Norm-Attainable Operators in Operator Ideals: Characterizations, Properties, and Structural Implications.

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Abstract

This paper explores the interplay between norm-attainable operators and operator ideals in the context of Hilbert spaces, providing a comprehensive characterization of their structural and geometric properties. We investigate norm-attainability within common operator classes, including compact operators, Schatten (p)-class, trace-class, and weakly compact operators. Foundational lemmas establish the existence and basic properties of norm-attainable operators, which are extended through propositions detailing their behavior under inclusion in specific operator ideals. Key theorems characterize conditions for norm-attainability, highlighting connections to compactness, spectral properties, and finite-rank approximations. The results elucidate practical implications, such as operator approximations and eigenvalue relationships. These findings have direct applications in quantum mechanics, signal processing, and numerical analysis, where operator approximations are crucial for efficient computation and system modeling. Furthermore, we outline potential extensions of this work to the more general settings of unbounded operators and Banach spaces, opening avenues for future research and broadening the scope of applicability. This study advances understanding of norm-attainable operators in operator theory, offering new insights into their algebraic and geometric significance within operator ideals.

keywords{Norm-Attainable Operator, Operator Ideals, Compact Operators, Schatten Class, Weakly Compact Operators}

Introduction

The study of norm-attainable operators within the framework of functional analysis and operator theory provides deep insights into the structure and behavior of bounded linear operators on Hilbert spaces[1,4,6]. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be norm-attainable if there exists a unit vector $x \in \mathcal{H}$ such that $\|Tx\| = \|T\|$. This property not only highlights key aspects of the operator's action on the space but also has significant implications for various classes of operators. Compact operators, which form a cornerstone of $\mathcal{K}(\mathcal{H})$, are known to always achieve their norm. Similarly, finite-rank operators $\mathcal{F}(\mathcal{H})$ and self-adjoint trace-class operators $\mathcal{S}_1$ are norm-attainable, demonstrating the fundamental role of spectral properties in norm attainment.The Schatten p-class $\mathcal{S}_p$ introduces a broader context for examining norm-attainable operators, particularly in relation to Schatten p-norms, with the inequality $\|T\|_p \geq \|T\|$ for $p \geq 1$[3,5,7,9]. Notably, the class of weakly compact operators $\mathcal{W}(\mathcal{H})$ includes norm-attainable operators, though not all weakly compact operators possess this property. The equivalence between the norm-attainability of T and its adjoint T^* enriches the theoretical understanding of this concept. Norm-attainable operators further exhibit intricate relationships with orthonormal bases, spectral theory, and operator approximations[2,8,11,13]. For compact, norm-attainable operators, an orthonormal basis $\{e_n\}$ exists such that $\|Te_n\| = \|T\|$ for some n[10,12,14]. In the Hilbert-Schmidt class $\mathcal{S}_2$, norm-attainability is characterized by approximation in the Hilbert-Schmidt norm by finite-rank norm-attainable operators. Self-adjoint operators $T \in \mathcal{B}(\mathcal{H})$ are norm-attainable if and only if $\|T\|$ corresponds to an eigenvalue, linking norm attainment to spectral properties.This paper explores these intricate relationships, provides new insights into the norm-attainability of various operator classes, and elucidates their implications in both theoretical and applied contexts within operator theory.

Preliminaries

Let $\mathcal{H}$ denote a Hilbert space, and let $\mathcal{B}(\mathcal{H})$ represent the set of all bounded linear operators on $\mathcal{H}$. Key subsets of $\mathcal{B}(\mathcal{H})$ include:

- **Compact Operators** $\mathcal{K}(\mathcal{H})$: Operators $T \in \mathcal{B}(\mathcal{H})$ such that T maps bounded sets to relatively compact sets.
- **Finite-Rank Operators** $\mathcal{F}(\mathcal{H})$: Operators $T \in \mathcal{B}(\mathcal{H})$ whose range is finite-dimensional.
- **Weakly Compact Operators** $\mathcal{W}(\mathcal{H})$: Operators $T \in \mathcal{B}(\mathcal{H})$ such that the image of the unit ball under T is relatively compact in the weak topology of $\mathcal{H}$.
- **Schatten p-class** $\mathcal{S}_p$: For $p \geq 1$, the set of compact operators $T \in \mathcal{K}(\mathcal{H})$ such that $\|T\|_p = \left(\sum_{n=1}^{\infty} s_n(T)^p\right)^{1/p} < \infty$, where $\{s_n(T)\}$ are the singular values of T.

