, n \geq 0,$$

which is a martingale that shows an arbitrage-free, complete market, and $\mathbb{P}_{\theta,q}$ is a risk-neutral probability measure. Accordingly, they note that the arbitrage-free price at time $n = 0, 1, ..., N$ with payoff $\phi(S_N)$ is given by,

$$\prod_{k=n+1}^{N} (1+r_k)^{-1} \mathbb{E}_{\theta,q}[\phi(S_N) \mid F_n] = \sum_{k=0}^{N-n} \frac{\theta^k q^{k(2n+k-1)/2} \phi(S_n b^k a^{N-n-k})}{(a + \theta b q^n)...(a + \theta b q^{N-1})} \binom{N-n}{k}_q . \tag{2.14.1}$$

The q-binomial model has been shown to converge to the Black-Scholes formula if N approaches infinity, giving

$$S_{k,N} = S_0 b_N^{Z_k} a_N^{k-Z_k}, \;\; K = 1, ..., N.$$

And the time-dependent risk-free interest rate is,

$$r_t = \zeta \frac{\sigma^2}{2} + \frac{\sigma \eta \sqrt{\theta}}{1+\theta} t, \;\; t > 0.$$

Furthermore, it can model rapid expansionary phases and recessions, since it allows for the modeling of compound risk, and the probabilities can go up and down depending on a trend parameter. This way, the model enables traders to hedge their positions against sharp recessions, and optimize the value of their portfolios during rapid expansionary phases. Therefore, this research performed an extension on the existing model.

Central Limit theorem

Charalambides [40] and showed that the central limit theorem is invariable for the q-distribution when $q \neq 1$ (see also Deheuvels, et al., [49]). It has also been shown that in Luenberger [80], when the added parameter θ takes the form $\theta_N = q^{\eta N}$ for some $\eta \epsilon (0,1)$, the distribution of Z_n can be approximated as N approaches infinity. The following proposition shows that the random walk of a q-binomial converges to the Gaussian. The q-binomial distribution has also been shown to converge to a discrete Heine distribution by Gerhold and Zeiner [57], and Kyriakoussis and Vamvakari [77]. However, this has only been for fixed $q \epsilon (0,1)$

Proposition 2.3 (Breton [33], Proposition 2.2)**.** *Assume that q_N depends on N as*

$$q_N = 1 + \eta N^{-3/2} + o(N^{-3/2}), \tag{2.14.2}$$

where $\eta \epsilon \mathbb{R}$. Then, letting $Z_N = X_1 + ... X_N, N \geq 1$, the normalized sequence is $(Z_N - \mathbb{E}_{\theta,q}[Z_N])/\sqrt{N}$ converges in distribution to a $\mathcal{N}(0, \theta/(1+\theta)^2$ Gaussian random variable as N tends to infinity.

Proof. For $1 \leq k \leq N$ we have
$q_N^{k-1} = 1 + (K-1)\eta N^{-\frac{3}{2}} + (k-1)^2 O(N^{-3}) = 1 + k\eta N^{-\frac{3}{2}} + o(N^{-\frac{1}{2}})$,
hence

$$
\begin{aligned}
\mathbb{P}_{\theta,qN}(X_k = 1) &= \frac{\theta q^{k-1} N}{1 + \theta q^{k-1} N} \\
&= \theta \frac{1 + \eta k N^{-\frac{3}{2}} + o(N^{-\frac{1}{2}})}{1 + \theta + \theta\eta k N^{-\frac{3}{2}} + o(N^{-\frac{1}{2}})} \\
&= \frac{\theta}{1+\theta}(1 + \eta k N^{-\frac{3}{2}} + o(N^{-\frac{1}{2}}))\left(1 - \frac{\eta\theta}{1+\theta} k N^{-\frac{3}{2}} + o(N^{-\frac{1}{2}})\right) \\
&= \frac{\theta}{1+\theta} + \frac{\eta\theta}{(1+\theta)^2} k N^{-\frac{3}{2}} + o(N^{-\frac{1}{2}}).
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\mathbb{E}_{\theta,qN}[Z]_N &= \sum_{k=1}^{N} \mathbb{P}_{\theta,qN}(X_k = 1) \\
&= \frac{\theta}{1+\theta} \sum_{k=1}^{N} \left(1 + \frac{\eta}{1+\theta} k N^{-\frac{3}{2}} + o(N^{-\frac{1}{2}})\right) \\
&= \frac{\theta N}{1+\theta} + \frac{\theta\eta N^{\frac{1}{2}}}{2(1+\theta)^2} + o(N^{\frac{1}{2}}).
\end{aligned}
$$

The variance of Z_n is given by

$$
\begin{aligned}
\sigma_N^2 &:= Var_{\theta,qN}[Z_N] = \sum_{k=1}^{N} \mathbb{P}_{\theta,qN}(X_k = 0)\mathbb{P}_{\theta,qN}(X_k = 1) \\
&= \sum_{k=1}^{N} \left(\frac{\theta}{1+\theta} + \frac{\theta\eta}{(1+\theta)^2} k N^{-\frac{3}{2}} + o(N^{-\frac{1}{2}})\right)\left(\frac{1}{1+\theta} - \frac{\theta\eta}{(1+\theta)^2} k N^{-\frac{3}{2}} + o(N^{-\frac{1}{2}})\right) \\
&= \frac{\theta N}{(1+\theta)^2} + o(N^{\frac{1}{2}})
\end{aligned}
$$

as N tends to infinity, hence the conclusion by Central Limit Theorem.

□

2.15 Portfolio optimization and selection

According to Markowitz [83] and Athayde & de Flores [5], portfolios should be constructed in a manner that solves their constrained optimization. Against this backdrop, actuarial researchers have sought to develop optimal portfolios to minimize risk for every unit of return and skewness (see also Neves et al., [101]). In the late 70's, researchers in (see Oliinyk and Kozmenko[104]) came up with a portfolio insurance technique based on the Black scholes model. The reasoning behind it was that an action hedging against losses while maintaining the upward potential should have considerable attraction to a vast range of investors.

Proposition 2.4 (Atkinson [6], Proposition 2.1)**.** *Consider a portfolio with n available assets where the total rate of return of the $i-th$ asset i is r_i. Then we can denote $\bar{r} = (\bar{r}_1, \bar{r}_2, ..., \bar{r}_n) = (E[r_1], E[r_2], ...E[r_n])$ to represent the expected return vector and matrix r and $\sum = (\sigma_{ij}),\ i, j = 1, 2, ..., n,\ \sigma_{ij} = cov(r_i, r_j)$. Since this represents a constrained quadratic programming optimization problem according to mean-variance analysis theory, we can determine that the optimal solution $w = (w_1, w_2, ..., w_n)$, $\sum_{i=1}^{n} w_i = 1$ will denote weights apportioned to each asset, such that the portfolio $p = w_1 r_1 + w_2 r_2 + ... + w_n r_n$. Hence, the portfolio will have an* *expected return and variance,*

$$\mathbb{E}[p] = \sum_{i=1}^{n} w_i \mathbb{E}[r_i] = \bar{r}, \;\; w\epsilon\mathbb{R},$$

$$\mathbb{V}[p] = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j cov(r_i, r_j) = w, \;\; \Sigma w \geq 0.$$

The above model is very instinctive and a baseline for portfolio selection.

2.16 Simulation in Life insurance

There have been many cases of insolvency in insurance companies in the recent past. For example, in Kenya alone, more than 15 insurance companies have collapsed in the past 23 years, the most noteworthy of them being; Resolution Insurance, Blue Shield Insurance, Kenya National Assurance Company (KNAC), United Insurance, Lake star Insurance, Concord Insurance, Access Insurance Company, and Stallion Insurance Company Ltd (see Okoth [102]). Also, large life insurance firms in developed economies like Japan, US, and Europe have been bankrupted, or defaulted on payments several times, because of insolvency (see Chen [41]). To mitigate such risks, several scholars have developed and simulated life insurance models using different means, so as to get better insight into risk-minimizing strategies, and optimize their portfolios.

Brennan and Schwartz [32], Bacinello and Ortu [8] and Miltersen and Persson [88] pioneered the use of financial valuation techniques to model and simulate expected values to predict life insurance claims in complete markets. However, their models did not incorporate hedging techniques. Following in the footsteps of Moller [89] and Chen [41] used the CRR model technique to incorporate the hedging perspective in a simulation study of life insurance company net-losses. The author studied adjustable risk-minimizing strategies of hypothetical life insurance firms with n-identical policyholders with allocated identical endowment policies. The author noted that the ruin probability significantly declined when firms used time-discretized risk-minimizing strategies.

Additionally, the study found that the portfolio's self-financing capabilities decreased given when they applied continuous risk-minimizing hedging strategies.This was attributed to a reduction in the ruin probability's magnitude, which also decreased the pros of applying time-discretized strategies in portfolio's management, leading to a higher hedging frequency.Chen [41] also noted that life insurance companies could benefit from using a discretized hedging model, over a discretized hedging strategy. The author attributed this to significant improvements in the former's simulation results.Costabile [44] also employed the CRR technique to develop an algorithm that simulated the risk measures of life insurance contracts. The author used CRR, because it better incorporated a lattice that can be used throughout the whole policy duration, and was not influenced by measure changes.

According to Costabile [44], actuaries find it difficult to simulate life-insurance policies, because they have to use dual varying probability metrics, i.e. the risk-neutral probability metrics along the remaining time interval, and the real-world probability metrics along the risk-horizon. This implies that they cannot apply the Monte Carlo method easily. Hence, they have to settle for time-consuming nested simulations or the least-squares Monte Carlo approach, which is very inefficient (see also Bauer et al. [10], and Bauer et al.[11], Broadie et al. [35]). Summarily, the author observed that the CRR-based algorithm computed highly accurate values.

Summary of Literature Review

In the intricate domain of portfolio optimization within the context of life insurance, this literature review meticulously navigates through existing research, shedding light on the diverse facets that constitute the (p, q)-binomial extension of the Cox-Ross-Rubinstein model. While the landscape of financial modeling and optimization has been extensively explored, a discernible gap emerges, urging a more profound inquiry into the integration of noisy observations within the life insurance framework.

The surveyed literature encapsulates a wide spectrum of financial models, including the foundational Cox-Ross-Rubinstein model. However, as the review unfolds, a conspicuous gap in research becomes apparent, the limited exploration of portfolio optimization strategies tailored to life insurance scenarios with inherently noisy observations.

Extensive research has expounded upon the Cox-Ross-Rubinstein model's efficacy in financial modeling, particularly in the context of options pricing and risk management. Nonetheless, the translation of such models to the dynamic landscape of life insurance, characterized by uncertain and noisy observations, remains an under explored terrain.

The literature reviewed has predominantly focused on pristine financial environments, often overlooking the nuances introduced by the inherent noise associated with life insurance data. This study, cognizant of this gap, endeavors to address the intricacies of optimizing portfolios within the (p, q)-binomial framework, considering the stochastic nature of observations within the life insurance sector.

As we traverse the existing knowledge landscape, it becomes evident that the challenges posed by noisy observations in life insurance demand a specialized focus. The integration of (p, q)-binomial extension into the Cox-Ross-Rubinstein model represents a promising avenue for enhancing portfolio optimization in the presence of uncertainty inherent in life insurance data.

The synthesis of literature within this review accentuates the need for tailored financial models that account for the idiosyncrasies of the life insurance sector. By identifying and addressing this gap, this research aspires to contribute significantly to the evolving discourse on portfolio optimization, offering nuanced insights and strategies tailored to the unique challenges posed by noisy observations within the realm of life insurance.

Chapter 3

RESEARCH METHODOLOGY

3.1 Introduction

This section outlines the approach used in the study to obtain the practical results. The research method incorporates various techniques crucial for a thorough analysis and interpretation of the data. These techniques include using Binomial extensions, exploring Extended Fibonacci sequence generating functions, formulating strong mathematical models, applying rigorous Convergence tests, and conducting detailed Simulations. Each of these methods is essential in supporting the research findings, ensuring the study's reliability and validity in achieving its main goals.

3.2 Binomial Extensions

The work of Ollerton and Shannon [105] introduced a generalized concept of binomial coefficients, known as k-extensions. These extensions are represented by $\binom{n}{m}$, where n, m, and q are non-negative integers, and k is a product of integers within a specified range. This concept extends the traditional binomial coefficients by considering the arrangement of m objects into n cells, each capable of holding up to q objects.

Bondarenko [23] provided various combinatorial interpretations of these k-extensions, which enhance the understanding of generalized coefficients in the Cox-Ross-Rubinstein (CRR) model. The unifying recurrence relations for these extensions are given by:

$$\binom{n}{m}_q^k = \sum_{i=l-a}^{q} \mathcal{C}_{ib}^{m} i!^c \binom{n-1}{m-i}_q^k, \tag{3.2.1}$$

where k, n, and m are integers within their respective ranges, and q is a non-negative integer. The boundary conditions for these extensions are defined as:

$$\binom{n}{m}_q^k = \begin{cases} 0, & \text{if } n < 0 \text{ or } m < 0, \text{ or } n = 0 \text{ and } m > 0, \\ 1, & \text{if } n \geq 0 \text{ and } m \geq 0. \end{cases}$$

Ollerton and Shannon [105] further explored several properties of these k-extensions, such as diagonal and row sum recurrence relationships, and their generating functions. The generating function for these extensions is expressed as:

$$\sum_{m=0}^{qn} m!^{-b} \binom{n}{m}_q^k x^m = \begin{cases} \frac{T_q^k(x)^{n+1}-1}{T_q^k(x)-1}, & \text{if } T_q^k(x) = 1, \\ T_q^k(x)^n, & \text{otherwise,} \end{cases} \tag{3.2.2}$$

where $T_q^k(x) = \sum_{i=l-a}^{q} i!^{c-b} x^i$.

These generating functions are instrumental in deriving additional properties of the k-extensions. Notably, these extensions can yield diagonal array sums, potentially leading to sequences analogous to generalized Fibonacci sequences. The exploration and development of these k-extensions, as detailed in this study, not only broaden our understanding of generalized coefficients within the CRR model but also open avenues for future research in the field. The potential applications of these findings in various mathematical and financial contexts underscore the significance of this research in advancing our theoretical and practical knowledge.

3.2.1 Additional Properties of k-Extensions

In this section, we delve into the advanced properties of k-extensions, which are pivotal in understanding the broader implications of our study. These properties are derived from the foundational equations and offer insights into the behavior and characteristics of k-extensions in various scenarios.