- **Trace-Class Operators** $\mathcal{S}_1$: The class $\mathcal{S}_p$ with $p = 1$; operators T such that $\|T\|_1 = \sum_{n=1}^{\infty} s_n(T) < \infty$.
- **Hilbert-Schmidt Operators** $\mathcal{S}_2$: The class $\mathcal{S}_p$ with $p = 2$; operators T such that $\|T\|_2 = \left(\sum_{n=1}^{\infty} s_n(T)^2\right)^{1/2} < \infty$.

The **operator norm** $\|T\|$ for $T \in \mathcal{B}(\mathcal{H})$ is defined as

$$\|T\| = \sup_{\|x\|=1} \|Tx\|.$$

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be **norm-attainable** if there exists a unit vector $x \in \mathcal{H}$ such that $\|T\| = \|Tx\|$. For operators in the Schatten p-class, the Schatten p-norm $\|T\|_p$ satisfies the inequality $\|T\|_p \geq \|T\|$ for $p \geq 1$. The relationship between these norms is critical in characterizing norm-attainable operators in various classes. A self-adjoint operator $T \in \mathcal{B}(\mathcal{H})$ is characterized by its eigenvalues, and $\|T\|$ is attained if it coincides with the absolute value of one of its eigenvalues. These foundational concepts and norms are essential for discussing norm-attainable operators and their properties in compact, finite-rank, and Schatten p-class settings.

Main Results and Discussions

Lemma 1. *For any compact operator $K \in \mathcal{K}(\mathcal{H})$, the norm of K is achieved, i.e., K is norm-attainable.*

Proof. Let $K \in \mathcal{K}(\mathcal{H})$ be a compact operator. By the properties of compact operators, we know that the image of the unit ball under K, denoted $K(B)$, is relatively compact in $\mathcal{H}$. This means that $K(B)$ is totally bounded, and therefore, there exists a sequence of vectors $\{x_n\} \subset B$ such that $\|Kx_n\|$ converges to $\|K\|$. In particular, by the compactness of the operator, we can find an element $x \in \mathcal{H}$ such that $\|Kx\| = \|K\|$. Thus, the norm of K is attained at the vector x, and K is norm-attainable. □

Lemma 2. *If $T \in \mathcal{B}(\mathcal{H})$ is norm-attainable and belongs to the Schatten p-class $\mathcal{S}_p$, then $\|T\|_p \geq \|T\|$ for any $p \geq 1$, where $\|T\|_p$ denotes the Schatten p-norm.*

Proof. Let $T \in \mathcal{B}(\mathcal{H})$ be norm-attainable and let $T \in \mathcal{S}_p$ for some $p \geq 1$. Since T is norm-attainable, there exists a vector $x_0 \in \mathcal{H}$ such that $\|Tx_0\| = \|T\|$. The Schatten p-norm of T is given by:

$$\|T\|_p = \left(\sum_{n=1}^{\infty} \sigma_n(T)^p\right)^{1/p},$$

where $\sigma_n(T)$ are the singular values of T. Since T is norm-attainable, the singular values satisfy $\sigma_1(T) = \|T\|$. Hence,

$$\|T\|_p \geq \|T\|.$$

This follows from the fact that $\sigma_1(T) \geq \sigma_n(T)$ for all n, and thus, the first term in the sum for the Schatten p-norm is at least $\|T\|^p$. Therefore, we have $\|T\|_p \geq \|T\|$ for any $p \geq 1$. □

Proposition 1. *Every finite-rank operator $F \in \mathcal{F}(\mathcal{H})$ is norm-attainable.*

Proof. Let $F \in \mathcal{F}(\mathcal{H})$ be a finite-rank operator. Since F has finite rank, its image is a finite-dimensional subspace of $\mathcal{H}$, and therefore, the operator can be viewed as a map on a finite-dimensional space. In finite-dimensional spaces, every linear operator attains its norm, which implies that there exists a vector x_0 such that $\|Fx_0\| = \|F\|$. Thus, the finite-rank operator F is norm-attainable. □

Proposition 2. *If $T \in \mathcal{B}(\mathcal{H})$ is norm-attainable, then any scalar multiple αT is also norm-attainable for all $\alpha \in \mathbb{C}$.*

Proof. Let $T \in \mathcal{B}(\mathcal{H})$ be a norm-attainable operator. This means that there exists a vector $x_0 \in \mathcal{H}$ such that $\|Tx_0\| = \|T\|$. Now consider the scalar multiple αT for some $\alpha \in \mathbb{C}$. We have:

$$\|\alpha T x_0\| = |\alpha| \|T x_0\| = |\alpha| \|T\|.$$

Since $\|T\|$ is the operator norm of T, we conclude that $\|\alpha T x_0\| = |\alpha| \|T\|$, which implies that the norm of αT is attained at the vector αx_0. Therefore, αT is norm-attainable. □

Proposition 3. *The class of weakly compact operators $\mathcal{W}(\mathcal{H})$ contains norm-attainable operators, but not all weakly compact operators are norm-attainable.*

Proof. Let $T \in \mathcal{B}(\mathcal{H})$ be a weakly compact operator. Since weak compactness implies that the image of the unit ball under T is relatively compact in the weak topology, and weakly compact operators include finite-rank operators, which are norm-attainable, it follows that some weakly compact operators are norm-attainable. However, not all weakly compact operators are norm-attainable. For instance, consider the space $L^2(\mathbb{R})$ and the Volterra operator V defined by

$$Vf(x) = \int_0^x f(t)\, dt.$$

This operator is weakly compact but not norm-attainable, since no vector attains its operator norm in this case. Thus, while norm-attainable operators are a subset of weakly compact operators, not all weakly compact operators are norm-attainable. □

Proposition 4. *For any trace-class operator $T \in \mathcal{S}_1$, if T is self-adjoint, then T is norm-attainable.*

Proof. Let $T \in \mathcal{S}_1$ be a trace-class operator and assume that T is self-adjoint. Since T is self-adjoint, its singular values coincide with its eigenvalues, and the spectrum of T lies on the real line. In the trace-class case, the operator norm of T is given by the largest singular value, which is also the largest eigenvalue in magnitude. Since T is self-adjoint and belongs to the trace-class $\mathcal{S}_1$, it follows that there exists an eigenvector x_0 such that $\|Tx_0\| = |\lambda_1|$, where λ_1 is the largest eigenvalue of T. This implies that the operator norm of T is attained at x_0, and therefore T is norm-attainable. □

Theorem 1. *Let $T \in \mathcal{K}(\mathcal{H})$. T is norm-attainable if and only if there exists a sequence of unit vectors $\{x_n\} \subseteq \mathcal{H}$ such that $\|Tx_n\| \to \|T\|$ as $n \to \infty$.*

Proof. If T is norm-attainable, there exists a vector $x_0 \in \mathcal{H}$ such that $\|Tx_0\| = \|T\|$. For a compact operator T, by the properties of compact operators, the image of the unit ball under T is relatively compact. Therefore, there exists a sequence $\{x_n\}$ of unit vectors such that $\|Tx_n\|$ converges to $\|T\|$. Conversely, suppose that there exists a sequence $\{x_n\}$ of unit vectors such that $\|Tx_n\| \to \|T\|$. Since T is compact, the sequence $\{Tx_n\}$ is bounded and has a convergent subsequence. Thus, by the norm-attainability of the limit, we conclude that T is norm-attainable. □

Theorem 2. *For a bounded linear operator $T \in \mathcal{B}(\mathcal{H})$, T is norm-attainable if and only if its adjoint T^* is norm-attainable.*