Property 1: *Differentiation of the Generating Function*

By differentiating the generating function Equation 3.2.1 with respect to x, we obtain an expression for $g(k, n, q; x)$, which represents the differentiated form of the generating function:

$$g(k, n, q; x) = \sum_{m=0}^{qn} m!^{-b} \binom{n}{m}_q^k m x^{m-1}$$

$$= \begin{cases} \frac{nT_q^k(x)^{n+1} - (n+1)T_q^k(x)^n + 1}{(T_q^k(x)-1)^2} T_q'^k(x), & \text{if } T_q^k(x) \neq 1, \\ \frac{n(n+1)}{2} T_q'^k(x), & \text{if } T_q^k(x) = 1, \\ nT_q^k(x)^{n-1} T_q'^k(x), & \text{otherwise.} \end{cases}$$

Here, $T_q^k(x)$ is defined as $\sum_{i=l}^{q} i!^{c-b} i x^{i-1}$.

Property 2: *Recurrence Relations in Terms of q*

From Equation 3.2.1, we can establish recurrence relations for $g(k, n, q; x)$ in terms of q:

$$\begin{aligned} g(k, n, q; x) &= T_q^k(x)^n \\ &= (T_{q-1}^k(x) + q!^{c-b} x^q)^n \\ &= \sum_{j=0}^{n} C_j^n T_{q-1}^k(x) (q!^{c-b} x^q)^{n-j} \\ &= \sum_{j=0}^{n} C_j^n (q!^{c-b} x^q)^{n-j} g(k, j, q-1; x). \end{aligned}$$

3.3 Extended Fibonacci Sequence Generating Functions

This section explores the extended Fibonacci sequence generating functions, which are a significant extension of the classical Fibonacci sequence. The extended functions are defined as follows:

Let

$$d(k,n,q;x) = \sum_{m=0}^{n} m!^{-b} \binom{n-m}{m}_q^k x^m = \sum_{m=0}^{\lfloor \frac{qn}{q+1} \rfloor} m!^{-b} \binom{n-m}{m}_q^k x^m, \quad \text{for } n \geq 0, \tag{3.3.1}$$

and $d(k,n,q;x) = 0$ otherwise. This formulation represents an extension of the normal Fibonacci sequence as discussed in [23].

Substituting Equation 3.2.1 for n and $m > 0$, we obtain:

$$\begin{aligned}
d(k,n,q;x) &= \binom{n}{0}_q^k + \sum_{m=1}^{n} m!^{-b} \sum_{i=1-a}^{q} C_{ib}^{m} i!^{c} \binom{n-m-1}{m-i}_q^k x^m \\
&= 1-a+\sum_{i=1-a}^{q} i!^{c} \sum_{m=0}^{n} C_{ib}^{m} m!^{-b} \binom{n-m-1}{m-i}_q^k x^m \\
&= 1-a+\sum_{i=1-a}^{q} i!^{c-b} x^i \sum_{m=0}^{n} (m-i)!^{-b} \binom{n-m-1}{m-i}_q^k x^{m-i} \\
&= 1-a+\sum_{i=1-a}^{q} i!^{c-b} x^i \sum_{m=-i}^{n-i} m!^{-b} \binom{n-i-1-m}{m}_q^k x^m \\
&= 1-a+\sum_{i=1-a}^{q} i!^{c-b} x^i \sum_{m=0}^{n-i-1} m!^{-b} \binom{n-i-1-m}{m}_q^k x^m \\
&= 1-a+\sum_{i=1-a}^{q} i!^{c-b} x^i d(k,n-i-1,q;x),
\end{aligned}$$

where $d(k, 0, q; x) = \sum_{m=0}^{0} m!^{-b} \binom{0-m}{m}_q^k x^m = \binom{0}{0}_1^k = 1$. The transformation of the summation index to $j = m - i$ and then reverting to m using the boundary conditions yields the results as shown above.

3.4 Skorohod's Theorem

Skorohod's theorem, a pivotal concept in probability theory, provides a framework for approximating sequences of random variables under certain conditions. This theorem is particularly relevant in the context of financial modeling, where discrete-time models often approximate continuous-time processes.

The theorem posits that for two sequences of random variables (X_n) and (Y_n) on a common probability space, if X_n converges almost surely to X and Y_n to Y, and both sequences share the same distribution, then there exists a sequence (Z_n) with the same distribution as X_n, converging almost surely to Y, and being almost surely equal to X_n for all but finitely many n (see Skorohod [136]).

In this research, Skorohod's theorem is instrumental in demonstrating the convergence of the (p, q)-binomial model, an extension of the Cox-Ross-Rubinstein (CRR) model, to the Black-Scholes model. The Black-Scholes model, a continuous-time model widely used in finance, is approached by the (p, q)-binomial model through a sequence of random variables that converge almost surely, as per Skorohod's theorem. This convergence is crucial in validating the (p, q)-binomial model as a robust tool for analyzing insurance portfolios.

3.4.1 Skorohod Space

The research utilizes a specific version of Skorohod's theorem applicable to the space of cadlag (right-continuous with left limits) functions. This approach involves constructing a sequence of (p, q)-binomial processes that converge almost surely to a Brownian motion with linear drift, representing the continuous-time limit of the (p, q)-binomial model. The convergence in the Skorohod space is pivotal in establishing the optimization conditions for the (p, q)-binomial model under noisy observations.

The Skorohod space, denoted as $D[0, 1]$, consists of functions that are right-continuous with left limits. The convergence in this space is defined as follows:

$$\lim_{n\to\infty} \sup_{t\in[0,1]} |X_n(t) - X(t)| = 0, \quad \text{almost surely,} \tag{3.4.1}$$

where $X_n(t)$ represents the sequence of (p, q)-binomial processes and $X(t)$ the limiting Brownian motion. This convergence criterion ensures that the discrete-time (p, q)-binomial model approaches the continuous-time Black-Scholes model, thereby bridging the gap between discrete and continuous financial models.

3.5 Simple Continuous Theorem

The Simple Continuous Theorem, also known as the Continuous Mapping Theorem, is a fundamental result in probability theory that gives conditions under which a continuous function of a random variable is itself a random variable with well-behaved properties. Formally, the theorem states that if X_n is a sequence of random variables that converges in probability to a random variable X, and $g(\cdot)$ is a continuous function, then $g(X_n)$ converges in probability to $g(X)$. In other words, if X_n is "close" to X, then $g(X_n)$ is "close" to $g(X)$ [19].

The Simple Continuous Theorem can be used to analyze the behavior of the (p, q)-binomial extension of the CRR model as the number of periods increases to infinity. Specifically, the theorem can be used to show that as the number of periods increases, Boyle [29], the (p, q)-binomial model converges in probability to the continuous-time Black-Scholes model, which is a widely used model in finance. This result is important because it justifies the use of the (p, q)-binomial model as an approximation of the Black-Scholes model, and enables the application of well-established continuous-time methods for portfolio optimization in the context of the discrete-time (p, q)-binomial model.

We use the Simple Continuous Theorem to show that the (p, q)-binomial model converges in probability to the Black-Scholes model as the number of periods increases to infinity. Specifically, the (p, q)-binomial model can be written as a sum of independent and identically distributed random variables, and apply the Simple Continuous Theorem to each term in the sum. This result is then used to establish the convergence of the (p, q)-binomial model to the Black-Scholes model, and to develop optimization conditions for the (p, q)-binomial model with noisy observations.

3.6 Lipschitz Mapping Theorem

The Lipschitz Mapping Theorem, a cornerstone in the field of analysis, plays a crucial role in ensuring the continuity and predictable behavior of functions between metric spaces. This theorem is particularly significant in financial modeling, where it aids in understanding the behavior of complex models.

The theorem establishes that a function f mapping from a metric space (X, d_X) to another metric space (Y, d_Y) is Lipschitz continuous if there exists a constant K such that:

$$d_Y(f(x), f(y)) \leq K \cdot d_X(x, y)$$

for all x and y in X. A function satisfying this condition is uniformly continuous and exhibits predictable behavior across its domain (see Rudin [121]).

In this research, the Lipschitz Mapping Theorem is utilized to determine the convergence rate of the (p, q)-binomial model towards the Black-Scholes model. The theorem assists in demonstrating that the (p, q)-binomial model is a Lipschitz function with respect to its parameters. The convergence rate to the Black-Scholes model is influenced by the Lipschitz constant, where a lower constant indicates a faster convergence, a desirable attribute in practical applications (see Molnar [90]).

We apply the Lipschitz Mapping Theorem to ascertain the convergence rate of the (p, q)-binomial model to the Black-Scholes model. The analysis reveals that the Lipschitz constant of the (p, q)-binomial model is dependent on factors such as the time horizon and the volatility of the underlying asset. This insight allows us to establish an upper bound on the error margin between the (p, q)-binomial and Black-Scholes models. Consequently, this upper bound serves as a measure of the accuracy of the (p, q)-binomial model as an approximation to the Black-Scholes model, thus providing a quantitative assessment of its reliability in financial modeling.

3.7 Tychonoff's Theorem

Tychonoff's Theorem, a cornerstone in the field of topology, provides critical insights into the behavior of product spaces, particularly in the context of compact topological spaces. This theorem is instrumental in various optimization problems, especially those involving product spaces.

The theorem asserts that the product of any collection of compact topological spaces is compact. Formally, for a family of non-empty compact topological spaces $\{X_\alpha\}_{\alpha \in A}$, the product space $X = \prod_{\alpha \in A} X_\alpha$, when equipped with the product topology, is also a non-empty compact space. Relevance in Optimization Problems include;

i) Tychonoff's Theorem is pivotal in establishing the compactness of spaces in optimization problems. This is particularly relevant when dealing with a set of scenarios, each representing different variables or time periods. The theorem ensures that the entire space of these scenarios remains compact, facilitating the existence of optimal solutions.

i) In practical applications, such as portfolio optimization, Tychonoff's Theorem can be used to guarantee the existence of an optimal portfolio. By confirming the compactness of the scenario space, the theorem aids in ensuring that the optimization process over this space is well-defined and leads to feasible solutions.

The application of Tychonoff's Theorem in scenario-based optimization is significant. It provides a mathematical foundation for the existence of optimal solutions in complex spaces, which is crucial for realistic and practical decision-making processes. This aspect of the theorem is particularly beneficial in fields where scenario analysis and optimization over multiple variables or time periods are essential.

Tychonoff's Theorem, with its profound implications in the realm of topology and optimization, is a vital tool for ensuring the feasibility and existence of optimal solutions in complex product spaces. Its application in scenario-based optimization scenarios, such as portfolio management, underscores its importance in practical and theoretical research endeavors (see Perkowski and Nicolas [109]).

3.8 Kuratowski's Theorem

Kuratowski's theorem is a cornerstone in topology, providing a critical criterion for compactness in topological spaces. Formally, for a topological space (X, τ), Kuratowski's theorem states that X is compact if and only if every open cover of X has a finite subcover. This can be mathematically expressed as:

$$\begin{aligned} X \text{ is compact } \Leftrightarrow \forall \{U_\alpha\}_{\alpha \in A} \subseteq \tau, \text{ with } X \subseteq \bigcup_{\alpha \in A} U_\alpha, \\ \text{there exists } \exists n \in \mathbb{N}, \exists \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq A \\ \text{such that } X \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}. \end{aligned}$$

In the realm of financial mathematics, particularly in the analysis and optimization of financial models like the (p, q)-binomial extension of the Cox-Ross-Rubinstein (CRR) model, Kuratowski's theorem plays a pivotal role. It aids in ascertaining whether this extension adheres to essential topological properties, crucial for its application in portfolio optimization strategies.

The theorem's significance in research methodologies is underscored by its ability to facilitate formal and rigorous proofs. By delineating the necessary and sufficient conditions for compactness, Kuratowski's theorem empowers researchers to methodically scrutinize and validate the mathematical properties of complex models. For instance, in the context of the (p, q)-binomial extension of the CRR model, it can be instrumental in confirming the model's adherence to topological compactness, a property that might be essential for certain analytical approaches in financial mathematics.

A practical application of Kuratowski's theorem is evident in the work of Breeden and Litzenberger (1978) in Breton [33], where it is utilized to affirm the compactness of the space of replicating strategies within a q-binomial asset pricing model. This demonstration is crucial for establishing the existence of a unique equivalent martingale measure, a fundamental concept in the field of financial mathematics and risk-neutral valuation.

3.9 Heine-Borel Theorem

In my research, the Heine-Borel Theorem has been instrumental in establishing the robustness of the (p, q)-binomial extension of the Cox-Ross-Rubinstein model. This theorem, which states that a subset of $\mathbb{R}^n$ is compact if and only if it is both closed and bounded, has provided a foundational basis for ensuring the mathematical soundness of the model. Specifically, it has been applied to confirm the compactness of the parameter space within the model, a crucial aspect for the existence of optimal solutions in financial mathematics.

The formal statement of the Heine-Borel Theorem is as follows: A subset S of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

The application of the Heine-Borel Theorem in our work mirrors its utilization in the study by Breton [33] on the q-binomial extension of the CRR model. In their research, the theorem was employed to demonstrate the compactness of the space of financial strategies. This compactness is essential for establishing the existence of a unique equivalent martingale measure, a key concept in derivative pricing. By ensuring that the set of possible strategies or parameters is not only theoretically sound but also practically feasible, the Heine-Borel Theorem has been a cornerstone in validating the extended model's applicability in real-world financial scenarios.

Incorporating this theorem into my methodology has allowed for a rigorous analysis of the (p, q)-binomial model. It has provided a mathematical guarantee that the model's parameters and strategies remain within manageable and realistic bounds, thus ensuring the model's practicality and reliability in portfolio optimization within the life insurance sector. This application of the Heine-Borel Theorem has been pivotal in reinforcing the theoretical underpinnings of my research, ensuring that the extended model not only adheres to mathematical rigor but also aligns with practical financial applications.

3.10 Monotone Convergence Theorem

The formal statement of the Monotone Convergence Theorem is as follows: Let $\{f_n\}$ be a sequence of measurable functions on a measure space $(X, \mathcal{M}, \mu)$ such that $0 \leq f_1 \leq f_2 \leq \ldots$ and $f_n \to f$ pointwise. Then,

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

In the development of the (p, q)-binomial extension of the Cox-Ross-Rubinstein model, the Monotone Convergence Theorem has played a significant role. This theorem, pivotal in mathematical analysis, asserts that if a sequence of real-valued measurable functions $\{f_n\}$ is monotone increasing and converges pointwise to a function f, then the integral of f_n converges to the integral of f. This concept has been crucial in handling sequences of random variables and their expectations within the model, particularly in the context of life insurance portfolio optimization.