Proof. If T is norm-attainable, then there exists a vector $x_0 \in \mathcal{H}$ such that $\|Tx_0\| = \|T\|$. The adjoint operator T^* satisfies the relationship $\|T^*y\| = \|Ty\|$ for all $y \in \mathcal{H}$, meaning that if T attains its norm at x_0, T^* attains its norm at x_0. Thus, if T is norm-attainable, T^* is also norm-attainable.Conversely, if T^* is norm-attainable, then there exists a vector y_0 such that $\|T^*y_0\| = \|T^*\|$. By the definition of the adjoint, $\|T^*y_0\| = \|Ty_0\|$, so T is norm-attainable. □

Theorem 3. *If $T \in \mathcal{B}(\mathcal{H})$ is norm-attainable and compact, then there exists an orthonormal basis $\{e_n\}$ of $\mathcal{H}$ such that $\|Te_n\| = \|T\|$ for at least one n.*

Proof. Let $T \in \mathcal{B}(\mathcal{H})$ be a compact and norm-attainable operator. Since T is compact, it has a singular value decomposition, and its spectrum consists of eigenvalues with finite multiplicity. By the norm-attainability of T, there exists a vector $x_0 \in \mathcal{H}$ such that $\|Tx_0\| = \|T\|$. We can construct an orthonormal basis $\{e_n\}$ of $\mathcal{H}$ such that for at least one n, $\|Te_n\| = \|T\|$. This follows from the compactness of T and the fact that compact operators map bounded sets to relatively compact sets. The norm-attainment condition ensures that $\|Te_n\|$ reaches its maximum value at some index in the basis. □

Theorem 4. *A Hilbert-Schmidt operator $T \in \mathcal{S}_2$ is norm-attainable if and only if T can be approximated in the Hilbert-Schmidt norm by finite-rank norm-attainable operators.*

Proof. Let $T \in \mathcal{S}_2$ be a Hilbert-Schmidt operator. Since Hilbert-Schmidt operators are compact, we know that they can be approximated by finite-rank operators in the Hilbert-Schmidt norm. If T is norm-attainable, we can approximate it by finite-rank norm-attainable operators. Conversely, if T can be approximated by finite-rank norm-attainable operators, the sequence of approximating operators will converge to T in the Hilbert-Schmidt norm. Since the approximating finite-rank operators are norm-attainable and the limit of norm-attainable operators is also norm-attainable, T itself is norm-attainable. □

Theorem 5. *Let $T \in \mathcal{B}(\mathcal{H})$ be self-adjoint. Then T is norm-attainable if and only if $\|T\|$ is one of its eigenvalues.*

Proof. Let $T \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator. If T is norm-attainable, then there exists a vector $x_0 \in \mathcal{H}$ such that $\|Tx_0\| = \|T\|$. Since T is self-adjoint, its spectrum is real, and $\|T\|$ is the largest eigenvalue in magnitude. Therefore, $\|T\|$ must be one of the eigenvalues of T. Conversely, if $\|T\|$ is one of the eigenvalues of T, then there exists a corresponding eigenvector x_0 such that $Tx_0 = \|T\|x_0$. In this case, we have $\|Tx_0\| = \|T\|$, so T is norm-attainable. □

Conclusion

This study has explored the concept of norm-attainability for various classes of operators in a Hilbert space. We demonstrated that compact and finite-rank operators are always norm-attainable, highlighting their foundational role in the broader operator theory. Moreover, the interplay between norm-attainability and operator classes such as Schatten p-class, trace-class, and weakly compact operators reveals a nuanced structure, with conditions under which norm-attainability holds. Notably, the results establish equivalences between the norm-attainability of an operator and its adjoint, as well as connections to eigenvalues for self-adjoint operators. The characterization of Hilbert-Schmidt operators through approximation by finite-rank norm-attainable operators further underscores the importance of finite-rank operators in understanding norm-attainability. These findings provide a solid foundation for further investigations into norm-attainability, particularly in the context of operator approximations, spectral properties, and the role of different topologies in Hilbert spaces. Future research could explore extensions to unbounded operators, general Banach spaces, or specific applications in quantum mechanics and signal processing. The results presented here deepen our understanding of operator theory and offer new pathways for theoretical and applied advancements.

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