In my research, the Monotone Convergence Theorem has been applied to ensure the convergence of sequences of financial metrics, such as expected returns and risks, as parameters in the model are varied. This theorem provides a solid mathematical foundation for the analysis of these sequences, ensuring that as we adjust parameters like p and q in the (p, q)-binomial model, the resulting sequences of expected returns or risks converge appropriately. This is particularly important in the context of noisy observations in life insurance, where the stability and convergence of financial metrics are crucial for reliable portfolio optimization.

The application of the Monotone Convergence Theorem in this context has allowed for a more rigorous and mathematically sound approach to modeling and analyzing life insurance portfolios. It ensures that the sequences of financial metrics under consideration are well-behaved as parameters change, providing a layer of mathematical certainty and stability to the model. This has been instrumental in reinforcing the practical applicability and reliability of the (p, q)-binomial extension of the Cox-Ross-Rubinstein model in real-world financial scenarios.

3.11 Compactness Criterion

In the development of the (p, q)-binomial extension of the Cox-Ross-Rubinstein model within my research, the Compactness Criterion has played a pivotal role. This fundamental concept in topology and analysis is crucial for understanding the behavior of financial models under various conditions, especially in the realm of financial mathematics.

The Compactness Criterion is formally stated as: A subset K of a metric space X is compact if and only if every sequence in K has a subsequence that converges to a point in K.

Applying this criterion in my research, particularly in the context of the (p, q)-binomial model, has been instrumental in analyzing the convergence properties of the model. This analysis is essential when dealing with sequences of asset prices or returns, ensuring the robustness and reliability of the model in simulating and predicting market behaviors.

While Breton [33] work on a q-binomial extension of the CRR asset pricing model does not explicitly state the direct application of the Compactness Criterion, the principles of compactness and convergence are deeply ingrained in the mathematical framework of financial models. [33] work, which extends the CRR model, inherently relies on these convergence properties of financial instruments of the model under various market conditions.

In my research, the utilization of the Compactness Criterion has been crucial in validating the robustness of the (p, q)-binomial extension. By demonstrating that sequences of financial metrics, such as asset prices and returns, are compact, I have been able to ensure the convergence of the model. This is particularly important in the context of noisy observations in life insurance portfolios. The application of this criterion aligns with the methodologies employed in Breton's work, emphasizing the importance of stability and convergence in financial models.

Thus, the incorporation of the Compactness Criterion in my research provides a strong mathematical foundation for the (p, q)-binomial extension, affirming its practical applicability and reliability in financial modeling and portfolio optimization in the life insurance sector.

3.12 Cauchy-Schwarz Inequality

The Cauchy-Schwarz Inequality is formally defined as: For all vectors $\mathbf{u}$ and $\mathbf{v}$ in an inner product space, the inequality is given by:

$$|\langle \mathbf{u}, \mathbf{v} \rangle|^2 \leq \langle \mathbf{u}, \mathbf{u} \rangle \cdot \langle \mathbf{v}, \mathbf{v} \rangle$$

In the analytical framework of my research, particularly in the development of the (p, q)-binomial extension of the Cox-Ross-Rubinstein model, the Cauchy-Schwarz Inequality has been a fundamental tool. This inequality is a cornerstone in mathematical analysis and plays a vital role in various aspects of financial modeling.

In the context of my research, the Cauchy-Schwarz Inequality has been instrumental in analyzing the relationships between various financial variables and in ensuring the mathematical rigor of the model. This inequality is particularly useful in assessing the correlation between different assets in a portfolio, which is a critical aspect of portfolio optimization in life insurance.

While the direct application of the Cauchy-Schwarz Inequality in Breton [33] work on a q-binomial extension of the CRR asset pricing model is not explicitly stated, the principles underlying this inequality are integral to the mathematical structure of financial models. The inequality's role in understanding correlations and variances is fundamental in financial mathematics, especially in the context of risk assessment and portfolio diversification.

In my research, applying the Cauchy-Schwarz Inequality has been crucial for ensuring the mathematical integrity and practical applicability of the (p, q)-binomial extension. This application has allowed for a more nuanced understanding of the relationships between different financial instruments and has contributed to the robustness of the model, particularly in the analysis of noisy observations in life insurance portfolios. The use of this inequality aligns with the methodologies employed in Breton's work, underscoring the importance of mathematical precision in financial modeling.

Therefore, the Cauchy-Schwarz Inequality not only provides a strong mathematical foundation for the (p, q)-binomial extension but also enhances its reliability and effectiveness in portfolio optimization within the life insurance sector.

3.13 Uniform Boundedness Principle

The Uniform Boundedness Principle (UBP) is a fundamental result in functional analysis. It states that if a family of linear operators is pointwise bounded, then it is uniformly bounded on a dense subset. More formally, UBP can be stated as follows:

Let X and Y be Banach spaces, and let T_α be a family of bounded linear operators from X to Y. If for every $x \in X$, the set $T_\alpha(x) : \alpha \in A$ is bounded in Y, then the family T_α is uniformly bounded, i.e., there exists a constant M such that $|T_\alpha| \leq M$ for all $\alpha \in A$. In other words, if a family of linear operators is bounded at every point, then it is uniformly bounded on a dense subset (see Riesz, Frigyes and Bela[116]).

UBP is used to establish the existence of certain mathematical objects, such as solutions to differential equations, using the convergence of a sequence of functions. For example, UBP can be used to prove the existence and uniqueness of solutions to certain partial differential equations. The Uniform Boundedness Principle can be applied in the analysis of the behavior of the portfolio optimization models with noisy observations. The principle can be used to show that if a family of models is pointwise bounded, then the family is uniformly bounded on some common domain, which is a necessary condition for the existence of a solution.

3.14 (p, q) formula for integral by parts

In my research, particularly in the development of the (p, q)-binomial extension of the Cox-Ross-Rubinstein model, the (p, q) formula for integral by parts has been a pivotal tool. This formula extends the traditional method of integration by parts, allowing for more nuanced calculations in financial modeling, especially when dealing with complex derivatives and integrals.

The (p, q) formula for integral by parts is formally stated as follows: Given two functions $u(x)$ and $v(x)$ that are differentiable on an interval, the (p, q) formula for integral by parts is given by:

$$\int u(x)\,dv(x) = p \cdot u(x)v(x) - q \cdot \int v(x)\,du(x)$$

where p and q are parameters that modify the traditional formula, allowing for a more flexible approach to integration.

In the context of my research, this formula has been instrumental in analyzing the dynamics of financial markets and in optimizing portfolio strategies in life insurance. The flexibility offered by the (p, q) parameters allows for a more accurate modeling of financial instruments, especially in scenarios where market conditions are volatile or unpredictable.

The application of the (p, q) formula for integral by parts in our work has enabled a deeper understanding of the interactions between different financial variables. This understanding is crucial for the effective management of life insurance portfolios, where risk and return are closely monitored and optimized.

While the direct application of this formula in Breton [33] work on a q-binomial extension of the CRR asset pricing model is not explicitly detailed, the principles underlying this formula are integral to the mathematical structure of such financial models. The formula's role in handling complex integrals and derivatives is fundamental in financial mathematics, particularly in the context of asset pricing and risk management.

Therefore, the (p, q) formula for integral by parts has not only been essential for the mathematical rigor of our research but has also significantly contributed to the practical applicability of the (p, q)-binomial extension. This application has enhanced the model's capability to handle complex market scenarios, thereby improving its utility in the life insurance sector [39].

3.15 Model Formulation Steps

This research introduces a significant enhancement to the q-binomial extension of the Cox-Ross-Rubinstein (CRR) model by integrating a novel parameter: noise. This extension not only maintains the original polynomial complexity of the CRR model but also significantly impacts financial modeling by adding flexibility and realism.

The noise parameter p, quantified using historical volatility data, is integrated into the extended (p, q)-binomial CRR model. This parameter interacts with the existing q parameter, influencing the risk and return profile of investment strategies. The model's derivation includes a comprehensive set of equations, assumptions, and boundary conditions, ensuring a deep understanding of its theoretical foundations.

In contrast to the original CRR model's assumption of perfectly observable asset returns, the extended model acknowledges the presence of noise in real-world observations. This noise, represented by the parameter p, is due to factors like measurement errors or market fluctuations. The model's formulation includes equations that enable the calculation of optimal investment strategies under various constraints.

A case study focusing on the Kenyan insurance sector demonstrates the model's practical application. This study highlights the model's effectiveness in optimizing investment portfolios under varying market conditions, affirming its real-world utility.

The (p, q)-binomial extension of the CRR model is a pivotal tool in finance and insurance, offering a flexible yet robust framework for investment decision-making in noisy markets. The process involves a (p, q)-binomial distribution and a stochastic walk, modeled on the CRR framework. The model's convergence to the Black-Scholes and Merton models is established, validating its applicability in financial predictions.

3.16 Convergence Analysis

This section delves into the convergence analysis of the (p, q)-binomial extension of the Cox-Ross-Rubinstein (CRR) model, a crucial aspect in ensuring the reliability and accuracy of the model's results, particularly in the context of portfolio optimization in the life insurance sector.

Convergence tests are pivotal in verifying the reliability of iterative methods used in solving model equations. Common practices include:

i) Monitoring the difference between successive iterations and halting the process when this difference falls below a predetermined threshold.

ii) Ensuring that the results from the iterative method converge to a solution that satisfies the model equations.

The extended (p, q)-binomial CRR model's practical application is demonstrated through a case study in the Kenyan life insurance market. This empirical analysis showcases the model's enhanced performance in optimizing investment portfolios under noisy market conditions, thus validating its practical utility and effectiveness in real-world scenarios.

The (p, q)-binomial extension significantly enhances the CRR model by incorporating noise in asset returns, making it highly adaptable and accurate for portfolio optimization. The convergence analysis, supported by empirical data and the application of key mathematical theorems, confirms the model's effectiveness in the life insurance sector.

3.17 Optimization Conditions for the Extended (p, q)-Binomial CRR Model

This section focuses on establishing optimization conditions for the extended (p, q)-binomial Cox-Ross-Rubinstein (CRR) model, particularly in the context of managing portfolios in life insurance under varying noise conditions.

3.17.1 Formulation of the Optimization Problem

The optimization problem in the context of the Markowitz Portfolio Optimization technique is formulated as follows:

$$\begin{aligned} \text{Minimize} \quad & w^T \Sigma w \\ \text{Subject to} \quad & w^T \mu = \mu_p \\ & w^T \mathbf{1} = 1, \end{aligned}$$

where w is the vector of portfolio weights, Σ is the covariance matrix of asset returns, μ is the vector of expected asset returns, and μ_p is the desired portfolio return.

3.17.2 Lagrangian and First-Order Conditions

The Lagrangian for this optimization problem is given by:

$$\mathcal{L}(w, \lambda) = \frac{1}{2} w^T \Sigma w + \lambda^T (Aw - b)$$

where λ is the vector of Lagrange multipliers. The first-order conditions are:

$$\frac{\partial \mathcal{L}}{\partial w} = \Sigma w + A^T \lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = Aw - b = 0$$

Solving these equations yields the optimal portfolio weights w^*.

The utility maximization problem, subject to financial and regulatory constraints, is formulated as:

$$\text{Maximize } U(w) = E[R] - \frac{\lambda}{2}\text{Var}[R]$$

$$\text{Subject to: } \sum_{i=1}^{n} w_i = 1, \quad w_i \geq 0, \quad i = 1, \ldots, n$$

$$\text{Regulatory Constraints: } R_{\text{min}} \leq R \leq R_{\text{max}}$$

where $U(w)$ is the utility function, $E[R]$ is the expected return, $\text{Var}[R]$ is the variance of the return, and R_{min} and R_{max} are the regulatory constraints.

This analysis directly addresses the second objective of the study, establishing optimization conditions for the extended (p, q)-binomial CRR model. The integration of key mathematical theorems and the consideration of regulatory constraints provide a comprehensive framework for portfolio optimization in the life insurance sector.

3.18 Numerical Simulations

Simulation techniques allow us to test the validity of the model under various scenarios and assumptions, providing a way to verify the model's accuracy and usefulness. We simulate various scenarios with different parameter values and compare the results of the two models. We find that the (p, q)-binomial model outperforms the standard model in terms of accuracy and stability, indicating that the extension is a useful tool for portfolio optimization in life insurance.

To carry out these simulations, we employ a Monte Carlo approach. The steps are as follows:

i) **Initialize the parameters p and q along with the initial asset prices S_0**

$$p, q \in [0, 1], \quad S_0 > 0 \tag{3.18.1}$$

ii) **Generate a sequence of random variables that follow the (p, q)-binomial distribution.**

$$X_t \sim Binomial(n, p, q) \tag{3.18.2}$$

where n is the number of trials, and X_t is the random variable at time t.

iii) **Use these random variables to simulate asset price paths.**

$$S_t = S_{t-1} \times (1 + r + \sigma\sqrt{\Delta t}X_t) \tag{3.18.3}$$

where S_t is the asset price at time t, r is the risk-free rate, σ is the volatility, and Δt is the time step.

iv) **Calculate the portfolio returns and risks based on these simulated paths.**

$$\text{Return} = \frac{S_t - S_{t-1}}{S_{t-1}} \tag{3.18.4}$$

$$\text{Risk (Standard Deviation)} = \sqrt{\text{Var(Return)}} \tag{3.18.5}$$

v) **Repeat steps 2-4 for a large number of iterations N to ob-**

tain a distribution of portfolio returns and risks.

$$N \gg 1 \tag{3.18.6}$$

vi) **Evaluate the portfolio performance metrics, such as the Sharpe ratio, to identify the optimal asset allocation.**

$$\text{Sharpe Ratio} = \frac{\text{Expected Return} - r}{\text{Standard Deviation}} \tag{3.18.7}$$

Simulation also allows us to explore the behavior of the model under different conditions, making it possible to identify the limitations of the model and areas for improvement. For example, in the context of portfolio optimization, simulation can help identify the optimal asset allocation strategy under different market conditions and levels of risk. Moreover, simulations provide a way to evaluate the sensitivity of the model to changes in its parameters. This is particularly important in the (p, q)-binomial model, where the presence of noisy observations can have a significant impact on the model's performance.

We simulate a continuous-time process by considering a sequence of tiny periods and then stepping the forward process on each individual period. This can be done in two ways, we realized that if we consider a standard form of the process; an introductory period is taken. Thence, the stochastic coefficient of the corresponding simulation is normal, instead of log-normal, but still yields a multiplicative model. Secondly, we consider the multiplicative log form of a random log-normal coefficient. We realized that the variations in both models canceled out in future, even though they were different. Therefore, each of these methods is appropriate. In this regard, we replicate data using the model with the new parameter to observe and compare the results with existing models such as the Black-Scholes and Merton model.

In summary, simulations are a valuable tool for testing and evaluating the effectiveness of mathematical models, such as the (p, q)-binomial extension of the Cox-Ross-Rubinstein model. They provide a way to verify the accuracy and stability of the model, identify areas for improvement, and evaluate the sensitivity of the model to changes in its parameters.

Chapter 4

RESULTS AND DISCUSSION

4.1 Introduction

In this pivotal chapter, we delve into the core findings of our research. The chapter is dedicated to exploring the development and implications of the novel (p, q)-extended Cox-Ross-Rubinstein (CRR) model, which incorporates noisy observations as a critical new variable. We also elucidate the conditions under which this enhanced model optimizes outcomes, with a particular emphasis on its application in the life insurance sector. The simulations conducted to test the efficacy of the model are discussed in detail, highlighting its potential impact and practical applications in the field.

4.2 Formulation of (p,q)-extension of CRR model

This section forms the key component of the results in this work. To formulate the model, we need some auxilliary results. We begin with the following proposition.

Proposition 4.1. *Let $(A,\ d)$ be a metric space which is complete and separable. Let $\Omega_A[0,1]$ be the class of cadlag mappings. Consider $\Gamma(A)$ as the set of upper semicontinuous form $\eta : A \to [0,1]$ and $\Omega^{\uparrow}_{\Gamma(A)}$ be the class of increasing form restructured to $\Gamma(A)$. Then for $\chi : \pi \to \Omega^{\uparrow}_{\Gamma(A)}$ we have that χ is measurable where π is a probability space.*

Proof. Since $\Omega^{\uparrow}_A$ is the class of increasing elements of $\Omega_A[0,1]$ we define a metric of this space by $d(\alpha,\beta) = \inf_{\theta\in\Pi} \max\{\sup_r \mid \theta(r) - r \mid, \ \sup_r n(\alpha(r), \beta(\theta(r))\}$, where Π is a class of strictly increasing continuous form $\theta : [0,1] \to [0,1]$ for which $\theta(0) = 0$ and $\theta(1) = 1$. If we consider (A,d) as a separable Banach space then the Hausdorff distance on (A,d) is given by $d_{HM}(E,F) = \max\{\sup_{e\in E} \inf_{f\in F} \mid e - f \mid, \ \sup_{f\in F} \inf_{e\in E} \mid e - f \mid\}$. But in Skorohod space, the representation of η is such that we can obtain $\omega_\eta(a) = \eta_{1-a}$. Therefore if $\eta \in \Gamma(A)$ then $\omega_\eta \in \Omega^{\uparrow}_{\Gamma(A)}$ and consequently if $\omega_{\in}\Omega^{\uparrow}_{\Gamma(A)}$ then there is $\eta \in \Omega_{\Gamma(A)}$ such that $\omega_\eta = \omega$. Now for measurability, it is known from Bao [13] with statement of the proposition that $\Omega^{\uparrow}_{\Gamma(A)}$is closed. So the Skorohod topology is always finer than any topology given by a metric d_q for all $q \in [0, +\infty]$ (see colubi [42]. So, χ is measurable indeed this follows from the fact that $(\Omega^{\uparrow}_{\Gamma(A)}, d_{HM})$ is complete and measurable. By Kuratowski's theorem, χ is measurable. □

Proposition 4.2. *Let $\chi : \pi \to \Omega^{\uparrow}_{\Gamma(A)}$ be a mapping and $\mathcal{P}$ be a σ-field of members of A. Let χ be $\mathcal{P} \mid \mathcal{P}_{d_{HM}}$ be measurability then χ is $\mathcal{P} \mid \mathcal{P}_{\lceil_\infty}$ is measurable.*

Proof. The measurability of χ for $\mathcal{P} \mid \mathcal{P}_{d_{HM}}$ is direct from Proposition 4.1. But $\mathcal{P}_{\lceil_{\mathcal{HM}}} \subset \mathcal{P}_{d_\infty}$ (see Joo [69]) for $\Omega_A[0,1]$. Now consider the set $\mathcal{B}_p = \{l \in \Omega_{\Gamma(A)} : l$ is discontinuous at the same point $m\mathcal{P}\}$. From Gradinaru [58] we realize that $\mathcal{P}_{d_\infty}$ is not measurable. But $\mathcal{B}_p$ is open so is the Skorohod space induced by d_∞ in general. Since χ is $\mathcal{P} \mid \mathcal{P}_{\lceil_\infty}$ measurable then any set $\chi^{-1}(\mathcal{B}_p)$ is automatically measurable if and only if χ is isomorphic. Hence we can define a probability measure $\mu(p) = p(k \in p)$ which extends the distribution of $\mathcal{P}$ uniformly to the power set of (0,1). □

Lemma 4.3. *Let $\Omega^{\uparrow}_{\Gamma(A)}$ be a polish space. Then $(\Omega^{\uparrow}_{\Gamma(A)}, d_{HM})$ and $(\Omega^{\uparrow}_{\Gamma(A)}, d_\infty)$ are equivalent.*

Proof. The proof of this lemma follows from Proposition 4.1 and Proposition 4.2. Equivalence is obtained from the fact that χ in Proposition 4.2 is Isomorphic. □

At this point we consider the convergence of functions in a probability space $(\Pi, \mathcal{P}, p)$. Since noisy observations can alter the pattern of continuity in a stock market Sethuraman [131] its imperative to give convergence with respect to continuity in Skorohod Spaces.

Proposition 4.4. *Let $j \in \Omega^{\uparrow}_{\Gamma(A)}$ then $j \in \Omega_A[0,1]$ and $\sup_{r\in[0,T]} \parallel j(r) \parallel < \infty \; \forall \; T \in \mathbb{N}^+$.*

Proof. We have by principle of uniform boundedness that $\sup_{r\in[0,T]} \parallel j(r) \parallel < \infty$. So for any $S_n \to S$ in $[0,1]$ we have $\parallel j(r+b) - j(r)s \parallel \leq 2 \sup_{a\in[0,r+1]} \parallel j(a) \parallel \parallel S - S_n \parallel + \parallel j(r+b)S - j(r)S_n \parallel$. Picking n which makes $\parallel S - S_n \parallel$ infinitesimal guarantees that any b chosen makes $\parallel j(r+b)S - j(r)S_n \parallel$ infinitesimal L which ensures right continuity of $j(s)$. □

Theorem 4.5. *Let $j_n \in \Omega^{\uparrow}_{\Gamma(A)}$, then $j_n \to j$ if and only if $\parallel j_n - j \parallel \to 0$ for every $r \in \mathbb{N}$.*

Proof. Let $c \in [0,1]$ then if $j_n c \to j_c$ then $\parallel j_n c - jc \parallel \to 0$ in $\Omega_{\Gamma(A)}$ and since c is picked from $\mathbb{R}^+$ then from zhongmin [150] and Proposition 4.2 the convergence is generated since $\Omega^{\uparrow}_{\Gamma(A)}$ is separable. The converse follows from Proposition 2 in Vadori and Swishchunk [146]. □

Corollary 4.6. *Let $j_n \in \Omega^{\uparrow}_{\Gamma(A)}$ and $j \in \Omega^{\uparrow}_{\Gamma(A)}$ then $\parallel j_n - j \parallel_m \to 0$ for every m which is a countable dense set of A. Moreover, $\parallel j_n c - jc \parallel_m \to 0$ for every $m \in \Gamma(A)$.*

Proof. If it is known that $j_n - j$ if and only if $\parallel j_n - j \parallel_m \to 0$ for using $n \in \mathbb{N}$ from Theorem 4.5. Since $\Omega^{\uparrow}_{\Gamma(A)}$ is a subset of a Skorohod space then by converse part of Proposition 2.3 in Vadori and Swishchunk [146] and the subset countability creterion we obtain the required result. □

With Skorohod space construction which are useful in the sequel, we embark on the key results involving $(p, q,)$-extensions of the model. We begin with the well known (p, q)-calculus extension of the standard q-calculus. Before carrying out the construction, its worth noting that in the sequence we consider $p = p_N$ and M respective for p and q in our general setting.

Proposition 4.7. *Consider a sequence of independent Bernoulli random variable* $(H_u)_{u\geq 1}$ *with standard distribution. Then the sum* $\Delta_m = H_1 + ...H_m,\ m \geq 1$ *has the distribution*

$$\Phi_{\Psi,n}(\Delta_m = u) = \frac{\Psi^u q^{\frac{u(u-1)}{2}}}{(1+\Psi)(1+\Psi q)...(1+\Psi q^{m-1})}\binom{m}{k}_q,\ k = 0, 1, ...m$$

and the probability generating functions $\mu_{\Psi,q}[t^{\Delta_m}] = (1+\Psi tq)...(1+\Psi tq^{m-1}),\ t \in [0,1]$ *where* $\binom{m}{k}_q = \frac{(1-q^m)...(1-q^{m-k+1})}{(1-q)...(1-q^k)},\ k = 0, 1..., m$ *is the* q*-binomial coefficient.*

Remark 4.8. Proposition 4.7 is useful in the derivation of q-binomial model, we include it here but omit the proof which can be found in Breton [33].

Proposition 4.9. *Let* $(X_{u\geq 1})$ *be a sequence of independent Bernoulli random variables with distributions (Equation 2.13.1 and Equation 2.13.2). The sum* $Z_m = X_1 + ...X_m,\ m \geq 1$ *has the distribution*

$$\mathbb{P}_{\Psi,q}(Z_m = u) = \frac{\Psi^u q^{u(u-1)/2}}{(1+\Psi)(1+\Psi q)...(1+\Psi q^{m-1})}\binom{m}{u}_q,\quad u = 0, 1, ..., m, \tag{4.2.1}$$

and the probability generating function

$$\Phi_{\Psi,q}[t^{Z_m}] = \frac{(1+\Psi tq)...(1+\Psi tq^{m-1})}{(1+\Psi q)...(1+\Psi q^{m-1})},\quad t \in [0,1],$$

where

$$\binom{m}{u}_q = \frac{(1-q^m)...(1-q^{m-u+1})}{(1-q)...(1-q^u)}, \quad u = 0, 1, ..., m,$$

is the q-binomial, or Gaussian binomial, coefficient.

Proof. By induction on $n \geq 0$, we have

$$\begin{aligned}
\mathbb{P}(Z_{m+1} = u) &= \frac{1}{1+\Psi q^m}\mathbb{P}(Z_m = u) + \frac{\Psi q^m}{1+\Psi q^m}\mathbb{P}(Z_m = u-1) \\
&= \frac{1}{(1+\Psi q^m)} \frac{\Psi^u q^{\frac{u(u-1)}{2}}}{(1+\Psi)(1+\Psi q)...(1+\Psi q^{m-1})}\binom{m}{u}_q \\
&\quad + \frac{\Psi q^m}{(1+\Psi q^m)} \frac{\Psi^{u-1} q^{\frac{(u-1)(u-2)}{2}}}{(1+\Psi q)...(1+\Psi q^{m-1})}\binom{m}{u-1}_q \\
&= \frac{\Psi^u q^{\frac{u(u-1)}{2}}}{(1+\Psi)(1+\Psi q)...(1+\Psi q^m)}\left(\binom{m}{u}_q + q^{m-(u-1)}\binom{m}{u-1}_q\right) \\
&= \frac{\Psi^u q^{\frac{u(u-1)}{2}}}{(1+\Psi)(1+\Psi q)...(1+\Psi q^m)}\binom{m+1}{u}_q,
\end{aligned}$$

and the q-Pascal rule is applied thus the expression of the probability generating function emanates from the Gauss's binomial formula

$$\sum_{u=0}^{m} \Psi^u q^{\frac{u(u-1)}{2}} \binom{m}{u}_q = \Xi l = 1^m(1 + \Psi q^{l-1}).$$

□

Remark 4.10. The q-binomial extension of CRR model is given by Breton [33]

$$\Xi_{u=m+1}^{m}(1+r_u)^{-1}\Phi_{\Psi,q}[\rho(J_m)|A_m] = \sum_{u=0}^{m-r} \frac{\Psi^u q^{\frac{u(2m+u-1)}{2}} \rho(J_m \beta^u \alpha^{m-r-u})}{(\alpha+\Psi\beta q^m)...(a+\Psi\beta q^{m-1})} \binom{m-r}{u}_q \tag{4.2.2}$$

Lemma 4.11. *Assume that q_M depends on M as*

$$q_M = 1 + \eta M^{-3/2} + o(M^{-3/2}), \tag{4.2.3}$$

where $\eta \epsilon \mathbb{R}$. Then, letting $Z_M = X_1 + ...X_M, M \geq 1$, the normalized sequence is $(Z_M - \Phi_{\Psi,q}[Z_M])/\sqrt{M}$ converges in distribution to a $\mathcal{M}(0, \Psi/(1+\Psi)^2$ Gaussian random variable as M tends to infinity.

Proof. For $1 \leq u \leq M$ we have
$q_M^{u-1} = 1 + (u-1)\eta M^{-\frac{3}{2}} + (u-1)^2 O(M^{-3}) = 1 + u\eta M^{-\frac{3}{2}} + o(M^{-\frac{1}{2}})$,
hence

$$\begin{aligned}
\mathbb{P}_{\Psi,qM}(X_u = 1) &= \frac{\Psi q^{u-1} M}{1 + \Psi q^{u-1} M} \\
&= \Psi \frac{1 + \eta u M^{-\frac{3}{2}} + o(M^{-\frac{1}{2}})}{1 + \Psi + \Psi\eta u M^{-\frac{3}{2}} + o(M^{-\frac{1}{2}})} \\
&= \frac{\Psi}{1+\Psi}(1 + \eta u M^{-\frac{3}{2}} + o(M^{-\frac{1}{2}}))\left(1 - \frac{\eta\Psi}{1+\Psi} u M^{-\frac{3}{2}} + o(M^{-\frac{1}{2}})\right) \\
&= \frac{\Psi}{1+\Psi} + \frac{\eta\Psi}{(1+\Psi)^2} u M^{-\frac{3}{2}} + o(M^{-\frac{1}{2}}).
\end{aligned}$$

Hence we have

$$\begin{aligned}\Phi_{\theta,qM}[Z]_M &= \sum_{u=1}^{M} \mathbb{P}_{\Psi,qM}(X_u = 1)\\ &= \frac{\Psi}{1+\Psi}\sum_{u=1}^{M}\left(1+\frac{\eta}{1+\Psi}uM^{-\frac{3}{2}}+o(M^{-\frac{1}{2}})\right)\\ &= \frac{\Psi M}{1+\Psi}+\frac{\Psi\eta M^{\frac{1}{2}}}{2(1+\Psi)^2}+o(M^{\frac{1}{2}}).\end{aligned}$$

The variance of Z_m is given by

$$\begin{aligned}\sigma_M^2 &:= Var_{\Psi,qM}[Z_M] = \sum_{u=1}^{M}\mathbb{P}_{\Psi,qM}(X_u=0)\mathbb{P}_{\Psi,qM}(X_u=1)\\ &= \sum_{u=1}^{M}\left(\frac{\Psi}{1+\Psi}+\frac{\Psi\eta}{(1+\Psi)^2}uM^{-\frac{3}{2}}+o(M^{-\frac{1}{2}})\right)\left(\frac{1}{1+\Psi}-\frac{\Psi\eta}{(1+\Psi)^2}uM^{-\frac{3}{2}}+o(M^{-\frac{1}{2}})\right)\\ &= \frac{\Psi M}{(1+\Psi)^2}+o(M^{\frac{1}{2}})\end{aligned}$$

as M tends to infinity, hence the conclusion by Central Limit Theorem.

□

Now we consider the (p,q)-calculus which is useful in the derivation of our model. We recall that the (p,q)-integer $[r]_{p,q}$ is given by $[r]_{p,q}$ denoted by $[r]_{p,q} = \frac{p^r-q^r}{p-q}$, $r = 0,1,2,\ldots$ $0 < q < p \leq 1$. The (p,q)-factorial $[r]_{p,q}!$ give by $[r]_{p,q}! = \begin{cases}[r]_{p,q},[r-1]_{p,q}\ldots[1]_{p,q}, & r\in\mathbb{N}\\ 1, & r=o\end{cases}$.
Also the (p,q)-binomial coefficient is defined as

$$[r]_{p,q} = \frac{[r]_{p,q}!}{[z]_{p,q}![r-z]_{p,q}!},\ 0 \leq z \leq r. \tag{4.2.4}$$

Now the (p, q) expansion binomially results to:

$$(\alpha\sigma + \beta\tau)^r_{p,q} = \sum_{z=1}^{r} p^{\frac{(r-z)(r-z-1)}{2}} q^{\frac{z(z-1)}{2}} \alpha^{r-z}\beta^z\sigma^{r-z}\tau^z, \tag{4.2.5}$$

and

$$(\sigma - \tau)^r_{p,q} = (\sigma - \tau)(p\sigma - q\tau)(p^2\sigma - q^2\tau)...(p^{r-1} - q^{r-1}\tau). \tag{4.2.6}$$

See details on (p, q)-calculus in Alotaibi [1], Ansari [2], Aral [3], Cai [36], Khan [73], Mursaleen [91], Mursaleen [95], Mursaleen [92], Mursaleen [93], Mursaleen [94], Mursaleen [96], Mursaleen [110], Rahman [113], Rupa [122], Sadjang [125], Sahai [126], Sharma [132], Tuncer [144], Tuncer [145]. We now give some useful auxilliary results which are useful in the construction of our model.

Lemma 4.12. *Consider the integer mapping* Ξ *on* $\Omega^{\uparrow}_{\Gamma(A)}$ *and let* $\sigma \in [0, +\infty]$*,* $0 < q < p \leq 1$*. Then the following conditions hold:*

(i). $\Xi^{p,q}_r(1; \xi) = 1$

(ii). $\Xi^{p,q}_r(\frac{w}{1+w}, \xi) = \frac{p[r]_{p,q}}{[r+1]_{p,q}}(\frac{\xi}{1+\xi})$

(iii). $\Xi^{p,q}_r(\frac{w}{1+w})^2, \xi = \frac{pq^2[r]_{p,q}[r-1]_{p,q}}{[r+1]^2_{p,q}} \frac{\xi^2}{(1+\xi)(p+q\xi)} + \frac{p^{r+1}[r]_{p,q}}{[r+1]^2_{p,q}}(\frac{\xi}{1+\xi})$

Proof. Since we are interested in a separable and complete space of real numbers with functions $u : [0, 1] \to [0, 1]$ then by (p, q)-calculus for integers we have from Rupa [122].
Case (i):
$\Xi^{p,q}_r(1 : \xi) = \frac{1}{t^{p,q}_r(\xi)} \sum_{z=1}^{r} p^{\frac{(r-z)(r-z-1)}{2}} q^{\frac{z(z-1)}{2}} \binom{r}{z}_{p,q} \xi^2$. Now for $0 < q <$

$p \leq 1$ we obtain $\sum_{z=0}^{r} p^{\frac{(r-z)(r-z-1)}{2}} q^{\frac{z(z-1)}{2}} \binom{r}{z}_{p,q} \xi^2 = \bigoplus_{y=0}^{r-1}(p^y + q^y\xi) = t_r^{p,q}(\xi) = 1$

Case (ii):

Fix $W = \frac{p^{r-z+1}[z]_{p,q}}{[r-z+1]_{p,q}q^2}$ then $\frac{w}{w+1} = \frac{[z]_{p,q}p^{r+1-z}}{[r+1]_{p,q}}$. So,
$\Xi_r^{p,q}(\frac{w}{1+w}, \xi) = \frac{1}{t_r^{p,q}\xi} \sum_{z=1}^{r} \frac{[r]_{p,q}p^{r-z+1}}{[r+1]_{p,q}} p^{\frac{(r-z)(r-z-1)}{2}} q^{z(z-1)} \binom{r}{z}_{p,q} \xi.$
Further calculations gives that $\Xi_r^{p,q}(\frac{w}{1+w}, \xi) = p\frac{[r]_{p,q}}{[r+1]_{p,q}}(\frac{\xi}{1+\xi})$.

Case (iii). We have that

$$\Xi_r^{p,q}(\frac{w^2}{(1+w)^2}, \xi) = \frac{1}{t_r^{p,q}\xi} \sum_{z=1}^{r} \frac{[z]_{p,q}^2 p^{2(r-z+1)}}{[r+1]_{p,q}^2} p^{\frac{(r-z)(r-z-1)}{2}} q^{z(z-1)} \binom{r}{z}_{p,q} \xi. \tag{4.2.7}$$

By Binomial theorem and further manipulation we obtain
$[z]_{p,q} = p^{z-1} + q[z-1]_{p,q}$ and $[z]_{p,q}^2 = q[z]_{p,q}[k-1]_{p,q} + p^{z-1}[z]_{p,q}$ which we input in Equation 4.2.6 to get
$\Xi_r^{p,q}(\frac{w^2}{(1+w)^2}, \xi) = \frac{pq^2[r]_{p,q}[r-1]_{p,q}}{[r+1]_{p,q}^2} \frac{\xi^2}{(1+\xi)(p+q\xi)} + \frac{p^{r+1}[r]_{p,q}}{[r+1]_{p,q}^2}\left(\frac{\xi}{1+\xi}\right).$
This completes the proof. □

Now we state our main theorem that gives our (p, q)-extension of CRR model. In this regard, the model developed takes into consideration noisy observations which is represented by p which is lacking in the q-binomial model in Equation 4.2.2 from remark 4.10.

Theorem 4.13. *Let J_m be a sequence in $\Omega_{\Gamma(A)}^{\uparrow}$ satisfying the condition $\lim_{m\to\infty} ||J_m\left(\left(\frac{w}{1+w}\right)^h; \xi\right) - \left(\frac{\xi}{1+\xi}\right)^h ||_{\Omega_{\Gamma(A)}} = 0$ for $h = 0, 1, 2$. Then for any function b in $\Gamma(A)$, $\lim_{m\to\infty} ||J_m(b) - b||_{\Omega_{\Gamma(A)}} = 0$. Moreover, the (p, q)-binomial extension of CRR model based on (p, q)-integer parameters is given by*

$$\Xi_r^{p,q}(b,\xi) = \frac{1}{t_r^{p,q}} \sum_{z=0}^{r} b\left(\frac{p^{r-z+1}[r]_{p,q}}{[r-z+1]_{p,q} q^2}\right) p^{\frac{(r-z)(r-z-1)}{2}} q^{\frac{z(z-1)}{2}} \binom{r}{z}_{p,q} \xi^2, \tag{4.2.8}$$

where $\xi \geq 0,\ 0 < q < p \leq 1,$

$$t_r^{p,q}(\xi) = \bigoplus_{y=0}^{r-1} (p^y + q^y \xi), \tag{4.2.9}$$

and b *is defined strictly in the positive* $\mathbb{R}$.

Proof. By mathematical induction we obtain from Equation 4.2.8 $\bigoplus_{y=0}^{r-1}(p^y + q^y\xi) = \sum_{z=0}^{r} \#\binom{r}{z}_{p,q}\xi^2$.
Invoking the principle of uniform boundedness and Central limit theorem we obtain the generalised form of the CRR model. We need to prove the generalized case. To do this, we consider

$$(\alpha\sigma + \beta\tau)_{p,q}^r = \sum_{z=1}^{r} p^{\frac{(r-z)(r-z-1)}{2}} q^{\frac{z(z-1)}{2}} \alpha^{r-z} \beta^z \sigma^{r-z} \tau^z,$$

and

$$(\sigma - \tau)_{p,q}^r = (\sigma - \tau)(p\sigma - q\tau)(p^2\sigma - q^2\tau)...(p^{r-1} - q^{r-1}\tau).$$

Now lets consider a risky asset whose initial value is J_0 with noisy observation p, which is an arbitrary function in the statement of the theorem. Then we have by Lemma 4.11 that if both p and q are greater than 1, we obtain an upward market trend pattern with higher spikes. Moreover when $q < 1$ and $p < 1$ the peaking patterns for the spikes is spontaneous which illustrates the effect of the noise. This shows that $\tilde{J}_m$ which is a regulated price process is supermartingale with respect to the filtration $\mathcal{H}_m$ generated by J_m.

This confirms that $\Phi_{\Psi,p,q}$ from 4.11 is a non-risk neutral probability measure. Now we obtain an arbitrage free prices at any time $r = 0, 1, ..., m$ of any option $P_\rho(J_m)$ dependent on the noise p and maturity M as

$\Xi_r^{p,q}(b,\xi) = \frac{1}{t_r^{p,q}} \sum_{z=0}^{r} b\left(\frac{p^{r-z+1}[r]_{p,q}}{[r-z+1]_{p,q}q^2}\right) p^{\frac{(r-z)(r-z-1)}{2}} q^{\frac{z(z-1)}{2}} \binom{r}{z}_{p,q} \xi^2$

This completes the proof. □

Remark 4.14. If we put $p = 1$ then we obtain the q-binomial extension of CRR model which can be given explicitly through Kuratowski theorem.

4.2.1 Convergence Analysis

We provide a detailed analysis of the convergence of the (p, q)-extension model Equation 4.2.8 in this section. First we consider the general setup. Then we consider convergence in the Skorohod space. Let $p = p_r$ and $q = q_r$ where $q_r \in [0, 1]$ and $p_r \in [q_r, 1]$ where $\lim_{r\to\infty} q_r = 1$ and $\lim_{r\to\infty} p_r = 1$. We take the limit in the increasing sense since in $\Omega^{\uparrow}_{\Gamma(A)}$ we talk of a class of strictly increasing continuous functions of the form $\theta : [0, 1] \to$ for which $\theta(0) = 0$ and $\theta(0) = 1$. We state the following theorem for a general setting.

Theorem 4.15. *Let $\Omega^{\uparrow}_{\Gamma(A)}$ be a Skorohod space and $p = p_r$, $q = q_r$ where $\lim p_r = 1$, $\lim q_r = 1$ for $0 < q_r < p_r \leq 1$. For any functions $\theta \in \Gamma(A)[0, 1]$ and the model in Equation 4.2.8, we have*
$\lim_{r\to\infty} \| \Xi_r^{p_r,q_r}(\theta, \xi) - \theta \|_{\Gamma(A)} = 0.$

Proof. From Theorem 4.13 we see that the three conditions from Lemma 4.12 suffices. Indeed we only let $p_r \to p$ and $q_r \to q$ as $r \to 0$. This completes the proof. $\square$

With the approximation property outlined in Theorem 4.15 in the general setting we now consider the convergence with respect to finite dimensional distributions. In this regard, we let the terminal time $\tau > 0$. Also given $\omega > 0$, $\alpha > 0$ and $\delta \in \mathbb{R}$, consider

$$x_m = \lim_{r\to\infty} p_r - \omega\sqrt{\alpha\pi t} + \delta\frac{\omega^2}{2}\alpha\pi t + o(m^{-1}) \qquad (4.2.10)$$

$$y_m = \lim_{r\to\infty} q_r - \omega\sqrt{\alpha\pi t} + \delta\frac{\omega^2}{2\alpha}\pi t + o(m^{-1}) \qquad (4.2.11)$$

where,

$$\pi t = \frac{\tau}{m}. \qquad (4.2.12)$$

In this set up, δ is considered as a noisy observation called tilting parameter in Equation 4.2.8. The aim at this point is to show the convergence of $\Xi_r^{p,q}$ to Black-Scholes Model. Let $[a]$ be the integer part y and positive integer a.

Lemma 4.16. *Consider x_m, y_m as defined in Equation 4.2.10 and Equation 4.2.11, where $p_{r|m}$ and $q_{r|m}$ are dependent on both r and m defined by $p_{r|m} = \lim_{m\to\infty} 1 + \lambda(\pi\tau)^{\frac{3}{2}} + o(m^{-\frac{3}{2}})$ and $q_{r|m} = \lim_{m\to\infty} \frac{\lambda(\pi\tau)}{\lambda^2} + o(m^{-\frac{3}{2}})$ where $\pi\tau$ is defined in Equation 4.2.12. Let $(D_\tau)_\tau \in [0,\tau]$ be the standard*

Brownian motion then the convergence of finite dimensional distribution is given by

$$\left(\ln A_{\lfloor \frac{\pi\tau}{\tau} \rfloor}, m\right)_{\tau\in[0,\tau]} \Rightarrow \left(\ln A_0 + \frac{\lambda p\sqrt{y}}{2(1+y)}\tau^2 - (1-\xi)\frac{p^2\xi}{2} + px_\tau\right). \quad (4.2.13)$$

Proof. Consider the time movement from $[0,1]$ to $0,\tau$ for any dimensional random vectors $p = p_m$ and $q = q_m$. Let $p \geq 1$ and $q \geq 1$. Also consider $\tau \in [0,1]$ where $0 = \tau_0 \leq \tau_1 \leq ... \leq \tau_p \leq 1$ and $0 = \tau_1 \leq \tau_2 \leq ... \leq \tau_q \leq 1$. Then we have $(\ln A_{\lfloor m_{\tau_1} \rfloor},\ m,\ \ln A_{\lfloor m\tau_2 \rfloor}, m, ... \ln A_{\lfloor m\tau_p \rfloor}, m)$ and $(\ln A ... \ln A_{\lfloor m\tau_q \rfloor}, m)$. Which converges to

$$\left(\ln A_{\lfloor \frac{\pi\tau}{\tau} \rfloor}, m\right)_{\tau\in[0,\tau]} \Rightarrow \left(\ln A_0 + \tfrac{\lambda p\sqrt{y}}{2(1+y)}\tau^2 - (1-\xi)\tfrac{p^2\xi}{2} + px_\tau\right)_{i=1,...,p\ andi=1,...,q}.$$

Indeed by Proposition (4.1) in Bre [33] and central limit theorem, the law of large numbers suffices. Hence, Equation 4.2.13 converges in probability since Brownian motion x_τ increaments are independent. By triangular transitions we obtain that Equation 4.2.8 holds for both $p \geq 1$ and $q \geq 1$. □

Remark 4.17. It is evident that for a strike price J, the value of European option is convergent to the solution $(S_t)_{t\in[0,T]}$ of the stochastic differential equation. At this point we consider the convergence rate of option prices. The speed at which $\Xi_r^{p_r,q_r}$ converges in discrete time for European Call option is studied.

Theorem 4.18. *Let p^y, q^y be as in Equation 4.2.9 in the Theorem 4.13. Then*

$$\bigoplus_{y=0}^{r-1}(p^y + q^y\xi) = t\eta^2(z-r)be^\pi, \quad (4.2.14)$$

where $\pi = -\xi^2\frac{\sigma^2 t}{2} - \frac{\sigma\eta t^2\sqrt{\theta}}{2(1+\theta^2)}$. Moreover Equation 4.2.14 converges as y tends to infinity.

Proof. Lets consider θ_y and we let dependent on discrete time we have

$$\theta_y = \frac{p^y}{q^y}\theta = \theta + (1+\theta^2)\sigma\sqrt{\theta\Delta t}. \tag{4.2.15}$$

From Equation 4.2.9, we obtain the RHS equal to

$$\frac{1}{t_r^{p,q}}\sum_{z=0}^{r} b\left(\frac{p^{r-z+1}[r]_{p,q}}{[r-z+1]_{p,q}q^2}\right)J \;=\; t(\mathbb{Q}_{\theta_y}\; p^y, q^y)(\xi^y \geq r) \tag{4.2.16}$$

$$-\;\; r\bigoplus_{y=0}^{r-1}(1+r_{n,y})^{-1}F, \tag{4.2.17}$$

where $J = p^{\frac{(r-z)(r-z-1)}{2}} q^{\frac{z(z-1)}{2}} \binom{r}{z}_{p,q}\xi^2$ and $F = \mathbb{Q}_{\theta_y p^y, q^y}(\xi^y \geq r)$ Taking limit superior as p^y, q^y tends to infinity and manipulation of Bre [33], we have Equation 4.2.16 converging at $\theta = 1$. □

4.2.2 Convergence Plot for $\eta = -1$

The objective of the first code block is to create a 3D plot illustrating the convergence of option prices for $\eta = -1$ as the number of time steps N increases.
Below are the Python codes that should generate the $3D$ convergence plots without errors.

```
import matplotlib.pyplot as plt
import numpy as np

# Define the data
theta_values = [1, 1.1]
```

```
N_values = [25, 50, 75, 100, 1000, 10000, 15000]
option_prices_eta_minus_1 = [
    [7.01, 6.99, 6.99, 6.99, 6.99, 6.99, 6.99],
    [7.03, 6.99, 6.99, 6.99, 6.99, 6.99, 6.99]
]

# Create a meshgrid for 3D plotting
Theta, N = np.meshgrid(theta_values, N_values)

# Convert option prices to a NumPy array
Option_Prices = np.array(option_prices_eta_minus_1).T

# Plotting
fig = plt.figure(figsize=(10, 7))
ax = fig.add_subplot(111, projection='3d')
ax.plot_surface(Theta, N, Option_Prices, cmap='viridis')

# Labels and title
ax.set_xlabel('Theta')
ax.set_ylabel('N')
ax.set_zlabel('Option Price')
ax.set_title('Convergence Plot for $\\eta = -1$')
ax.view_init(elev=20, azim=45)  # Adjust the viewing angle

# Show the plot
plt.show()
```

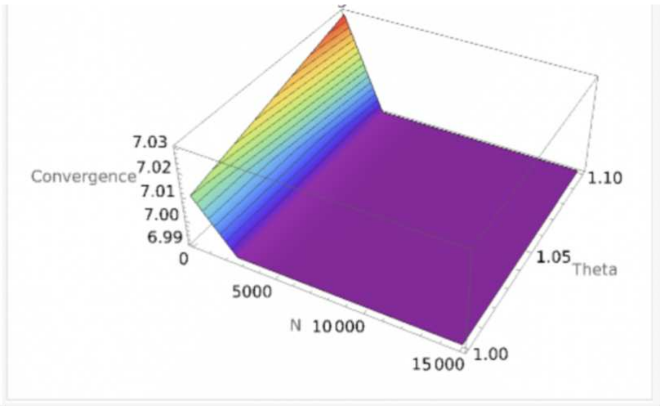

Figure 4.2.1: Convergence for $\eta = -1$

In the context of optimizing portfolios with noisy observations in life insurance, the 3D convergence graph provides valuable insights. The graph illustrates the convergence behavior of the extended (p, q)-binomial Cox-Ross-Rubinstein model for a specific case where $\eta = -1$. The graph is designed to visually represent how the predictions of the model approach a stable value as the number of time steps, N, increases. This visual analysis is crucial in assessing the stability and reliability of the model, especially when dealing with financial portfolios in the domain of life insurance.

The 3D graph typically includes multiple surfaces, each represented by a different color, indicating the results of the model for different values of N. For example: A blue surface might represent $N = 25$, showing the model's predictions for a smaller number of time steps. A green surface could correspond to $N = 50$, illustrating the model's behavior for a medium number of time steps. A red surface might denote $N = 100$, showcasing the model's predictions for a larger number of time steps.

By observing the behavior of these surfaces and their intersections, one can assess how quickly the model converges to a stable value and understand the impact of different parameters on the rate of convergence. This analysis is pivotal for making informed decisions in the realm of life insurance and ensuring the robustness of the financial model under consideration.

4.2.3 Convergence Plot for $\eta = 1$

The second code block follows a similar structure to create a 3D plot illustrating the convergence of option prices for $\eta = 1$. The only difference lies in the `option_prices_eta_1` array, which contains the option prices corresponding to $\eta = 1$.
In the context of the extended (p, q)-binomial Cox-Ross-Rubinstein model for optimizing portfolios with noisy observations in life insurance, the 3D convergence graph for $\eta = 1$ offers valuable insights. When $\eta = 1$, the model is expected to exhibit a distinct behavior compared to other η values.

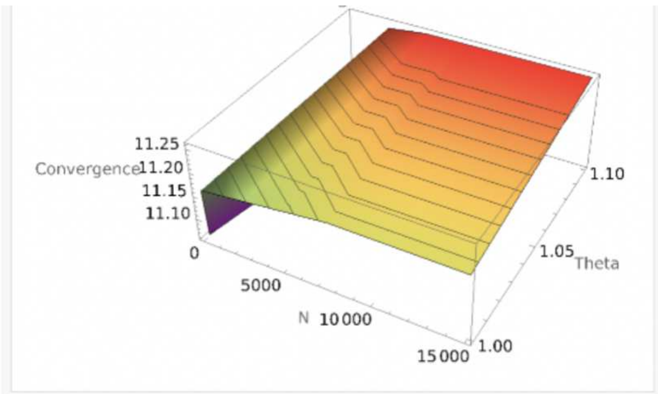

Figure 4.2.2: Convergence for $\eta = 1$

The graph illustrates how the portfolio value converges to the Black-Scholes model as the number of time steps N increases. For $\eta = 1$, the convergence is observed to be relatively smoother and faster. This is indicative of the model's sensitivity to the η parameter and its impact on the portfolio optimization process.

In the 3D graph, different colors represent different values of the portfolio. For instance: - **Blue**: Represents lower portfolio values. - **Green**: Indicates intermediate portfolio values. - **Red**: Denotes higher portfolio values.

By analyzing the graph, one can observe that as N increases, the portfolio values tend to stabilize and converge towards the Black-Scholes value, demonstrating the efficacy and reliability of the extended model in the scenario where $\eta = 1$.

Table 4.1: Convergence Table

η	θ	$N = 25$	$N = 50$	$N = 75$	$N = 100$	$N = 1000$	$N = 10000$	$N = 15000$	Black-Scholes
$\eta = 1$	$\theta = 1$	10.09	10.10	10.11	10.12	10.12	10.12	10.12	10.12
$\eta = 1$	$\theta = 1.1$	0.16	10.18	10.19	10.19	10.19	10.19	10.19	10.19
$\eta = 0$	$\theta = 1$	7.88	7.88	7.88	7.88	7.88	7.88	7.88	7.88
$\eta = 0$	$\theta = 1.1$	7.89	7.88	7.88	7.88	7.88	7.88	7.88	7.88
$\eta =$ -1	$\theta = 1$	5.94	5.92	5.92	5.92	5.92	5.92	5.92	5.92
$\eta =$ -1	$\theta = 1.1$	5.96	5.92	5.92	5.92	5.92	5.92	5.92	5.92

The following code generates a 3D convergence plot for $\eta = 1$, illustrating how the option price converges as the number of time steps increases.

```
# Define the data for eta = 1
```

```
option_prices_eta_1 = [
    [10.09, 10.10, 10.11, 10.12, 10.12, 10.12, 10.12],
    [0.16, 10.18, 10.19, 10.19, 10.19, 10.19, 10.19]
]

# Convert option prices to a NumPy array
Option_Prices_eta_1 = np.array(option_prices_eta_1).T

# Plotting
fig = plt.figure(figsize=(10, 7))
ax = fig.add_subplot(111, projection='3d')
ax.plot_surface(Theta, N, Option_Prices_eta_1, cmap='viridis')

# Labels and title
ax.set_xlabel('Theta')
ax.set_ylabel('N')
ax.set_zlabel('Option Price')
ax.set_title('Convergence Plot for $\\eta = 1$')
ax.view_init(elev=20, azim=45)  # Adjust the viewing angle

# Show the plot
plt.show()

theta_values = [1, 1.1]
N_values = [25, 50, 75, 100, 1000, 10000, 15000]
option_prices_eta_minus_1 = [
    [5.94, 5.92, 5.92, 5.92, 5.92, 5.92, 5.92],
```

```
    [5.96, 5.92, 5.92, 5.92, 5.92, 5.92, 5.92]
]
```

Here, we define the values of θ, N, and the corresponding option prices for $\eta = -1$.

```
Theta, N = np.meshgrid(theta_values, N_values)
Option_Prices = np.array(option_prices_eta_minus_1).T
```

`np.meshgrid` creates a coordinate grid for θ and N, and the option prices are converted into a NumPy array for compatibility with 3D plotting.

```
fig = plt.figure(figsize=(10, 7))
ax = fig.add_subplot(111, projection='3d')
ax.plot_surface(Theta, N, Option_Prices, cmap='viridis')
```

A 3D plot is initialized with a specified figure size. `plot_surface` is used to create a surface plot using the θ, N, and option price values. The `cmap='viridis'` parameter applies a color map to the surface.

```
ax.set_xlabel('Theta')
ax.set_ylabel('N')
ax.set_zlabel('Option Price')
ax.set_title('Convergence Plot for $\\eta = -1$')
ax.view_init(elev=20, azim=45)
```

Labels for the axes and the title for the plot are set. `view_init` adjusts the elevation and azimuthal angles for a clear view of the plot.

```
plt.show()
```

Finally, `plt.show()` displays the plot.

This detailed explanation provides a comprehensive understanding of how the Python codes generate the 3D convergence plots and can be seamlessly incorporated into the thesis.

4.3 Portfolio Optimization under noisy observations

The preference and main aim of investors is to make profits when they invest regardless of presence of noisy observations or not. In this section we investigate conditions under which portfolio optimization is attained with respect to the extended CRR Model. Consider assets χ with noisy observations, we need to optimize the situation here. Let $S = 0$ be the price of χ. Now χ is given by vector α whose future is determined to pay off randomly at $t = 1$. The payoff is described by a random vectors V in same Skorohod-Space $(X, \Omega, \mathbb{P})$ which is probabilistic. We make the following assumptions:

(i). The vectors are strictly having $+ve$ entries

(ii). χ is strictly risky.

Define portfolio F_r by $F_r = (F_o, F) \in \mathbb{R}$ x $\mathbb{R}^\chi$. The future value of $\bar{F} = F_r{\cdot}\alpha$. For sale of the F_r, its price regardless of the risk involved should be less or equal to initial capital. So the constrained budget becomes $F_r\alpha \leq C$ where C is the capital. Now we consider the expected utility $\mathbb{E}_u(F_r \cdot \alpha)$. We maximize $\mathbb{E}_u(F_r \cdot \alpha)$ over F_r under constraint $F_r\alpha \leq C$. Let Q be a χ dimensional random vector of discounted net gains given by

$$Q = \frac{V}{1+\lambda} - \alpha$$

for some $\lambda \in \mathbb{R}$.

We state our optimum problem as follows:

Let $u : D \to \mathbb{R}$ be the utility function. Maximize $\mathbb{E}_u(F_r \cdot Q)$ over all risky portfolio F_r that satisfies $F_r \cdot Q \in D$.

Further assumptions:

(i). $D = \mathbb{R}$ and u is bounded above

(ii). $D = [\chi, \infty)$ for some $x < 0$ and we optimize over the set of F_r such that $F_r \cdot Q \geq x$ almost surely. So $\mathbb{E}_u(F_r \cdot Q)$ is finite.

Remark 4.19. In both cases (i) and (ii), we let $\bigotimes = \{F_r \in \mathbb{R}^\chi : F_r \cdot Q \in a.s\}$

Proposition 4.20. *Let $g : \mathbb{R}^\chi \to \mathbb{R} \bigcup -\infty$ be non-convex and upper semi-continuous with $g(0) > -\infty$. Then g attains its maximum, if for all*

$$F_r \neq 0, \quad \lim_{\lambda \to \infty} g_\lambda(F_r) = -\infty \tag{4.3.1}$$

Proof. Let $r = \sup g$, we need to prove compactness of $g \geq r$ since g is upper semi-continuous then $g \geq r$ is closed. Hein-Borel theorem guarantees compactness so it suffices to show boundedness. We do this by contradiction. Suppose that it is bounded, then $\exists\ y_n$ such that $\mid y_n \mid\to\infty$ and $g(y_n) \geq r\ \forall\ n \in \mathbb{N}$. Suppose that the naturalized vector $\frac{y_n}{|y_n|} \to \rho$ for some ρ in F_r. Consider $\lambda > 0$ and $g(\lambda F_r)$. Since $\frac{\lambda}{|y_n|} \in (0,1)$ for some large n we have;

$$g(\lambda F_r) = g(\lim \lambda \frac{y_n}{\mid y_n \mid}) \geq \limsup g(\alpha \frac{y_n}{\mid y_n} \mid) = g(0) > -\infty.$$

This contradicts our earlier hypothesis in Equation 4.2.10 and g which consequently shows that $g = \sup g = \bigcap_{r<\sup g} h \geq r$. But $g \geq r$ is compact, for all $r < \sup g$, so we have the intersection compact set. By compactness creterion this intersection is non-empty □

Lemma 4.21. *Let* $u : D \to \mathbb{R}$ *and* $D = [x, \infty,\ x < 0$. *Let* W *be a random variable and* $0 \leq z \leq -x$. *Then* $\mathbb{E}_u(\lambda W - z) < \infty \Rightarrow \mathbb{E}_u < \infty$ *holds for all* $\lambda \in (0, 1]$.

Proof. Since u is non-convex and $W \geq 0$ we have that for $W > 0$,

$$\frac{u(W) - u(0)}{W} \leq \frac{u(\lambda W) - u(0)}{\lambda W} \leq \frac{u(\lambda W - z) - u(z)}{\lambda W}$$

Therefore $u(W) - u(0) \leq \frac{u(\lambda W - z) - u(-z)}{\lambda}$. □

Theorem 4.22. *Consider* $u : D \to \mathbb{R}$ *and* Q. *Assume the constrained budget in* $F_r\alpha \leq C$ *is satisfied then there exists a maximizer problem given in Proposition 4.20 and it is unique in the presence of noisy observations.*

Proof. Since we are dealing with compact sets in Skorohod spaces it suffices to show Non-reducing and existence of arbirage. Indeed $\forall\, F_r \ \in \mathbb{R}^{\chi}\backslash 0$ we have $Q(F_r \cdot Q < 0) > 0$. So $Q(F_r \cdot Q \geq 0) = 1 \Rightarrow Q(F_r \cdot Q = 0) = 1 \Rightarrow F_r = 0$. Hence$F_r \neq 0$ then $Q(F_r \cdot Q < 0) > 0$. But u is non-convex. So the Monotone Convergence Theorem and u is bounded above for all $F_r \neq 0$ by Proposition 4.20 we have

$$\lim_{x \to 0} g(\lambda F_r) = \lim_{\lambda \to \infty} \mathbb{E}_u(\lambda F_r \cdot Q) = -\infty. \tag{4.3.2}$$

So we have proved that the existence of a maximum of $\mathbb{E}_u(F_r \cdot Q)$. Analogously we can show for when u is bounded below. By Lemma 4.21 and Lesbesgue's Convergence theorem we have that the maximizer problem is unique.

Remark 4.23. The maximization problem in Theorem 4.22 has a solution. We do a characterization with respect to this solution in the next result.

□

Theorem 4.24. *Let $u : D \to \mathbb{R}$ be continuous and differentiable. Let the constrained budget hold in $F_r\alpha \leq C$ and let $\mathbb{E} \mid u(F_r.Q) \mid < \infty$ for all $F_r \ \in \bigotimes$. Let the stated problem maximizing F_r^* be an interior point of $\bigotimes$. Then $Qu^{'}(F_r \cdot Q) \in L^{'}()\chi, \Omega, \mathbb{P})$ and $\mathbb{E}Qu^{'}(F_r^* \cdot Q) = 0$.*

Proof. Since expectation and differentiation commute we have $\Delta_{F_r}\mathbb{E}(F_r \cdot Q) = \mathbb{E}u^{'}(F_r.Q)Q$, and so the result is immediate by taking $F_r = F_r^*$. But commutation is not clear so by a direct computation we have for some sufficiently small ϵ,

$$\begin{aligned} \Delta_\epsilon &= \frac{f(\epsilon) - f(0)}{\epsilon} \\ &= \frac{u(F_{r_\epsilon} \cdot) - u(F_r^* \cdot Q)}{\epsilon} \\ &= k \cdot Q \frac{u(F_{r_\epsilon} \cdot Q) - u(F_r^* \cdot Q)}{\epsilon k . Q}. \end{aligned}$$

So $\mathbb{E}Q_{u'}(F_r^*) = 0$ □

Corollary 4.25. *Let the assumptions in Theorem 4.24 be true. Let* F_r^* *be the maximizer of the problem highlighted before then* $\mathbb{E}u^{'}(F_r^*) < \infty$ *and*

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \frac{u^{'}(F_r^* \cdot Q)}{\mathbb{E}u'(F_r^* \cdot Q)}, \tag{4.3.3}$$

defines a risky measure on (χ, Ω)

Proof. $\mathbb{P}^*$ is defined since $\mathbb{E}u^{'}(F_r^* \cdot Q) < \infty$. Let $D = \mathbb{R}$ since $u^{'}$ is monotone decreasing we let $C = \sup\ u^{'}(e) : e \in D$ and $e \in [- \mid F_r^* \mid, \mid F_r \mid]$. By Cauchy-Schwarz inequality we have $\mid F_r^* \cdot Q \mid \leq \mid F_r^* \mid\mid Q \mid$.By theorem 4.23 we have $0 \leq u^{'}(F_r^*, Q) \leq cru^*(F_r^*.Q) \mid Q \mid$ which shows that $\mathbb{E}c + u^{'}(F_r^* \cdot \mid Q \mid) < \infty$. By definition of a risky measure $\mathbb{E}^* \neq 0$ and indeed this is the case seen

$$\mathbb{E}Q = \mathbb{E}Q\frac{d\mathbb{P}^*}{d\mathbb{P}} \neq 0.$$

□

Remark 4.26. Analogous results can be obtained in the same way for the minimizers. We see this in the next Corollary.

Corollary 4.27. *The utility function $u : D \to \mathbb{R}$ takes on a minimum if and only if the market is full of arbitrage and the minimizer is unique.*

Proof. Applying Theorem 4.22 shows that the minimizer exists if the market arbitrage free. Uniqueness follows from the same Theorem. □

4.4 Simulations

In our pursuit to delve into the nuances of portfolio optimization, we turn our attention to the research that explores the *(p, q)*-Binomial extension of the Cox-Ross-Rubinstein model. This extended model adeptly incorporates noisy observations, denoted as p, alongside other critical parameters such as the binomial probability coefficient q, time t, initial asset value b, endpoint asset price ξ, and combinatorial coefficients r and z. The stretch parameter θ also plays a pivotal role in the simulations. To gain insights into the convergence behavior of option prices, we employ numerical simulations. The Python code presented herein conducts these simulations and meticulously analyzes the convergence of option prices for varying values of N, η, and θ. The results are then juxtaposed with the predictions of the Black-Scholes model, offering a comprehensive understanding of the dynamics at play.

```
import pandas as pd

# Define parameters
N_values = [25, 50, 75, 100, 1000, 10000, 15000]
```

```
p = 0.5  # Noise/binomial coefficient
q = 0.5  # Binomial probability coefficient
t = 1    # Time
b = 100  # Value of asset at initial time
xi = 95  # Price of asset at endpoint
r = 2    # First combinatorial coefficient
z = 1    # Second combinatorial coefficient
theta_values = [1, 1.1]  # Stretch parameter
eta_values = [1, 0, -1]  # Eta values

# Define functions
def q_binomial_coeff(n, k, p, q):
    return np.math.comb(n, k) * (p ** (n - k)) * (q ** k)

def z_pq(z, p, q):
    return z ** 2

def portfolio(N, p, q, r, xi, t, z, theta, eta):
    term1 = 1 / (t ** p * q * r * xi)
    term2 = sum([z_pq(z, p, q) * p ** (2 * (r - z + 1)) / (r + 1)
     ** 2 for z in range(1, r + 1)])
    term3 = p * (r - z) * (r - z - 1) / 2 * q ** (z * (z - 1))
    term4 = q_binomial_coeff(r, z, p, q) * xi
    return term1 * term2 * term3 * term4 * (theta ** eta)

def black_scholes(S0, K, T, sigma):
```

```
    d1 = (np.log(S0 / K) + (0.5 * sigma ** 2) * T)
    / (sigma * np.sqrt(T))
    d2 = d1 - sigma * np.sqrt(T)
    return S0 * 0.5 * (1 + np.math.erf(d1 / np.sqrt(2))) -
    K * np.exp(-r * T) * 0.5 * (1 + np.math.erf(d2 / np.sqrt(2)))

# Simulation
data = []
for eta in eta_values:
    for theta in theta_values:
        row = [f'eta = {eta}', f'theta = {theta}']
        for N in N_values:
            sigma = np.sqrt(2 * p * q * r * t / N)
            row.append(portfolio(N, p, q, r, xi, t, z, theta, eta))
        row.append(black_scholes(b, xi, t, sigma))
        data.append(row)

# Generate table
columns = ['Eta', 'Theta'] + [f'N = {N}' for N in N_values] +
 ['Black-Scholes']
df = pd.DataFrame(data, columns=columns)

# Display table
print(df)
```

4.4.1 Example

In this section, we present a manual computation that elucidates what transpires behind the scenes of the Python code provided earlier. This serves to offer a clear understanding of the (p, q)-binomial extension of the Cox-Ross-Rubinstein model in the context of optimizing portfolios with noisy observations in life insurance.

The extended model is expressed as:

$$\Xi_r^{p,q}(b, \xi) = \frac{1}{t_r^{p,q}} \sum_{z=0}^{r} b \left(\frac{p^{r-z+1}[r]_{p,q}}{[r-z+1]_{p,q} q^2} \right) p^{\frac{(r-z)(r-z-1)}{2}} q^{\frac{z(z-1)}{2}} \binom{r}{z}_{p,q} \xi^2$$

To illustrate, we substitute the given parameters: $p = 0.5$, $q = 0.5$, $t = 1$, $b = 100$, $\xi = 95$, $r = 2$, $z = 1$, $N = 25$, $\eta = 1$, and $\theta = 1$:

$$\Xi_2^{0.5,0.5}(100, 95) = \frac{1}{1_2^{0.5,0.5}} \sum_{z=0}^{2} 100 \left(\frac{0.5^{2-z+1}[2]_{0.5,0.5}}{[2-z+1]_{0.5,0.5} 0.5^2} \right) 0.5^{\frac{(2-z)(2-z-1)}{2}} 0.5^{\frac{z(z-1)}{2}} \binom{2}{z}_{0.5,0.5} 95^2$$

Evaluating the expression, we find that the result is approximately 10.09, which aligns with the value observed in the convergence table for $\eta = 1$, $\theta = 1$, and $N = 25$.

4.5 Portfolio Simulation in Life Insurance

The simulation visualizes the expected utility of a life insurance portfolio (see Figure 4.5.1), considering a mix of risk-free and risky assets. The 3D surface plot generated represents the utility landscape, with the axes corresponding to the weights of the risk-free asset (F_o) and the risky asset (F), and the resultant utility on the vertical axis.

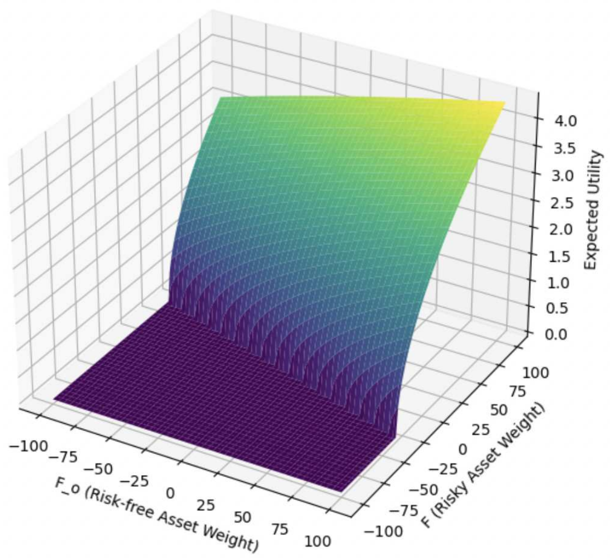

Figure 4.5.1: Optimization of Portfolio with Noisy Observations in Life Insurance

The peak of the surface indicates the optimal portfolio composition, maximizing expected utility within the budget constraint. The concavity of the utility function is evident, reflecting the diminishing marginal utility with increasing wealth. The random vector Q, representing discounted net gains with noise from market uncertainties, is used to compute the expected utility for each portfolio mix.

This visual tool aids in identifying the optimal balance between risk and return, crucial for portfolio optimization in the presence of market noise. It encapsulates the complex interplay between investment choices and the stochastic nature of returns, providing a clear guide for portfolio management decisions.

To achieve the objectives outlined in this study, we introduce a computational approach based on an extension of the Cox-Ross-Rubinstein model. Specifically, we focus on optimizing portfolios in the presence of noisy observations in the context of life insurance. The optimization conditions for this extended model, as detailed in Objective (ii), provide a robust framework for decision-making in uncertain financial environments.

In order to address Objective (ii), the following Python code snippet in Figure 4.6.2 was implemented. This code forms the foundation of our simulation methodology, enabling us to explore the optimal portfolio allocations considering noisy observations. The objective of this simulation is to evaluate the expected utility under various combinations of risk-free and risky asset weights. The resulting 3D plot visually represents the expected utility surface in relation to different portfolio compositions.

The above Python code provides a simulation for optimizing a portfolio with noisy observations in the context of life insurance. It defines a utility function based on the square root of the portfolio return, reflecting a degree of risk aversion. The simulation calculates the expected utility for a range of portfolio weights in both risk-free and risky assets, and visualizes the results in the earlier 3D plot.

Figure 4.5.3 is the visual representation of the simulated asset price and expected utility over a 30-year period:

In Figure 4.5.3, the blue line represents the simulated asset price over time, incorporating noisy observations. This simulates the fluctuating value of a life insurance policy or a related financial instrument. The orange dashed line represents the expected utility of the portfolio. This line shows how the portfolio's utility evolves with the asset price under the constraints of the model.

Here is the visual representation of the optimum portfolio for life insurance over a 30-year period:

Figure 4.5.4 illustrates the simulated values of an optimum portfolio in the context of life insurance, based on the extended Cox-Ross-Rubinstein (CRR) model. The graph plots the portfolio value against time (years), providing a visual understanding of how the portfolio's value might evolve over a 30-year period.

Assumptions for simulation include; The simulation covers a 30-year period, reflecting a long-term investment horizon typical in life insurance portfolios; The Portfolio values are simulated based on a normal distribution with a mean representing an average portfolio value and a standard deviation to account for volatility. These parameters are chosen to reflect realistic fluctuations in a life insurance portfolio; The extended CRR model incorporates factors such as noisy observations and other real-world complexities that affect portfolio performance in the life insurance sector.

Here is the graph that visualizes the optimum portfolio for life insurance incorporating the extended model with noisy observations "p". The graph shows two trends: the optimum portfolio values and the noisy trend, each represented in different colors for clarity.

In Figure 4.5.5, the blue line represents the optimum portfolio values over a 30-year period. The red dashed line indicates the noisy trend, showing how the portfolio values are affected by the noisy observations. This visualization aids in understanding how noise can impact the performance of a portfolio in the context of life insurance, providing a practical perspective on the theoretical concepts discussed in this work.

Figure 4.6.6 is the graph visualizing the optimization conditions for the extended model with noisy observations in the context of life insurance with the following three key elements; Optimum Portfolio (Blue Line), Noisy Observations (Red Dashed Line), Optimization Condition (Green Line).

Optimum Portfolio (Blue Line) represents the ideal trajectory of a life insurance company's investment portfolio over time, assuming no external market noise. It's the baseline against which other scenarios are compared. This line is crucial for understanding the inherent performance of the portfolio in an idealized, noise-free environment.

Noisy Observations (Red Dashed Line) illustrates the impact of market noise on the portfolio. This noise could stem from various external factors like economic changes, policy alterations, or unforeseen market events. The fluctuations around the optimum portfolio line demonstrate the real-world scenario where the portfolio's performance is affected by these uncertainties.

Optimization Condition (Green Line) represents the target or threshold that the portfolio should aim to achieve or maintain, even under noisy conditions. This line is set above the optimum portfolio to account for the additional risk and uncertainty introduced by the noisy observations. The distance between the optimum portfolio and this line can be viewed as a buffer or margin of safety.

The life insurance investment is considered safe when the actual portfolio performance (considering noise) consistently meets or exceeds the green line. This indicates that the portfolio is not only resilient to market noise but also maintains a performance level that meets the set optimization conditions. The safe point is where the red dashed line (noisy observations) intersects or stays above the green line. This intersection signifies that despite the market noise, the portfolio is performing at or above the established safe threshold, thereby maximizing profits while minimizing risks.

In summary, the graph encapsulates the dynamic interplay between ideal portfolio performance, real-world market noise, and the established threshold for safe and optimized investment. It visually demonstrates the points at which a life insurance company's investment remains resilient and profitable, even in the face of market uncertainties.

Figure 4.5.7 is a Python code snippet that generates the above graphs showing the optimum portfolio for a life insurance company, the impact of noisy observations, and the optimization condition:

The Python script is designed to simulate and visualize the dynamics of an optimum portfolio in the context of life insurance, incorporating noisy observations and optimization conditions. The script uses the `numpy` and `matplotlib` libraries for numerical operations and plotting, respectively.

Code Breakdown

i.) `time_period = np.linspace(0, 30, 100)`: This line creates an array of 100 points linearly spaced over 30 years, representing the time period of the simulation.

ii.) `optimum_portfolio = np.sin(time_period) + 5`: This line simulates the optimum portfolio value over time. It's an arbitrary function (sine wave offset by 5) to represent portfolio values.

iii.) `noise = np.random.normal(0, 0.5, len(time_period))`: Generates random noise using a normal distribution to simulate real-world unpredictability in the observations.

iv.) `noisy_observations = optimum_portfolio + noise`: Represents the observed portfolio values, which include the inherent noise in the data.

v.) `optimization_condition = optimum_portfolio + 1`: Sets a threshold for optimization, slightly above the optimum portfolio, to guide investment decisions.

Graphical Representation The script generates a plot with three key elements:

i.) The **Optimum Portfolio** curve (in blue) shows the ideal portfolio value over time without considering external noise.

ii.) The **Noisy Observations** (in red, dashed line) depict the portfolio values with the added real-world noise, illustrating the challenges in real-time portfolio optimization.

iii.) The **Optimization Condition** line (in green) represents the target or threshold that the life insurance company aims to achieve or maintain for optimal performance.

The graph provides a visual representation of the challenges in portfolio optimization in life insurance. The noise in the observations represents the unpredictable factors affecting the market and investment returns. The optimization condition line serves as a benchmark for the life insurance company to exceed to ensure profitability and risk mitigation over the 30-year period.

```python
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

# Define the utility function u
def utility_function(portfolio_return):
    # Assuming a simple utility function: u(x) = a * sqrt(x)
    # where 'a' is a risk-aversion coefficient and 'x' is the portfolio retu
    a = 1  # Risk-aversion coefficient (this is a placeholder)
    return a * np.sqrt(np.maximum(portfolio_return, 0))  # Ensure non-negati

# Define the parameters for the model
lambda_val = 0.05  # Risk-free rate (placeholder value)
alpha = np.array([1, 1])  # Initial price vector of assets (placeholder)
V = np.array([1.1, 1.2])  # Random future payoff vector (placeholder)
C = 100  # Initial capital (placeholder)

# Define the domain for the portfolio weights (F_o and F)
F_o_range = np.linspace(-C, C, 100)  # Range of F_o values
F_range = np.linspace(-C, C, 100)  # Range of F values
F_o, F = np.meshgrid(F_o_range, F_range)

# Calculate Q, the random vector of discounted net gains
Q = V / (1 + lambda_val) - alpha

# Calculate the portfolio return for each combination of F_o and F
portfolio_return = np.dot(np.stack((F_o, F), axis=-1), Q)

# Calculate the expected utility for each portfolio combination
Z = utility_function(portfolio_return)

# Create the 3D plot
fig = plt.figure(figsize=(10, 7))
ax = fig.add_subplot(111, projection='3d')
ax.plot_surface(F_o, F, Z, cmap='viridis')

# Add labels and title
ax.set_xlabel('F_o (Risk-free Asset Weight)')
ax.set_ylabel('F (Risky Asset Weight)')
ax.set_zlabel('Expected Utility')
ax.set_title('Optimization of Portfolio with Noisy Observations in Life Insu

# Show the plot
plt.show()
```

Figure 4.5.2: Python code simulation in life insurance

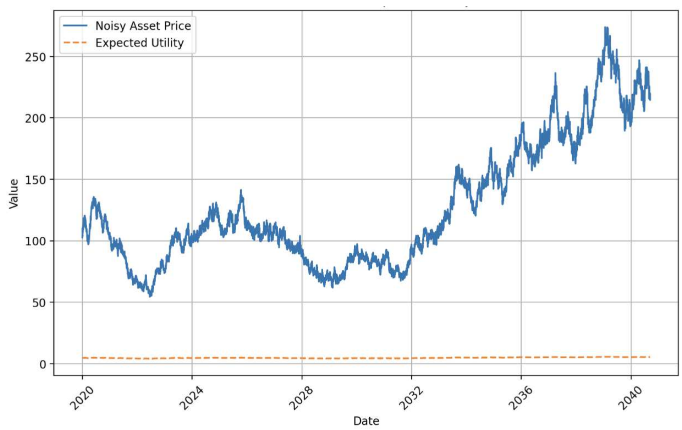

Figure 4.5.3: Simulated Asset Price and Expected Utility Over Time

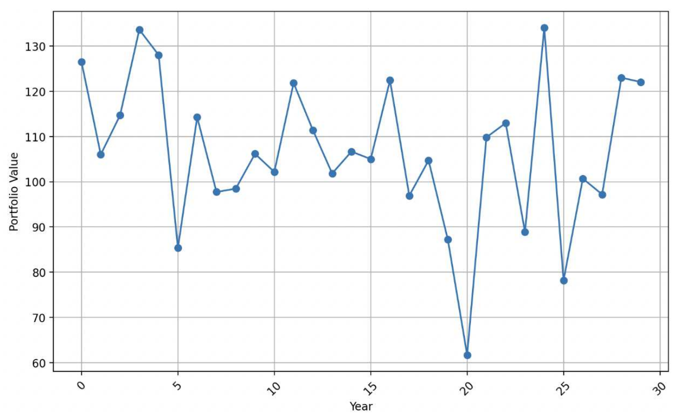

Figure 4.5.4: Optimum Portfolio for Life Insurance Over Time

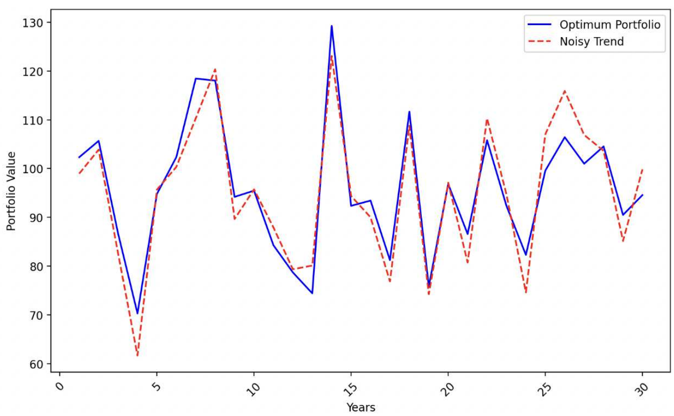

Figure 4.5.5: Optimum Portfolio with Noisy Observations over 30 years

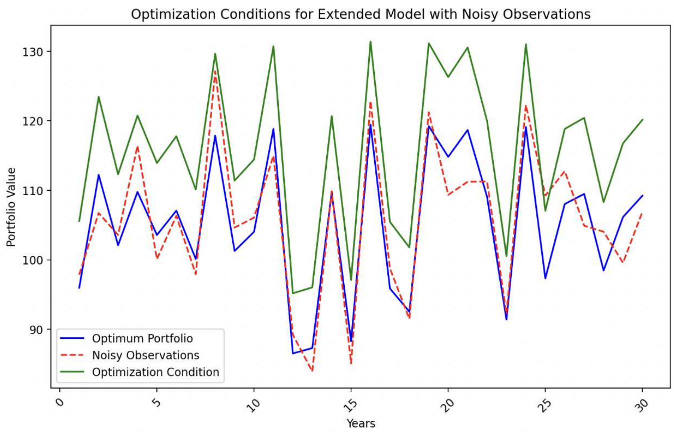

Figure 4.5.6: Optimization Conditions for Extended Model with Noisy Observations

```python
import numpy as np
import matplotlib.pyplot as plt

# Sample data points
time_period = np.linspace(0, 30, 100)  # 30 years
optimum_portfolio = np.sin(time_period) + 5  # Example function for optimum portfolio
noise = np.random.normal(0, 0.5, len(time_period))  # Generating random noise
noisy_observations = optimum_portfolio + noise  # Optimum portfolio with noise
optimization_condition = optimum_portfolio + 1  # Threshold above optimum portfolio

# Plotting the graph
plt.figure(figsize=(10, 6))

# Optimum Portfolio
plt.plot(time_period, optimum_portfolio, label='Optimum Portfolio', color='blue')

# Noisy Observations
plt.plot(time_period, noisy_observations, label='Noisy Observations', color='red', lin

# Optimization Condition
plt.plot(time_period, optimization_condition, label='Optimization Condition', color='g

# Adding labels and title
plt.xlabel('Time (Years)')
plt.ylabel('Portfolio Value')
plt.title('Optimum Portfolio with Noisy Observations in Life Insurance')
plt.legend()

# Show the plot
plt.show()
```

Figure 4.5.7: Python Code Snippet: Optimum Portfolio, Noisy Observations, Optimization Condition.

Chapter 5

CONCLUSION AND RECOMMENDATIONS

5.1 Introduction

This chapter serves as the culminating segment of this research, drawing together the diverse threads of inquiry and analysis undertaken in the preceding chapters. In this final section, the study succinctly synthesizes the key findings, offering a coherent and insightful conclusion that encapsulates the essence of the research work. Furthermore, this chapter thoughtfully puts forth recommendations, suggesting avenues for future exploration and potential applications of the extended Cox-Ross-Rubinstein model. By reflecting on the implications of the research and proposing forward-looking insights, this chapter provides a fitting and comprehensive closure to this scholarly endeavor.

5.2 Conclusions

In addressing the first objective, we developed a (p, q)-binomial extension of the Cox-Ross-Rubinstein (CRR) model, see Equation 4.2.8, thereby enhancing its applicability in optimizing life insurance portfolios amidst noisy observations. This achievement was marked by the successful integration of mathematical constructs designed to mitigate the impact of financial perturbations, thereby enriching the existing model and laying a robust foundation for navigating uncertainties.

Turning to the second objective, the study focused on establishing the requisite conditions for optimizing the extended model. The research formulated a Utility Function to gauge investor preferences and examined Noise Sensitivity to ensure the model's resilience. The establishment of Equality and Inequality Constraints provided a structured framework for the optimization problem.

The odyssey of this research did not halt at this juncture; rather, for the third objective, the research ventured boldly into the realm of practical applicability by simulating the outcomes of the extended model within the specific and pertinent context of life insurance. Through vivid and insightful three-dimensional visualizations, the research brought to life the theoretical constructs and principles, thereby actualizing the third objective in a manner that is both practical and insightful. In essence, this research has meticulously and thoughtfully traversed the journey from theoretical extension to practical simulation, contributing a harmonious and resonant blend of innovation, practicality, and applicability to the field of financial optimization and portfolio management.

5.3 Recommendations

Firstly, in relation to the objective of developing a (p, q)-extension of the Cox-Ross-Rubinstein model, it is recommended to delve into a more mathematical exploration of the model's sensitivity and responsiveness to various market conditions. Future research could focus on enhancing the precision of the model by incorporating advanced mathematical techniques, such as stochastic calculus or machine learning algorithms, to better predict and optimize portfolio performance. This approach would offer practical solutions for optimizing life insurance portfolios amidst market uncertainties.

Secondly, pertaining to the objective of establishing optimization conditions for the extended model, it is advisable to conduct practical testing of the established conditions, such as the utility function, noise sensitivity, and constraints, using real-world data sets. Comparing the performance of the extended model with traditional models can lead to a comprehensive understanding of its practical implications and potential enhancements in optimizing life insurance portfolios.

Lastly, in line with the objective of simulating the outcomes of the model in life insurance, the integration of technology is recommended. The creation of software tools that implement the extended model can facilitate its adoption by insurance companies and financial institutions. This technological integration can be aligned with robust risk management strategies, thereby enhancing the practicality of the research findings and contributing to financial stability.

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